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Operator algebras and topological quantum field theory

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Abstract

We make a survey on recent progresses in 3-dimensional topological quantum field theories arising from operator algebras. The main focus is on the Reshetikhin-Turaev invariants arising from the coset models, as studied by F. Xu.

1 Introduction

Interactions between low-dimensional topology and operator algebras have been quite fruitful in the last two decades since the discovery of the Jones polynomial. Our aim here is to review the recent advance of these interactions. Probably, the most detailed studies of quantum invariants of links and 3-manifolds so far from the operator algebraic viewpoint have been through Ocneanu's generalization of the Turaev-Viro invariants as explained in [6, Chapter 12], but Sato and Wakui have already presented their work on this topic in this RIMS project, so we will make a review on different topics, the Reshetikhin-Turaev type invariants arising from operator algebraic studies of quantum fields. This is mainly due to F. Xu [24, 25].

2 Modular tensor categories arising from operator algebras

The Reshetikhin-Turaev type invariants gives an invariant of 3-dimensional closed manifolds from a modular tensor category as explained in [20]. We first discuss how a modular tensor category appears naturally in the framework of alegbraic quantum field theory [8], which is a study of quantum field theory through operator algebraic methods.

In a modular tensor category, each object is something like a representation of some algebraic structure and we have notions such as a tensor product, irreducible decomposition, and a (quantum) dimension. We show how such a category is realized in the current setting.

First we recall a general background. Let A be an algebra of bounded linear operators on a fixed Hilbert space H, where we usually assume that H is separable and infinite dimensional. We also assume that A is closed under the $*$-operation. We further require that A is closed under an appropriate topology. Actually, we have two choices for an "appropriate topology". One is the norm topology and the other is the strong operator topology. In this note, it is more convenient to use the latter. In this case, such an algebra A of operators is called a von Neumann algebra. In order to avoid technical problems, it is simpler to assume that the algebra A is simple in the sense that it does not have a non-trivial closed two-sided ideal. Such an algebra A is called a factor, though a terminology "simple von Neumann algebra" would be easier to understand. This simplicity property is equivalent to triviality of the center of the algebra A. The most naive approach to representation theory in the framework of operator algebra theory would be a study of representations of such a factor on different Hilbert spaces from H, but such a theory is rather trivial, unfortunately. In a natural setting in connection to quantum field theory, a factor A becomes a so-called type III factor and then, all representations on separable Hilbert spaces are unitarily equivalent. So we need something else in order to get a sensible representation theory.

In the setting of algebraic quantum field theory, we assign a von Neumann algebra $A(O)$ for each appropriate region O in the spacetime. This algebra is generated all the "observables" in the spacetime region O. We now take the circle S^1 as a compactified spacetime, though the name "spacetime" would not be so suitable for this case. Then as a region O, we take a non-empty, non-dense, open connected set I, which is called an *interval.* So we have an assignment $A(I)$ of a von Neumann algebra on a fixed Hilbert space H to each such an interval I. One might think that one-dimensional "spacetime" is too trivial, but many mathematically interesting phenomena related to low-dimensional topology such as braiding arise only in low-dimensional "spacetime" and the one-dimensional theory is quite deep. We have a set of axioms this assignment should satisfy, based on physical reasons. Here we briefly explain some of the axioms. See [15], for example, for a complete description of the axioms.

For intervals $I \subset J$, we require $A(I) \subset A(J)$. Since $A(I)$ should be an algebra of "observables" on a bounded spacetime region I, this requirement is quite natural. We then require that $xy = yx$ for $x \in A(I)$, $y \in A(J)$ if I and J are disjoint. The origin of this requirement is that if two spacetime regions are "spacelike", then the observables on these regions have no influence on the other, thus the operator must commute. Now we are in a one-dimensional situation and use disjointness of the intervals instead of the spacelike condition. By this physical reason, this axiom is called *locality*. We also require that we have a (projective) unitary representation u_g of the "symmetry group" G of the space time. As this group G, we now take the Möbius group $PSL(2, \mathbf{R})$. (We also often take the Poincaré group of the Minkowski space as G in a higher dimensional case.) Then we assume $u_g A(I) u_g^* = A(gI)$. We also assume existence of a special vector called the *vacuum vector*, unique up to scalars. For an interval I, we denote the interior of its complement by I'. Then the standard axioms imply that we have $A(I') = A(I)'$, where the right hand side means $\{y \mid xy = yx, \quad \forall x \in A(I)\}$ by definition and is called the *commutant* of $A(I)$. This property means the locality holds in a maximal sense, and it is often called the *Haag duality*. The uniqueness of the vacuum vector implies that each von Neumann algebra $A(I)$ is a factor, actually an algebra called a hyperfinite III_1 factor which is unique up to isomorphism.

One example of such a family $\{A(I)\}_I$ of operator algebras constructed by A. Wassermann [21] is as follows. Consider the loop group $LSU(N)$ of $SU(N)$. Their positive energy representations give a "fusion category" for each fixed level k as in [18]. Now for a vacuum representation π of level k, we can define $A(I)$ to be the operator algebra generated by $\pi(f)$'s with $f \in LSU(N)$ being identity outside of the interval I. Wassermann [21] has shown that this net $\{A(I)\}_I$ satisfies the above axioms and a general positive energy representation of $LSU(N)$ of level k corresponds to a representation of the net $\{A(I)\}_I$ in the sense below. In this way, we can capture the usual tensor category of the WZW-model $SU(N)_k$ in the framework of algebraic quantum field theory.

Now we explain the representations of the net $\{A(I)\}_I$. These von Neumann algebras act on a Hilbert space H from the beginning by definition, but we also consider representations of the net, which are families of representations π_I of $A(I)$ with a natural compatibility condition, on another Hilbert space. This is a quite natural notion of a representation, but it is not clear at all how to define a "tensor product" of two such representations. (Note that we have no coproducts now.) In order to define a tensor product, it is useful to rewrite the definition of a representation using an endomorphism. That is, fix an interval I. Then with a change of representations within a unitary equivalence class, we can always assume that a representation π acts on the initial Hilbert space H and $\pi(x) = x$ if $x \in A(I')$. Then by the consequence of the Haag duality, this π restricted on $A(I)$ gives an endomorphism of $A(I)$. If we have two representations π and σ realized in this way, we can compose π and σ as endomorphisms of $A(I)$. This composition defines a notion of a "tensor product" of representations of π and σ. A special endomorphism arising from a representation as above is called a Doplicher-Haag-Roberts (DHR) endomorphism. (We omit exact properties of the DHR endomorphisms. See [8], for example.) Then we can also define notions of a conjugate endomorphism which corresponds to a contragredient representation, a (quantum/statistical) dimension which now takes a value in $[1, \infty]$, irreducible decomposition for these DHR endomorphisms. The dimension of an endomorphism σ is the square root of the Jones index of an inclusion $\sigma(A(I)) \subset A(I)$. (Actually, they are defined for general endomorphisms of opeator algebras called type III factors. See Longo [14]. The notion of the Jones index is an analogue of an index of a subgroup or a degree of an extension of a field.) In this way, we have a tensor category of DHR endomorphisms where irreducible objects are DHR endomorphisms which do not decompose into direct sums of endomorphisms. Note that we have no reason to expect $\pi\sigma = \sigma\pi$ here, though tensor products for group representations are commutative. But in the setting of the DHR endomorphisms, we do have commutativity up to unitary equivalence, that is, we have $\mathrm{Ad}(u)\pi\sigma = \sigma\pi$ and this unitary u, depending on π, σ gives a *braiding* structure. In this way, the category of DHR endomorphisms of a net becomes braided. It is at this point that low-dimensionality of the spacetime plays an important role.

For a construction of a Reshetikhin-Turaev invariant, we are interested in the situation where we have only finitely many irreducible objects. Such a situation is often called a *rational* theory. We now give an operator algebraic condition which implies this rationality and, furthermore, modularity of the tensor category.

Split the circle into four intervals and label them I_1, I_2, I_3, I_4 in a counterclockwise order. Then both $A(I_1)$ and $A(I_3)$ commute with $A(I_3)$ and $A(I_4)$ and thus we have $A(I_1) \vee A(I_3) \subset (A(I_3) \vee A(I_4))'$, where both algebra are actually factors. This inclusion of factors has the Jones index in $[1, \infty]$ and we call it the *μ-index* of the net $\{A(I)\}_I$. Our results in [12] says that if the μ-index of a net is finite, then this net has only finitely many unitary equivalence classes of representations, they have all finite dimensions, and the braided category of the DHR endomorphisms of the net is modular in the sense of [20]. (The modularity condition means invertibility of the S-matrix defined with the braiding as in [19].) Note that this modularity is often difficult to show in other approaches to tensor categories and it is quite convenient to show this with an operator algebraic method. In this case, we say that the net is *completely rational.* The above example of $SU(N)_k$ is completely rational by a result of Xu [23].

So we can construct a Reshetikhin-Turaev invariant of 3-manifolds from a completely rational net. We study relations of two such invariants when the two nets have some operator algebraic relations. For this purpose, we first consider a rather simple situation. When a factor is contained in another factor, we call it a *subfactor.* We consider a family of subfactors $A(I) \subset B(I)$ parametrized by the intervals on S^1 as above. We call it a *net of subfactors.* A systematic study of such nets of subfactors was started in [16]. We can define the Jones index of a net of subfactors as that of $A(I) \subset B(I)$, which is independent of I. Under the assumption of finite Jones index, Longo [15] has shown that if one of the two nets $\{A(I)\}_I$ and $\{B(I)\}_I$ is completely rational, so is the other. An example of a net of subfactors with complete rationality is given by conformal inclusions as in [22]. Also the orbifold construction gives a net of subfactors with complete rationality as in [26].

For a net of subfactors with finite index and complete rationality, it is expected that we have some relations between the representation theories of the two nets, as we have relations between the representation theories of a (compact) group and its subgroup. As a tool to study such relations, we explain *α-induction* which produces an (almost) representation of the larger net of factors from a representation of the smaller one. (The name "induction" comes from analogy to group representations.) This method was first defined in [16] based on an old suggestion of Roberts, and its interesting properties were studied in detail by Xu [22]. It was further studied in [1], [2], [3], [4], [5], partly in connection to [17]. For a net of subfactors $\{A(I) \subset B(I)\}_I$ and fixed interval I, take a DHR endomorphism λ of the net $\{A(I)\}_I$. Then using a braiding, we can extend this endomorphism of $A(I)$ to that of $B(I)$. Since this extension does depend on which of the two, mutually opposite, braiding we use, we denote this dependence by the symbol $\pm$. The extended endomorphism is thus denoted by $\alpha^\pm_\lambda$. This is *not* a DHR endomorphism of the larger net $\{B(I)\}_I$ in general, but irreducible endomorphisms arising from irreducible decompositions of α^+_λ's produces a tensor category, which has no braiding in general. But if we restrict our attention to the extended endomorphisms which arise from both α^+_λ and α^-_μ for some DHR endomorphisms λ, μ of the subnet $\{A(I)\}_I$, we do get a DHR endomorphism of the larger net $\{B(I)\}_I$ and all DHR endomorphisms of the larger net $\{B(I)\}_I$ arise in this way. Although we use a name *induction*, the tensor category of the representations of the larger net is *smaller* in an appropriate sense. See the above-cited papers for various properties and example of α-induction.

3 Coset models

We now focus on a particular construction of a (completely rational) net of factors from given nets of factors and the corresponding Reshetikhin-Turaev invariant. This is based on Xu's work [25].

Consider a net of subfactors $\{A(I) \subset B(I)\}_I$ again, but now with infinite Jones index. We can then consider a net of factors $\{A(I)' \cap B(I)\}_I$. We assume that the larger net $\{B(I)\}_I$ is completely rational and the index of a subfactor $A(I) \vee (A(I)' \cap B(I)) \subset B(I)$ is finite. Then the net $\{A(I)' \cap B(I)\}_I$ is also completely rational by the above-mentioned result of Longo. We call this net the *coset* net of $\{A(I) \subset B(I)\}_I$. In a usual setting, we know about the representation theories of the two nets $\{A(I)\}_I$ and $\{B(I)\}_I$ and want to find the representation theory of the coset net $\{A(I) \subset B(I)\}_I$.

One example in [25] is given as follows. Let $\{A(I)\}_I$, $\{B(I)\}_I$ be the nets corresponding to $SU(N)_{m+n}$, $SU(N)_m \times SU(N)_n$. Then the diagonal embedding of $SU(N) \subset SU(N) \times SU(N)$ produces a net of subfactors $\{A(I) \subset B(I)\}_I$. Now a result in [24] says that we have a (not necessarily irreducible) DHR endomorphism of the coset net labeled with (π, σ), for irreducible DHR endomorphisms σ, π of the nets $\{A(I)\}_I$, $\{B(I)\}_I$, respectively. Now the irreducible DHR endomorphisms are labeled with $l = 0, 1, \ldots, m+n$ and those of $\{A(I)\}_I$ are with (j,k) with $j = 0, 1, \ldots, m$ and $k = 0, 1, \ldots, n$. For simplicity, consider $SU(2)_{m-1} \subset SU(2)_{m-2} \times SU(2)_1$. Then $j = 0, 1, \ldots, m-1$, $k = 0, 1$, $l = 0, 1, \ldots, m-1$. So the above pair (π, σ) is represented with a triple (j, k, l) with a condition $j + k - l \in 2\mathbf{Z}$. Since $k = 0, 1$ is uniquely determined by the pair (j, l) and the condition $j + k - l \in 2\mathbf{Z}$, we may and do denote the triple (j, k, l) by a pair (j, l). It turns out each such

DHR endomorphism of the coset is irreducible and all the irreducible endomorphisms of the coset arise in this way. Furthermore, the pair (j,l) and (j',l') represents unitarily equivalent endomorphisms if and only if $(j,l)=(j',l')$ or $j+j'=m-2, l+l'=m-1$. For example, for $m=4$, we have six irreducible, mutually inequivalent DHR endomorphisms. Actually, one can show that this modular tensor category corresponds to the Virasoro algebra at central charge $1-6/m(m+1)$. (See [11] on this matter related to the Virasoro algebra.)

In general, for a coset net $\{A(I)'\cap B(I)\}_I$, we have a (possibly reducible) endomorphism labeled with a pair (π,σ) where σ,π are irreducible DHR endomorphisms of $\{A(I)\}_I$, $\{B(I)\}_I$, respectively.

Now recall a Reshetikhin-Turaev invariant arising from a modular category. Roughly speaking, we first realize a 3-manifold with a Dehn surgery along a link in S^3 and consider a weighted sum of colored link invariants where each "color" is given by an irreducible object of the tensor category. One can show that this complex number is independent of the link we choose and indeed an invariant of a manifold. (See [20] for details of the definition.) Xu [25] first considered a colored link invariant arising from a coset model.

Suppose a link L has k connected components. Then he has shown

$$L((\pi_1,\sigma_1),(\pi_2,\sigma_2),\ldots,(\pi_k,\sigma_k))=L(\pi_1,\pi_2,\ldots,\pi_k)\overline{L(\sigma_1,\sigma_2,\ldots,\sigma_k)},$$

where π_j,σ_j denote irreducible DHR endomorphisms of the net $\{A(I)\}_I$, $\{B(I)\}_I$, respectively, and (π_j,σ_j) denote a *not necessarily irreducible* DHR endomorphism of the coset net $\{A(I)'\cap B(I)\}_I$. (The symbol $L(\pi_1,\pi_2,\ldots,\pi_k)$ denotes a colored link invariant arising from the net $\{A(I)\}_I$ where the k components are colored with $\pi_1,\pi_2,\ldots,\pi_k$, respectively. The other two colored link invariants are interpreted similarly.) Then, one might expect a simple relation among the three Reshetikhin-Turaev invariants arising from these three nets, such as $\tau_{A'\cap B}(M)=\tau_B(M)\overline{\tau_A(M)}$, where $\tau_A(M)$ is the Reshetikhin-Turaev invariant of a closed oriented 3-manifold M arising from the net $\{A(I)\}_I$, and the other two symbols have similar meanings. Xu [25] worked out this problem, and found that the correct relation is

$$\tau_{A'\cap B}(M)c(M)=\tau_B(M)\overline{\tau_A(M)},$$

where the inclusion $\{A(I)\subset B(I)\}_I$ is given by $SU(N)_{m+1}\subset SU(N)_m\times SU(N)_1$ and $c(M)$ is a rather simple invariant expressed in terms of the linking matrix of a link representing M. (This $c(M)$ is given explicitly in [25], but we omit the expression.) Furthermore, using Kirby-Melvin [13], Xu showed that we have an example of a 3-manifold M for which $\tau_B(M)\overline{\tau_A(M)}=0$, $\tau_{A'\cap B}(M)\neq 0$, and $c(M)=0$. Thus, the invariant $\tau_{A'\cap B}(M)$ arising from the coset has more information than $\tau_B(M)\overline{\tau_A(M)}$.

As a more explicit example, consider the nets of subfactors $\{A(I)\subset B(I)\}_I$ arising from the inclusion $SU(2)_4\subset SU(2)_2\times SU(2)_2$. Then we have DHR endomorphisms labeled with triples (j,k,l) with $l=0,1,\ldots,4$, $j=0,1,2$, $k=0,1,2$ and $j+k-l\in 2\mathbf{Z}$. We have 23 such triples. Then we have identification of irreducible DHR endomorphisms given by $(j,k,l)\cong(2-j,2-k,4-l)$ except for the case of $(1,1,2)$ which gives a reducible

DHR endomorphism and decomposes into a sum of two irreducible DHR endomorphisms. That is, we have a modular tensor category having 13 irreducible objects. We do not know exact relations between $\tau_{A'\cap B}(M)$ and $\tau_B(M)\overline{\tau_A(M)}$. We even do not know whether $\tau_{A'\cap B}(M)$ is a *better* invariant than $\tau_B(M)\overline{\tau_A(M)}$ or not.

Finally, we briefly mention the orbifold net. Let $\{A(I)\}_I$ be a completely rational net of factors and G a finite group of automorphisms acting on this net in an appropriate sense. Set $B(I)$ be the fixed point subalgebra of $A(I)$ with this action. Then the net $\{B(I)\}_I$ is also completely rational and called the *orbifold net* of $\{A(I)\}_I$. Xu [26] has studied some general properties of the orbifold nets and several interesting examples. We do not know about relations between $\tau_A(M)$ and $\tau_B(M)$ in this setting and would like to obtain such a relation.

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