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京都大学
SECTIONAL KNOTS IN SEIFERT FIBERED 3-MANIFOLDS

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はじめに

本稿では、2001年12月に京都大学数理解析研究所で行われた研究集会「双曲空間及び
離散群の研究II」において、「Drilling surfaces and surface-automorphisms」という題目
で発表した共同研究の、もう一つの側面であるザイフェルト多様体内的双曲結び目に関
する結果を報告する。尚、研究集会で発表した、曲面に穴をあけるという操作による曲面
上の自己同相写像のNielsen-Thurston分類型の変化に関する結果については、プレプリ
ント[7]を参照のこと。

1. Introduction

The aim of this note is to take another side view of the result of [7]. The preprint [7]
mainly concerns Nielsen-Thurston type of a surface-automorphism and its behavior under
drilling a surface. In this note, as an application of [7], we consider the hyperbolicity of
a knot appearing as a section in a surface bundle over the circle $S^1$ admitting a Seifert
fibration.

We begin with recalling some fundamental definitions and results.

A compact, orientable 3-manifold is called Seifert fibered if it admits a foliation by
circles called Seifert fibers. A Seifert fibered 3-manifold can be regarded as a fiber bundle
over a 2-orbifold with circular fiber. Every Seifert fibered 3-manifold with non-empty
boundary and some of closed ones also admits a fibration over $S^1$ with surface fiber. In
this note, we will mainly deal with such Seifert fibered 3-manifolds. About Seifert fibered
3-manifolds, see [13] for a survey.

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tion, Culture, Sports, Science and Technology, Japan.
As usual, a knot will mean an embedding of $S^1$ or its image in a 3-manifold. We empirically know that ‘most’ knots are hyperbolic, that is, have the complements with a complete hyperbolic metric of finite volume, even in a non-hyperbolic 3-manifold. About hyperbolic knots, see [2] for a survey.

In a Seifert fibered 3-manifold which also fibers over $S^1$, we consider a knot appearing as a section of the fibration, which we call a sectional knot, and ask the next question.

**Question.** In a Seifert fibered 3-manifold fibering over $S^1$, which sectional knot is hyperbolic?

Although no Seifert fibered 3-manifold is hyperbolic, it is conjectured that there exist plenty of hyperbolic sectional knots. We will actually confirm this in some cases by describing when such a knot is hyperbolic in terms of the projection of a knot.

To state our theorems, we prepare some notations, which will be used throughout the article. Let $F$ be a closed, connected, orientable surface and $f$ an orientation preserving automorphism of $F$. Let $M_f$ be a mapping torus with the gluing map $f$, meaning that $M_f = (F \times I)/\{(x, 0) = (f(x), 1)\}$, where $I$ denotes the unit interval $[0, 1]$. This $M_f$ is obviously regarded as a fiber bundle over the circle $S^1$ with fiber $F$. Note that if $f$ is periodic, i.e., some power of $f$ is the identity map of $F$, then $M_f$ is foliated by circles. This gives the unique Seifert fibration of $M_f$ up to isotopy when the genus of the surface $F$ is greater than one [9].

Let $p : F \times I \to F$ be a natural projection and $q : F \times I \to M_f$ a natural quotient map. For a sectional knot $K$ in $M_f$, we call the curve appearing as $p \circ q^{-1}(K)$ on $F$ a projection of $K$. This definition, unlike the usual knot theory, yields a projection which is not a closed curve. To avoid this, if necessary, we isotope $f$ to have at least one fixed point $x_0$ and isotope a sectional knot to run through the point $q(x_0 \times \{0\})$ in $M_f$. Under this setting, every projection of a sectional knot is a (not necessarily simple) closed curve on $F$ containing $x_0$.

When the genus of the surface $F$ is less than two, it is shown that no sectional knots are hyperbolic in $M_f$. A brief observation about this will be given in the next section. Henceforth, except for Section 2, we will always assume that the genus of $F$ is greater than one.

The first theorem, which was essentially obtained by Kra [10], concerns the simplest case that $f$ is the identity map. In this case, $M_f$ is homeomorphic to $F \times S^1$. 
Theorem 1. A sectional knot in $F \times S^1$ is hyperbolic if and only if its projection is stably filling on $F$.

We say that a closed curve on $F$ is stably filling if any curve freely homotopic to it intersects every nontrivial embedded loop on $F$. The proof we will give is based on 3-manifold topology, and so it is quite different from the one induced from [10].

Next, we consider the case that $M_f$ is a small Seifert fibered 3-manifold. A Seifert fibered 3-manifold is called small if it is a circle bundle over a 2-orbifold whose underlying space is the 2-sphere $S^2$ and whose singular set consists of at most three cone points. Note that $M_f$ is small if and only if the gluing map $f$ is irreducible and periodic in the sense of Nielsen-Thurston. In this case, we have the following.

Theorem 2. Let $M_f$ be a small Seifert fibered 3-manifold fibering over $S^1$.

(1) Suppose that $M_f$ has no sectional Seifert fiber. Then every sectional knot in $M_f$ is hyperbolic.

(2) Suppose that $M_f$ has sectional fibers $t_0, t_1, \ldots, t_n$. Let $x_i$ be the point $p \circ q^{-1}(t_i)$ for $0 \leq i \leq n$. Set the point $x_0$ as the base point of projections of sectional knots. Then a sectional knot $K$ in $M_f$ is hyperbolic if and only if no projection of $K$ represents an element of $\pi_1(F, x_0)$ which has the form $[\gamma *(f \circ \gamma)]$ for a path $\gamma$ from some $x_i$ to $x_0$.

In the statement above, $\tilde{\gamma}$ denotes the path obtained from a path $\gamma$ by inverting its orientation. The product of two paths $\gamma_1$ and $\gamma_2$ is denoted by $\gamma_1 * \gamma_2$. For a closed curve $c$ with a base point $x_0$, $[c]$ denotes the element of $\pi_1(F, x_0)$ represented by $c$.

Note that for a sectional Seifert fiber $t$ in $M_f$, $p \circ q^{-1}(t)$ is a fixed point of $f$, and so the case (1) corresponds to the case that $f$ is irreducible and periodic without fixed points.

In the special case that $M_f$ has only one sectional Seifert fiber $t_0$, we immediately have the following corollary. In the following, $f_*$ denotes the automorphism of $\pi_1(F, x_0)$ induced from $f$.

Corollary 1. Let $M_f$ be a small Seifert fibered 3-manifold which fibers over $S^1$ and contains single sectional Seifert fiber $t_0$. Let $x_0$ be the point $p \circ q^{-1}(t_0)$ and set $x_0$ as the base point of projections of sectional knots. Then a sectional knot $K$ in $M_f$ is hyperbolic if and only if no projection of $K$ represents an element of $\pi_1(F, x_0)$ which has the form $[c']^{-1}f_*(c')$ for any closed curve $c'$ with base point $x_0$.

Also note that this corresponds to the case that $f$ is irreducible and periodic with single fixed point.
2. Small genera case

In this section, we give brief observations about the small genera cases.

Suppose that the genus of $F$ equals 0, that is, $F$ is homeomorphic to $S^2$. Then there exists only one $M_f$, that is, $S^2 \times S^1$. It is well-known that every sectional knot in $S^2 \times S^1$ is isotopic to be vertical, that is, the form $\{*\} \times S^1$. This means that every sectional knot in $S^2 \times S^1$ is isotopic to a sectional Seifert fiber and so no one is hyperbolic.

Next, suppose that the genus of $F$ equals 1, that is, $F$ is the torus. In this case, it is known that there exist just five $M_f$'s which are Seifert fibered ([6, Examples 12.3.]). All of these have sectional Seifert fibers. Moreover it is verified that every sectional knot is equivalent to be vertical; there is a self-homeomorphism of the ambient manifold which sends the given sectional knot to a sectional Seifert fiber. Again, in this case, no sectional knot is hyperbolic.

3. Sectional knots in $F \times S^1$

3.1. Outline of the proof of Theorem 1. Throughout this subsection, let $M$ be the mapping torus with trivial gluing map; $(F \times I)/\{(x, 0) = (x, 1)\}$, $K$ a sectional knot in $M$, $c$ a projection of $K$ on $F$ and $E(K)$ the exterior $M - \text{Int}N(K)$.

Let us first show that 'only if' part. For a contradiction, suppose that $K$ is hyperbolic and $c$ is freely homotopic to a closed curve $c'$ which avoids some nontrivial simple loop $\epsilon$. There is the sectional knot $K'$ in $M$ which has $c'$ as a projection. In $M - \text{Int}N(K')$, there is a vertical torus $T_\epsilon = q(\epsilon \times I)$. This $T_\epsilon$ is essential (i.e., incompressible and not boundary parallel) since $\epsilon$ is nontrivial on $F$. Note that the free homotopy between $c$ and $c'$ implies that an isotopy between $K$ and $K'$, and it extends to an ambient isotopy of $M$ which moves $K$ to $K'$. In particular, $E(K)$ is homeomorphic to $M - \text{Int}N(K')$. Therefore we find an essential torus in $E(K)$. This contradicts that $K$ is hyperbolic.

Next, let us consider the 'if' part. Suppose that $c$ is stably filling on $F$ with the base point $x_0$. By the Thurston's Uniformization Theorem [14], it suffices to show that (1) $E(K)$ contains no essential tori and (2) $E(K)$ is not Seifert fibered.

Suppose for a contradiction to (1) that $E(K)$ contains an essential torus $T$.

Let $F_t$ be the surface $q(F \times \{t\})$ in $M$ for $t \in I$ and $\tilde{F}_t$ the surface $F_t \cap E(K)$ in $E(K)$. Since $\tilde{F}_0(= \tilde{F}_1)$ is incompressible in $E(K)$, by an isotopy of $T$ in $E(K)$, we assume that the intersection $T \cap F_0$ consists of non-empty nontrivial loops in both $T$ and $F_0$. Also we assume that the number of components of $T \cap F_0$ is minimal. The preimage $q^{-1}(T)$ of $T$ are the disjoint union of annuli $A_1, \ldots, A_n \subset F \times I$. 

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Claim 1. For each $A_i$ ($i = 1, \ldots, n$), one boundary component is in $F \times \{0\}$ and the other is in $F \times \{1\}$.

Proof. Remark that these annuli is incompressible in $F \times I - q^{-1}(K)$ since $T$ is essential. If the boundary of one of them is entirely contained in $F \times \{0\}$ or $F \times \{1\}$, then by [15, Corollary 3.2] it is boundary parallel, and so it contradicts the minimality of the number of components of $T \cap F_0$.

The following lemma therefore implies that each annulus $A_i$ ($i = 1, \ldots, n$) is also incompressible in $F \times I$.

Lemma 3.1.1. Let $F$ be a closed orientable surface and $K'$ an monotone arc in $F \times I$ connecting $(x_0, 0)$ and $(x_0, 1)$ for some point $x_0 \in F$. Let $A$ be an incompressible annulus in $E(K') = F \times I - \text{int}N(K')$ with one boundary component in $F \times \{0\}$ and the other in $F \times \{1\}$. If $A$ is compressible in $F \times I$, then $A$ is parallel to the frontier of $N(K')$ in $F \times I$.

Moreover the following holds in this case.

Claim 2. $p(A_1 \cap (F_0)) = p(A_1 \cap (F_1))$ holds. In particular, the number $n$ of annuli is equal to 1.

Let $c_1$ denote the curve $p(\partial A_1)$ for this single annulus $A_1$.

Claim 3. There is an isotopy of $F \times I$ such that $A_1$ is moved to the vertical annulus $c_1 \times I$ and it is identity on the surfaces $F \times \{0\}$ and $F \times \{1\}$.

For proofs of these claims, see [7].

Under the isotopy above, the arc $q^{-1}(K)$ is moved to an arc $k$ keeping the endpoints fixed. Thus there is a homotopy between $c = p(q^{-1}(K))$ and $p(k)$ on $F$. Since the original $A_1$ is disjoint from $q^{-1}(K)$, the annulus $c_1 \times I$ does not intersect $k$, and hence $p(k) \cap c_1 = \emptyset$. However this contradicts that $c$ is stably filling on $F$.

Suppose for a contradiction to (2) that $E(K)$ is Seifert fibered. Then there exists a Seifert fibration of $M$ in which $K$ is a fiber by the the next lemma.

Lemma 3.1.2. Let $M$ be an irreducible manifold and $k$ is a knot in $M$. If $M - \text{Int}N(k)$ is Seifert fibered, then $M$ admits a Seifert fibration in which $k$ is a fiber.

Here each Seifert fiber of $M = F \times S^1$, up to free isotopy, is the form $\{*\} \times S^1$, and so it is sectional.

Finally we use the following:
Lemma 3.1.3. Let $M_f$ denote the mapping torus $(F \times I)/\{(x, 0) = (f(x), 1)\}$ with the periodic gluing map $f$ which has a non-empty fixed point set $\{x_0, \ldots, x_n\}$. Let $K$ be a sectional knot $K$ in $M_f$ and $c$ a projection of $K$. Suppose that $K$ is isotopic to a sectional Seifert fiber of the form $((x_i, 1)/\{(x_i, 0) = (x_i, 1)\}$ if and only if $[c] = [\gamma \ast (f \circ \gamma)]$ in $\pi_1(F, x_0)$ for some path $\gamma$ from $x_i$ to $x_0$, where $\ast$ denotes the product of (possibly non-closed) paths.

In our case, the gluing map $f$ is the identity map, and so we have
$$[c] = \alpha^{-1} f_\ast(\alpha) = \alpha^{-1} \alpha = 1$$
for some $\alpha \in \pi_1(F, x_0)$. However, since a stably filling curve $c$ is nontrivial, it is absurd.

3.2. Examples of curves. The purpose of this subsection is to give concrete examples of stably filling curves on $F$. Recall that a closed curve $c$ on a closed orientable surface $F$ is called filling if every connected component of $F - c$ is an open disk. Note that $c$ is stably filling if and only if every closed curve freely homotopic to $c$ is filling.

In the following, we fix a hyperbolic metric on $F$. As shown in [1] implicitly, we have:

Lemma 3.2.1. A filling closed geodesic is stably filling.

Let us give examples of stably filling curves on a surface of genus two. Similarly, one can construct such examples for the higher genus case. See [7] for a detail.

By virtue of Lemma 3.2.1, it suffices to find filling geodesics. We start with four copies of a regular truncated triangle, equivalently, a right-angled rectangular hexagon, in $\mathbb{H}^2$. By gluing their edges suitably, we obtain two copies of a pair of pants $P_1$ and $P_2$ with equilong boundaries. On $P_1$, take three geodesic arcs each one of which is the shortest path connecting distinct boundary components. Remark that each boundary component of $P_1$ is bisected by the endpoints of these arcs. On $P_2$, take also three geodesic arcs each one of which is the return path of a boundary component. Remark that each boundary component of $P_2$ again is bisected by the endpoints of these arcs. Then $P_1$ and $P_2$ are glued so that the six geodesic arcs form single closed curve $c$. For these arcs match geodesically as they are all orthogonal to the boundary. It is easily checked that this $c$ is actually filling on the resultant surface of genus two.

We can find a simple closed curve $c'$ on the resultant surface which intersects $c$ exactly once. By performing the $m$-Dehn twist along $c'$, we obtain infinitely many curves $c_n$ which is also stably filling. These are all mutually non-isotopic. In fact, $c_m$ and $c_n$ $(m \neq n)$ are not homologous. For one can find a simple closed curve $c''$ intersecting $c'$
exactly once, and then the algebraic intersection number of \( c_m \) and \( c'' \) varies depending only on \( m \).

4. Sectional knots in a small Seifert fibered 3-manifold

4.1. Outline of the proof of Theorem 2. Note that Theorem 2 is an immediate corollary of the next proposition together with Lemma 3.1.3.

**Proposition 1.** Let \( K \) be a sectional knot in a small Seifert fibered 3-manifold which fibers over \( S^1 \). Then \( K \) is hyperbolic if and only if \( K \) is isotopic to none of sectional Seifert fibers.

*Proof.* Let \( M_f \) denote a small Seifert fibered 3-manifold, equivalently, we assume that \( f \) is an irreducible, periodic surface-automorphism. It is known that \( M_f \) is *atoroidal*, that is, \( M_f \) contains no essential tori [8]. Let \( K \) be a sectional knot in \( M_f \), \( c \) a projection of \( K \) and \( E(K) \) the exterior \( M_f - \text{Int}N(K) \).

If \( K \) is isotopic to a sectional Seifert fiber, then \( E(K) \) admits a Seifert fibration obviously, and so \( K \) is not hyperbolic.

Conversely suppose that \( K \) is not hyperbolic. If \( E(K) \) contains an essential torus, then it remains incompressible in \( M_f \) by Lemma 3.1.1. This contradicts that \( M_f \) is small, and so \( E(K) \) is atoroidal. Then, by the Thurston’s Uniformization Theorem [14], there is a Seifert fibration on \( E(K) \). By Lemma 3.1.2, the ambient manifold \( M_f \) also admits a Seifert fibration in which \( K \) is a fiber. Since the Seifert fibration of such \( M_f \) is unique up to isotopy [9], \( K \) is isotopic to a Seifert fiber which is sectional. \( \square \)

4.2. Examples of maps. In this subsection, we give examples of irreducible, periodic automorphisms corresponding to Theorem 2 (1) and Corollary 1.
To do this, we give an observation about the covering theory. See [3] for a detail. Let us denote an orbifold with the underlying space \( S^2 \) and three cone points \( b_1, b_2 \) and \( b_3 \) of indices \( p_1, p_2 \) and \( p_3 \) by \( S^2(p_1, p_2, p_3) \). Put \( n = \text{lcm}(p_1, p_2, p_3) \). Suppose that a surjective representation \( \rho : \pi_1(S^2 - \{b_1, b_2, b_3\}) \rightarrow \mathbb{Z}_n \) which sends a small loop around \( b_i \) to an element of order \( p_i \) is given. Then by taking an \( n \)-fold cyclic orbifold covering associated with \( \rho \), we obtain a closed orientable surface \( F \) of genus \( g = \frac{1}{2} n (1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3}) + 1 \) and a periodic automorphism \( f \) so that \( F/\langle f \rangle = S^2(p_1, p_2, p_3) \) ([3], [11]). This \( f \) so obtained is irreducible by [4, Theorem 3.1]. Algebraic conditions for an existence of such branched coverings (equivalently such representations) was given in [5].

We remark that an irreducible, periodic automorphism \( f : F \rightarrow F \) of period \( p \) fixes at most three points. For if we take a quotient of \( F \) by the action \( \langle f \rangle \), then we obtain an orbifold \( S^2(p_1, p_2, p_3) \) [4, Theorem 3.1]. If \( f \) fixes a point \( x_i \in F \), then \( x_i \) is injectively projected to a cone point \( b_i \) of index \( p \). This implies that \( f \) fixes at most three points on \( F \).

**Example 1.** Choose an orbifold \( S^2(m_1 + m_2, m_1(m_1 + m_2), m_1m_2) \), where \( m_1 \) and \( m_2 \) are coprime. Take a cyclic orbifold covering associated with a representation \( \rho : \pi_1(S^2 - \{b_1, b_2, b_3\}) \rightarrow \mathbb{Z}_{m_1m_2(m_1+m_2)} \) which sends a small loop around \( b_i \) to \( m_i \) (\( i = 1, 2 \)). Then we obtain a closed orientable surface \( F \) of genus \( \frac{1}{2}(m_1 + m_2)(m_1m_2 - 2) + 1 \) and an irreducible periodic automorphism \( f : F \rightarrow F \) of period \( m_1m_2(m_1 + m_2) \). Since each branching index is strictly smaller than the period, \( f \) has no fixed point. This gives an example of Theorem 2 (1).

The simplest case of \( (m_1, m_2) = (2, 3) \) was described in [11, Proposition 1]. In this case, the genus of the resultant surface is 11 and the period of the resultant automorphism is 30.

**Example 2.** Choose an orbifold \( S^2(m_2, m_1, m_1m_2) \), where \( m_1 \) and \( m_2 \) are coprime. Take a cyclic orbifold covering associated with a representation \( \rho : \pi_1(S^2 - \{b_1, b_2, b_3\}) \rightarrow \mathbb{Z}_{m_1m_2} \) which sends a small loop around \( b_i \) to \( m_i \) (\( i = 1, 2 \)). Then we obtain a closed orientable surface \( F \) of genus \( \frac{1}{2}(m_1 - 1)(m_2 - 1) \) and an irreducible periodic automorphism \( f : F \rightarrow F \) of period \( m_1m_2 \). The cone point \( b_3 \) has the branching index \( m_1m_2 \) which is equal to the covering index. Thus the preimage of \( b_3 \) consists of exactly one point and this is a unique fixed point of \( f \). This gives an example of Corollary 1.

In fact, the resultant automorphism is the monodromy of the surface bundle over \( S^1 \) obtained by 0-surgery along \( (m_1, m_2) \)-torus knot in \( S^3 \).
4.3. Examples of curves. In this subsection, we see that there are plenty of curves described in Corollary 1.

Let \( f : F \to F \) an orientation preserving periodic automorphism satisfying \( f(x_0) = x_0 \) for some point \( x_0 \in F \). We assume that the period \( p \) of \( f \) is greater than two. For the period two case, please see [7].

Hereafter, fix an \( \langle f \rangle \)-invariant hyperbolic metric on \( F \); for an existence of such a metric, see [13, Section 2]. Then each element in \( \pi_1(F, x_0) \) is represented by a geodesic closed path (i.e., a curve \( c : I \to F \) such that \( c(0) = c(1) = x_0 \) and \( c|_{(0,1)} \) is geodesic).

Actually we show the following.

**Proposition 2.** Let \( f : F \to F \) be a periodic automorphism of period \( p > 2 \). Then there exists a positive constant \( C_f \) depending only on \( f \) such that an element \( \gamma \) or \( \gamma^{-1} \) in \( \pi_1(F, x_0) \) cannot be written as \( \alpha^{-1} f_*(\alpha) \) for any element \( \alpha \in \pi_1(F, x_0) \) if the length of the geodesic closed path representing \( \gamma \) is greater than \( C_f \).

By this proposition, the element of \( \pi_1(F, x_0) \) which and whose inverse are both represented as \( \alpha^{-1} f_*(\alpha) \) with some \( \alpha \in \pi_1(F, x_0) \) must have the geodesic closed path with bounded length as a representative. Since the holonomical image of \( \pi_1(F, x_0) \) is discrete, there exist only finite number of such elements.

In the following, we outline the proof of Proposition 2. See [7] for a detail.

Let \( c \) be a geodesic closed path on \( F \). We denote by \( c^{-1} \) the geodesic closed path defined by \( c^{-1}(t) = c(1 - t) \) for \( t \in I \) and by \( \theta_{f,c} \) the angle from \( \dot{c}(1) \) to \( df_{x_0}(\dot{c}(0)) \) for a geodesic closed path \( c \), where \( -\pi < \theta_{f,c} \leq \pi \).

Set a positive constant \( \Theta_f \) depending only on \( f ; 0 < \Theta_f < \pi \), as follows.

\[
\Theta_f = \begin{cases} 
\pi - \theta_f & 0 < \theta_f \leq \pi/2 \\
\theta_f & \pi/2 < \theta_f < \pi \\
2\pi - \theta_f & \pi < \theta_f \leq 3\pi/2 \\
\theta_f - \pi & 3\pi/2 < \theta_f < 2\pi 
\end{cases}
\]

where \( \theta_f \) denotes the rotation angle \( (0 < \theta_f < 2\pi) \) of the action of \( df_{x_0} \) on \( T_{x_0}F \). Remark that \( \theta_f \neq \pi \) by the assumption that \( p > 2 \).

Then the next is proved by careful case-by-case arguments.

**Claim 4.** For any geodesic closed path \( c \), we have \( \min\{|\theta_{f,c}|, |\theta_{f,c^{-1}}|\} \leq \Theta_f \). \( \square \)

Moreover we have:
Claim 5. Suppose that $|\theta_{f,c}| \leq \Theta_{f}$ holds. If

$$\cosh l_c/2 \geq \frac{1}{\sin(\pi - \Theta_{f})/2}$$

holds, then the product $[c][f \circ c] \cdots [f^{p-2} \circ c][f^{p-1} \circ c]$ is not trivial in $\pi_{1}(F,x_{0})$, where $l_c$ denotes the length of $c$.

The key of the proof of this claim is to show the preimage of a geodesic closed path in the universal cover $\mathbb{H}^{2}$ of $F$ is not closed if the assumption satisfied. See [7] for a detail.

Finally, we use the next claim, which is proved by elementary algebraic calculations.

Claim 6. If $[c][f \circ c] \cdots [f^{p-2} \circ c][f^{p-1} \circ c] \neq 1$ in $\pi_{1}(F,x_{0})$, then $[c]$ cannot be written as $\alpha^{-1}f_{*}(\alpha)$ for any $\alpha \in \pi_{1}(F,x_{0})$.

Consequently, we obtain a desired constant $C_{f} = 2 \cosh^{-1}(\frac{1}{\sin(\pi - \Theta_{f})/2})$.

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