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Kyoto University
The Jørgensen number of the Whitehead link

Hiroki Sato
佐藤 宏樹 (静岡大学理学部) *

Abstract. In this paper we will sketch out the result obtained recently: the Jørgensen number of the Whitehead link is two. Furthermore we will represent points corresponding to the Whitehead link by using the coordinates introduced in Sato [7]. The details will appear in Sato [9].

1. In 1976 Jørgensen obtained the following important theorem called Jørgensen’s inequality, which gives a necessary condition for a non-elementary Möbius transformation group $G = \langle A, B \rangle$ to be discrete.

Theorem A (Jørgensen [1]). Suppose that the Möbius transformations $A$ and $B$ generate a non-elementary discrete group. Then

$$J(A, B) := |\text{tr}^2(A) - 4| + |\text{tr}(ABAB^{-1}) - 2| \geq 1.$$  

The lower bound 1 is best possible.

Definition 1. Let $A$ and $B$ be Möbius transformations. The Jørgensen number

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$J(A, B)$ of the ordered pair $(A, B)$ is defined as

$$J(A, B) := |\text{tr}^2(A) - 4| + |\text{tr}(ABA^{-1}B^{-1}) - 2|.$$ 

We denote by Möb the set of all Möbius transformations. Throughout this paper we will always write elements of Möb as matrices with determinant 1. We recall that Möb (= PSL(2, C)) acts on the upper half space $H^3$ of $\mathbb{R}^3$ as the group of conformal isometries of hyperbolic 3-space. A subgroup $G$ of Möb is said to be elementary if there exists a finite $G$-orbit in $\mathbb{R}^3$.

**Definition 2.** Let $G$ be a non-elementary two-generator subgroup of Möb. The Jørgensen number $J(G)$ for $G$ is defined as

$$J(G) := \inf\{J(A, B) \mid A \text{ and } B \text{ generate } G\}.$$ 

**Definition 3.** A non-elementary two-generator subgroup $G$ of Möb is a Jørgensen group if $G$ is a discrete group with $J(G) = 1$.

**Theorem B (Jørgensen-Kiukka [2]).** Let $\langle A, B \rangle$ be a non-elementary discrete group with $J(A, B) = 1$. Then $A$ is elliptic of order at least seven or $A$ is parabolic.

If $\langle A, B \rangle$ is a Jørgensen group such that $A$ is parabolic, then we call it a Jørgensen group of parabolic type. Here we only consider Jørgensen groups of parabolic type.

2. Let $\langle A, B \rangle$ be a marked two-generator group such that $A$ is parabolic. Then we can normalize $A$ and $B$ as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := B_{\sigma, \mu} = \begin{pmatrix} \mu \sigma & \mu^2 \sigma - 1/\sigma \\ \sigma & \mu \sigma \end{pmatrix},$$

where $\sigma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$. We denote by $G_{\sigma, \mu}$ the marked group generated by $A$ and $B_{\sigma, \mu} : G_{\sigma, \mu} = \langle A, B_{\sigma, \mu} \rangle$. We say that $(\sigma, \mu)$ is the point representing a marked group $G_{\sigma, \mu}$ and that $G_{\sigma, \mu}$ is the marked group corresponding to a point $(\sigma, \mu)$. 
In particular, we consider the case of $\mu = ik$ ($k \in \mathbb{R}$). Namely, we consider marked two-generator group $G_{\sigma,ik} = \langle A, B_{\sigma,ik} \rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := B_{\sigma,ik} = \begin{pmatrix} i\sigma & -k^2\sigma - 1/\sigma \\ \sigma & i\sigma \end{pmatrix},$$

where $\sigma \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{R}$.

3. Let $C$ be the following cylinder: $C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbb{R}\}$.

**Theorem C** (Sato [7]). If a marked two-generator group $G_{\sigma,ik}$ is a Jorgensen group, then the point $(\sigma, ik)$ representing $G_{\sigma,ik}$ lies on the cylinder $C$.

By Theorem C we consider marked two-generator groups $G_{\sigma,\mu} = \langle A, B_{\mu,\sigma} \rangle$ with $\sigma = -ie^{i\theta}$ ($0 \leq \theta < 2\pi$) and $\mu = ik$ ($k \in \mathbb{R}$). For simplicity we set $B_{\theta,k} := B_{\sigma,ik}$ and $G_{\theta,k} = \langle A, B_{\sigma,ik} \rangle$ for $\sigma = -ie^{i\theta}$.

4. There are infinite number of Jorgensen groups (see Jorgensen-Lascurain-Pignataro [3], Sato [7]). The following familiar groups are all Jorgensen groups: The modular group, the Picard group (Jorgensen-Lascurain-Pignataro [3], Sato [8, 9], Sato-Yamada [10]), the figure-eight knot group (Sato [7]), "the Gehring-Maskit group" (Sato [7]), where "the Gehring-Maskit group" is the group studied in Maskit [5]. Namely, we have the following theorem:

**Theorem D** (Jorgensen-Lascurain-Pignataro [3], Sato [7, 8], Sato-Yamada [10]).

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\theta,k} = \begin{pmatrix} ke^{i\theta} & ie^{-i\theta}(k^2e^{2i\theta} - 1) \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}$$

and let $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$ be the group generated by $A$ and $B_{\theta,k}$, where $0 \leq \theta < 2\pi$ and $k \in \mathbb{R}$. Then
(i) \( G_{\pi/2,0} \) is a Jørgensen group.

(ii) \( G_{\pi/2,1/2} \) is a Jørgensen group.

(iii) \( G_{\pi/6,\sqrt{3}/2} \) is a Jørgensen group.

(iv) \( G_{0,\sqrt{3}/2} \) is a Jørgensen group.

**Remark** (1) The groups \( G_{\pi/2,0} \), \( G_{\pi/2,1/2} \), \( G_{\pi/6,\sqrt{3}/2} \) and \( G_{0,\sqrt{3}/2} \) are conjugate to the modular group, the Picard group, the figure-eight knot group and "the Gehring-Maskit group", respectively.

(2) See Sato [7] for other Jørgensen groups of parabolic type.

5. Now it gives rise to the following problem.

**Problem.** Is the Whitehead link a Jørgensen group?

Here we can give the answer to the problem, that is, we have the following theorems.

**Theorem 1** (Sato [9]). *The Jørgensen number of the Whitehead link is two.*

**Corollary** (Sato [9]). The Whitehead link is not a Jørgensen group.

**Theorem 2** (Sato [9]). *The Whitehead link is conjugate to the marked two-generator group \( G_{\sigma,\mu} \) where \( \sigma = \sqrt{2}e^{3\pi i/4} \) and \( \mu = -1/2 \).

6. The proofs of the theorems will appear elsewhere. Here we only give sketches of the proofs.

**Theorem E** (cf. Wielenberg [11], Krushkal’, Apanasov and Gusevskii,[4]). *The Whitehead link \( G_W \) has the following presentation:*

\[
G_W = \langle A, B \mid (A^{-1}BAB^{-1})(ABA^{-1}B^{-1})(AB^{-1}A^{-1}B)(A^{-1}B^{-1}AB) = 1 \rangle,
\]
\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 - i & 1 \end{pmatrix}.
\]

**Proposition 1.** Let \( G_W \) be the Whitehead link defined in Theorem E. Then an element \( X \) of \( G_W \) has the following form:

\[
X = \begin{pmatrix} 1 + (1 - i)a & b_1 + (1 - i)b_2 \\ (1 - i)c & 1 + (1 - i)d \end{pmatrix}.
\]

where \( a, b_1, b_2, c, d \in \mathbb{Z} + i\mathbb{Z}, \ a + d - b_1c + (1 - i)(ad - b_2c) = 0. \)

**Proposition 2.** Let \( G_W \) be the Whitehead link defined in Theorem E and let \((X,Y)\) be a non-elementary subgroup generated by \( X \) and \( Y \), where \( X, Y \in G_W \). Then the Jørgensen number of \((X,Y)\) is greater than or equal to two: \( J(X,Y) \geq 2 \).

**Proposition 3.** Let \( A, B \) be the matrices in Theorem E. Set \( C = AB \). Then \( A \) and \( C \) generate the Whitehead link \( G_W \) and \( J(A,C) = 2 \).

Theorem 1 follows from Propositions 2 and 3.

6. Next we will give a sketch of the proof of Theorem 2.

Let \( P \) be the regular ideal octahedron in Ratcliffe [6, p.454]. Let the sides \( S_A, S_B, S_C, S_D, S_{A'}, S_{B'}, S_{C'} \) and \( S_{D'} \) be the sides of \( P \). Let \( f_A, f_B, f_C \) and \( f_D \) be the side pairing transformations of \( S_A \) to \( S_{A'} \), of \( S_B \) to \( S_{B'} \), of \( S_C \) to \( S_{C'} \), and of \( S_D \) to \( S_{D'} \), respectively.

**Proposition 4.** Let \( f_A, f_B, f_C \) and \( f_D \) be the side pairing transformations defined in the above. Then \( f_A, f_B, f_C \) and \( f_D \) generate the Whitehead link \( G_{W,R} \) in the sense of Ratcliffe.
Proposition 5. Let

\[ G_{W,R}^* = \langle A^*, B^* \mid A^*(B^*)^{-2} A^* B^* (A^*)^{-1}(B^*)^{-1}(A^*)^{-1}(B^*)^2 (A^*)^{-1}(B^*)^{-1} A^* B^* = 1 \rangle, \]

where

\[ A^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 1/2 + i/2 & 3/4 + i/4 \\ -1 + i & 1/2 + i/2 \end{pmatrix}. \]

Then \( G_{W,R}^* \) is conjugate to the Whitehead link \( G_{w,R} \) in the sense of Ratcliffe.

(ii) \( J(A^*, B^*) = 2 \).

Proposition 6. The marked group \( G_{W,R}^* = \langle A^*, B^* \rangle \) in Proposition 5 corresponds to the point \((-1 + i, -1/2)\).

Theorem 2 follows from Propositions 5 and 6.

References


Department of Mathematics
Faculty of Science
Shizuoka University
Ohya Shizuoka 422-8529
Japan
e-mail: smhsato@ipc.shizuoka.ac.jp