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Knots and links in spatial graphs

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Abstract We study the relations of knots and links contained in a spatial graph.

This is an survey article on the results about knots and links contained in a spatial graph. We do not intend to cover all results in this topic. We only treat some of them here.

The set of knots and links contained in a spatial graph is a naive invariant of spatial graph. However it is of course not a complete invariant in general. For example Kinoshita's theta curve in Fig. 1 is not trivial but contains only trivial knots as the trivial theta curve. See for other such examples [5], [20] and [15].

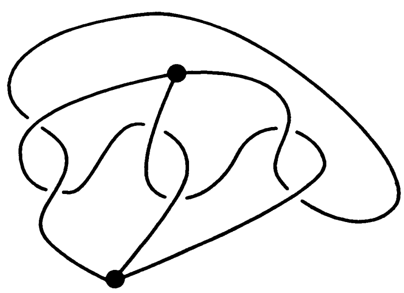

Fig. 1

Anyway we are interested in the set of knots and links contained in a spatial graph. In [6] it is shown that any given $n(n-1)/2$ knot types are realized by an embedding of θ_n at once. Here θ_n denotes the graph on two vertices and n edges joining them. For example, suppose that trefoil knot, figure eight knot and $(2,5)$-torus knot are given. Then there is an embedding of $\theta = \theta_3$ that contains all of them. See Fig. 2 for such an example.

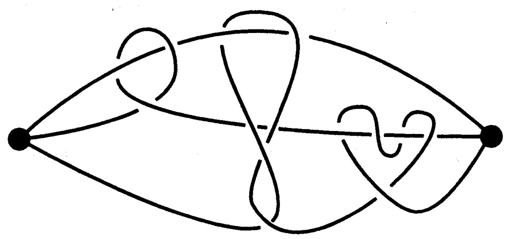

Fig. 2

Now we give a precise definition. Let G be a finite graph. We consider G as a topological space as well as a combinatorial object. Let Γ be a set of subgraphs of G. Suppose that for each $H \in \Gamma$, an embedding $\phi_H : H \to R^3$ is given. Then we say that the set of embeddings $\{\phi_H | H \in \Gamma\}$ is *realizable* if there is an embedding $\varphi : G \to R^3$ such that the restriction map $\varphi|_H$ is ambient isotopic to ϕ_H for each $H \in \Gamma$. The fundamental problem is whether or not given $\{\phi_H | H \in \Gamma\}$ is realizable.

Let $f : G \to R^3$ be an embedding. Then the Wu invariant $\mathcal{L}(f)$ of f is an element of an abelian group $L(G)$ associated to G. See [16] for their definitions. Let H be a subgraph of G. Then there is a natural homomorphism $h_H : L(G) \to H$. Let I_G be a subset of $L(G)$ that is defined by $I_G = \{\mathcal{L}(f) | f : G \to R^3 \text{ is an embedding}\}$. Then the following is known in [17] as a necessary condition of realizability.

Theorem 1. *Suppose that $\{\phi_H | H \in \Gamma\}$ is realizable. Then there is an element $x \in I_G$ such that $h_H(x) = \mathcal{L}(\phi_H)$ for each $H \in \Gamma$.*

From now on we only consider the case that $\Gamma = \Gamma(G)$ is the set of all cycles of G. Here a *cycle* is a subgraph of G that is homeomorphic to a circle. A cycle on n vertices is called an n-cycle. Let $\Gamma_n(G)$ be the set of all n-cycles of G. We say that a graph

G is *adaptable* if any set of embeddings $\{\phi_H | H \in \Gamma(G)\}$ is realizable. Then the result stated above is rephrased that θ_n is adaptable. In [21] it is shown that K_4 is adaptable. Here K_n denotes the complete graph on n vertices. Moreover in [22] it is shown that all proper subgraphs of K_5 are adaptable. In [22] Yasuhara establised a method of realization of knots and links in a spatial graph based on band description of knots. Now we are interested in whether or not K_5 is adaptable. The answer is 'No'. In fact we have the following theorem.

Theorem 2. *A set of embeddings $\{\phi_H | H \in \Gamma(K_5)\}$ is realizable if and only if there is an integer m such that*

$$\sum_{H \in \Gamma_5(K_5)} a_2(\phi_H(H)) - \sum_{H \in \Gamma_4(K_5)} a_2(\phi_H(H)) = \frac{m(m-1)}{2}.$$

We note that the 'only if' part of Theorem 2 is shown in [8] and the 'if' part of Theorem 2 is shown in [19]. We refer the reader to [19], [12], [13] and [11] for related results.

Now we are interested in the existence of nontrivial knots and links in a large complete graph. The following theorem in [1] is a milestone in this area.

Theorem 3. (1) *For any embedding $f : K_6 \to R^3$ the sum of the linking numbers of the links in $f(K_6)$ is an odd number.*

(2) *For any embedding $f : K_7 \to R^3$ the sum of the second coefficients of the Conway polynomials of the knots of 7-cycles in $f(K_7)$ is an odd number.*

In [9] it is shown that for any knot J there is a natural number n such that every linear embedding of K_n into R^3 contains a cycle that is ambient isotopic to J. See also [7] [10] etc. for related results.

In [3] it is shown that every embedding of K_{10} into R^3 contains a 3-component nonsplittable link. In [4] it is shown that for any natural number n there is a graph G such that every embedding of G into R^3 contains an n-component nonsplittable link.

In [2] it is shown that for any natural number n there is a natural number m such that every embedding of K_m contains a 2-component link whose absolute value of the linking number is greater than or equal to n. It is also shown in [2] that for any natural number n there is a natural number m such that every embedding of K_m contains a knot whose absolute value of the second coefficient of the Conway polynomial is greater than or equal to n. In the first result m is actually given by a polynomial of n whose degree is 2. In the second result m is actually given by a polynomial of n whose degree is 1. Recently the author and Shirai showed that in the first result m can be given by a polynomial of n whose degree is 1, and in the second result m can be given by a polynomial of n whose degree is 1/2. See [14] for mor details.

Let σ^n_{2n+3} be the n-skeketon of a $(2n+3)$-simplex. In [18] it is shown that for any embedding of σ^n_{2n+3} into the $(2n+1)$-sphere the sum of the linking numbers of the 2-component n-links contained in the embedding is an odd number. The case $n = 1$ is just Theorem 3 (1). Thus this result is a higher dimensional generalization of Theorem 3 (1).

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