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無限型リーマン面間の位相同型と Fuchs 群モデルの同型について

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1. INTRODUCTION

The purpose of this paper is to find conditions for an isomorphism between Fuchsian groups to be geometric. The original result along these lines is due to Fenchel and Nielsen (unpublished).

Theorem (The Fenchel-Nielsen isomorphism theorem). *Let G and H be finitely generated Fuchsian groups, operating on the unit disk D. Let $\psi : G \to H$ be an isomorphism with the following properties.*

(1) $\psi(g)$ is parabolic iff g is.

(2) $\psi(g)$ preserves an arc of discontinuity of H (i.e., a connected component of $\Omega(H) \cap S^1$) iff g preserves an arc of discontinuity of G.

Then there is a homeomorphism f of $\bar{D}$ onto itself so that $f \circ g(x) = \psi(g) \circ f(x)$ for all $g \in G$ and for all $x \in \bar{D}$.

An isomorphism with the property (1) will be called type-preserving. We will attempt to drop the condition that Fuchsians are finitely generated. We will show the following.

Theorem A. *Let G and H be Fuchsian groups with the property that for each arc of discontinuity there is an element ($\neq$id.) which preserves it. Suppose there is a type-preserving isomorphism $\psi : G \to H$ with the property that*

(A) $g_1, g_2 \in G$ have intersecting axes if and only if $\psi(g_1)$ and $\psi(g_2)$ also do.

Then there is a homeomorphism f of $\bar{D}$ onto itself so that $f \circ g(x) = \psi(g) \circ f(x)$ for all $g \in G$ and for all $x \in \bar{D}$. Here the restriction $f|_D$ is a real-analytic diffeomorphism.

We will call the condition (A) above the axes condition. The key tool is the following theorem due to Douady and Earle [1].

Theorem. *Given a homeomorphism $\phi : S^1 \to S^1$, we have an extension $E(\phi) = \Phi : \bar{D} \to \bar{D}$ which is continuous on S^1 and $\Phi|_D$ is a real-analytic diffeomorphism. Moreover, $\phi \mapsto \Phi$ is conformally natural, i.e.,*

$$E(g \circ \phi \circ h) = g \circ E(\phi) \circ h,$$

for all $g, h \in Aut(D)$.

To prove Theorem, we will construct a homeomorphism of S^1 compatible with the Fuchsians. Then, by the theorem of Douady and Earle, we will have a homeomorphism of $\bar{D}$ compatible with the Fuchsians.

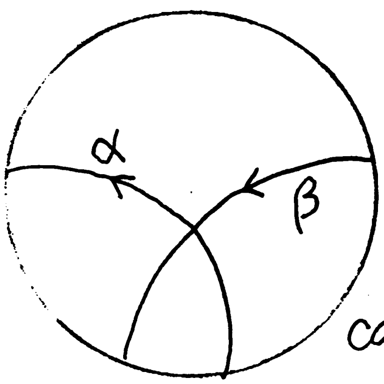
If we ignore the boundary correspondence $S^1 \to S^1$, we have the following theorem (Marden [2], with some restriction, Tukia [3]).

Theorem. *Let G and H be Fuchsian groups. Suppose there is a type-preserving isomorphism $\psi : G \to H$ with the axes condition. Then there is a homeomorphism $f : D \to D$ so that $f \circ g(x) = \psi(g) \circ f(x)$ for all $g \in G$ and $x \in D$.*

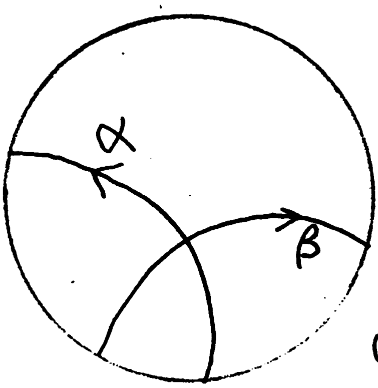
2. Preliminaries

Let $\psi : G \to H$ be a type-preserving isomorphism with axes condition. Put Λ_G be the limit set of G and $\Lambda_{\infty G}$ ($\subset \Lambda$) the set of fixed points of hyperbolic elements of G. For a hyperbolic element $g \in G$, denote by $a(g)$ the attractive fixed point and by $r(g)$ the repelling fixed point. Define $\phi : \Lambda_{\infty G} \to \Lambda_{\infty H}$ to be $\phi(a(g)) = a(\phi(g))$ and $\phi(r(g)) = r(\phi(g))$.

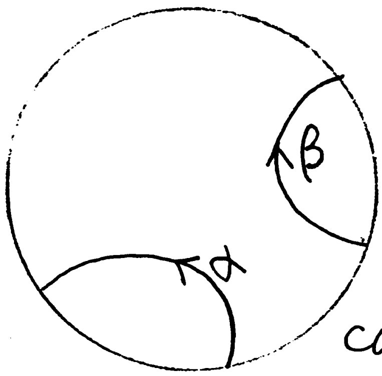
We orient all axes in the direction of the attractive fixed point. Let α and β be two axes of G. Then, the following four cases may occur for their relative position and directions.



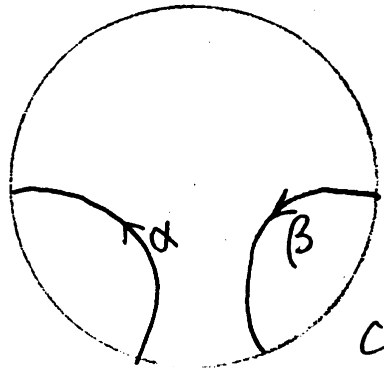
case 1



case 2



case 3



case 4

Let α' and β' be two axes of H corresponded to α and β respectively by ψ. We will say ψ is orientation preserving if α' and β' are of the same case as α and β, namely, ψ preserves their relative position and direction. Otherwise, we will say ψ is orientation reversing. The definition makes sense because Marden [2] showed that if ψ is orientation preserving with respect to a pair of axes it is also orientation preserving with respect to every other choice. (Although he assumed that G and H are finitely generated, his argument is valid for infinitely generated case, too.) From now on, we will only consider the case when Ψ is orientation preserving.Orientation reversing case can be treated similarly.

Definition. Let $\zeta_0, \zeta_1, \zeta_2 \in S^1$ be each other distinct points.We will say that a ordered triple of each other distinct points $(\zeta_0, \zeta_1, \zeta_2)$ is (resp. counter) clockwise oriented if $\zeta_2 \notin \widehat{\zeta_0\zeta_1}$ where $\widehat{\zeta_0\zeta_1}$ is (resp. counter) clockwise oriented arc from ζ_0 to ζ_1.

The following lemma is easily seen.

Lemma 1. *Let $\zeta_0, \zeta_1, \zeta_2 \in \Lambda_{\infty G}$. Then, $(\zeta_0, \zeta_1, \zeta_2)$ is (resp. counter) clockwise oriented if and only if $(\phi(\zeta_0), \phi(\zeta_1), \phi(\zeta_2))$ is (resp. counter) clockwise oriented.*

Let $\zeta_n \in S^1$ $(n = 0, 1, 2, \dots)$. We will say $(\zeta_n)_{n=0}^{\infty}$ is a (resp. counter) clockwise oriented sequence if for an arbitrary $n \in \mathbb{N}$, $(\zeta_0, \zeta_n, \zeta_{n+1})$ is (resp. counter) clockwise oriented.

Lemma 2. *Let $\zeta \in \Lambda_G$. Take a (resp. counter) clockwise oriented sequence $(\zeta_n)_{n=0}^{\infty} \subset \Lambda_{\infty G}$ converges to ζ. Then, $(\phi(\zeta_n))_{n=0}^{\infty}$ is also a (resp. counter) clockwise oriented sequence and it converges to a limit point ζ'. The limit point ζ' is independent of the choice of the sequence $(\zeta_n)_{n=0}^{\infty} \subset \Lambda_{\infty G}$.*

Proof. Lemma 1 implies that $(\phi(\zeta_n))_{n=0}^{\infty}$ is a (resp. counter) clockwise oriented sequence when $(\zeta_n)_{n=0}^{\infty}$ is a (resp. counter) clockwise oriented sequence. $(\phi(\zeta_n))_{n=0}^{\infty}$ converges since it is like a bounded monotone sequence. The uniqueness of the limit point is easy to see. □

Lemma 3. *Let $\zeta \in \Lambda_{\infty G}$. Suppose that there exist a counter clockwise oriented sequence $(\zeta_n)_{n=0}^{\infty} \subset \Lambda_{\infty G}$ converges to ζ and a clockwise oriented sequence $(z_n)_{n=0}^{\infty} \subset \Lambda_{\infty G}$ converges to ζ. Then,* $\lim_{n\to\infty} \phi(\zeta_n) = \lim_{n\to\infty} \phi(z_n) = \phi(\zeta)$.

Proof. Put $\zeta' = \lim_{n\to\infty} \phi(\zeta_n)$ and $z' = \lim_{n\to\infty} \phi(z_n)$. Suppose that $\zeta' \neq \phi(\zeta)$. Then there exists a discontinuous open arc $I \subset S^1$ with end points ζ' and $\phi(\zeta)$, and a hyperbolic element $h \in H$ which fixes ζ' and $\phi(\zeta)$. But there is no element in G which corresponds to $\psi(h)$. Therefore, $\zeta' = \phi(\zeta)$. By the same argument, we see $z' = \phi(\zeta)$. □

2. Proof of Theorem A

Let $\zeta \in \Lambda_G$. Let $(\zeta_n)_{n=0}^{\infty} \subset \Lambda_{\infty G}$ be a counter clockwise oriented sequence converges to ζ and $(z_n)_{n=0}^{\infty} \subset \Lambda_{\infty G}$ be a clockwise oriented sequence converges to ζ. If $\lim_{n\to\infty} \phi(\zeta_n) = \lim_{n\to\infty} \phi(z_n)$, we can define $\phi(\zeta)$ as $\lim_{n\to\infty} \phi(\zeta_n)$. And if, for an arbitrary $\zeta \in \Lambda_G$, the limit of counter clockwise oriented sequences ($\subset \Lambda_{\infty G}$) and that of clockwise oriented sequences ($\subset \Lambda_{\infty G}$) coincide, we observe that, for an arbitrary sequence $(\zeta_n)_{n=0}^{\infty} \subset \Lambda_G$ converges to ζ, $\lim_{n\to\infty} \phi(\zeta_n) = \phi(\zeta)$. This means that, on Λ_G, ϕ is defined without violating continuity. In general, this is not always the case. But in Theorem A, we assume that for each arc of discontinuity there is an element ($\neq$id.) which preserves it. Therefore, this is the case.

Now, we will construct ϕ on the set of arcs of discontinuity. Let ω be a fundamental domain for G. Let I_j be be a component of $\bar{\omega} \cap S^1$ which is a subset of an arc of discontinuity. We denote by $\tilde{I}_j$ the arc of discontinuity I_j belongs to. Then, by the assumption, there is a hyperbolic element $g_i \in G$ which preserves it. For $h_i = \psi(g_i)$, there exists an arc of discontinuity corresponding to it, say, $\tilde{I}'_j$. Take an arbitrary point $z \in \tilde{I}'_j$ and denote by I'_j the arc from z to $h_j(z)$. Then, we define a homeomorphism $\phi : I_j \to I'_j$, for instance, by linearity. On $g(I_j)$ $(g \in G)$, we define $\phi : g(I_j) \to \psi(g)(I'_j)$ by $\phi \circ g(\zeta) = \psi(g) \circ \phi(\zeta)$, $\zeta \in I_j$. Now, we have a homeomorphism $\phi : S^1 \to S^1$. It is easy to see that $\phi \circ g = \psi(g) \circ \phi$ holds for an arbitrary $g \in G$. Applying the theorem of Douady and Earle, we get a homeomorphism $f : \bar{D} \to \bar{D}$, which is our desired result. □

References

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[3] P. Tukia, *On discrete groups of the unit disk and their isomorphisms*, Ann. Acad. Sci. Fenn. AI **504** (1972).