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Quantum Theory of Angular Momentum: Hypergeometric Series and Polynomial zeros

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1. Introduction: In accord with the fundamental theorem of algebra, every orthogonal polynomial of degree n has exactly n roots, which are real and distinct. Conventional zeros (or roots) of orthogonal polynomials have many applications. In the Gaussian quadrature procedure a given definite integral is approximated by the expression:

$$\int_a^b f(x)\,dx = \sum_{k=0}^{n} w_k\, f(x_k)$$

where $w_k(k=0,1,\cdots,n)$ are the weighting coefficients and x_k are the associated $n+1$ points which are the positions of the roots of a Legendre polynomial of degree $n+1$. In potential theory, if unit charges are placed within a given interval $[-1,1]$, say, then the point of equilibrium of the physical system corresponds to the zeros of the Jacobi polynomial. These are two examples cited to emphasize that the zeros in these cases correspond to the roots of the polynomial and they are always with respect to the variable x.

In the case of the Gauss hypergeometric series, ${}_2F_1(a,b;c;x)$, Gauss emphasized the importance of treating the ${}_2F_1$ as a function not in one variable (x) but as a function in all the four variables (a,b,c,x). However, conventionally the Gauss function is considered as a function in the variable x and a,b are considered as its numerator parameters and c its denominator parameter. In quantum theory of angular momentum, the 3-j and 6-j coefficients are intimately related to the generalized hypergeometric functions ${}_3F_2$ and ${}_4F_3$ of *unit* argument, $(x=1)$. The *non trivial* zeros of these angular momentum coupling (3-j) and recoupling (6-j) coefficients were discussed for the first time by Koozekanani and Biedenharn (1974). Since these coefficients can be related to the Hahn and Racah polynomials, we choose to call these *non trivial* zeros as *polynomial zeros*, since they arise due to the zeros of the hypergeometric functions of unit argument.

The conditions on the parameters of the ${}_{p+1}F_p(1)$ relate the problem of the zeros of degree 1 to the solutions of the homogeneous Multiplicative Diophantine Equations, and the problem of zeros of degree 2 is related to the solutions of Pell's equation.

While the 3-j and the 6-j coefficients are related to sets of ${}_3F_2(1)$s and sets of ${}_4F_3(1)$s, and through them to the Hahn and Racah polynomials respectively; the 9-j recoupling coefficient, also called as the Russell-Saunders (or $LS-jj$) transformation coefficient, has been shown by Srinivasa Rao nad Rajeswari (1989) to be related to a triple hypergeometric series (at unit arguments). However, its connection to a polynomial is still not unambiguously established and remains an open problem. The identification of the 9-j

coefficeint with a triple hypergometric series has enabled us *to define for the first time* its polynomial zeros. All the polynomial zeros of degree 1 have been obtained as the solutions of a set of Multiplicative Diophantine Equations (of degree 3). In this article, the current status of this subject is presented and the open problems stated.

2. Angular Momentum: The quantum theory of angular momentum has been developed from the beginning of quantum physics and has become an essential tool in the hands of the theoretical physicist. The article of Smorodinskii and Shelepin (1972) entitled: *Clebsch-Gordan coefficients viewed from different sides* is an excellent review article revealing the close relation of Clebsch-Gordan coeffieints to combinatorics, finite differences, special functions, complex angular momenta, projective geometry, etc. In recent times, Askey – in his Preface to *Special Functions*, by George Andrews, Richard Askey and Ranjan Roy (1999) – points out that the 3-j, 6-j symbols that appear in quantum theory of angular momentum *are all hypergeometric functions and many of their elementary properties are best understood when considered as such*, though not recognized widely.

In quantum physics, the angular momenta $(\vec{L}, \vec{S}, \vec{J})$ are vector operators acting in Hilbert space. While $\vec{L}, \vec{S}$ act on a product space basis $|L\mu\rangle|S\nu\rangle$, $\vec{J}$ acts on the coupled basis state $|JM\rangle$, so that

$$\vec{J}|JM\rangle = \vec{L}|L\mu\rangle \otimes \mathbf{1}|S\nu\rangle \ + \ \mathbf{1}|L\mu\rangle \otimes \vec{S}|S\nu\rangle,$$

where μ, ν and M are scalar projection quantum numbers satisfying the additive property: $\mu + \nu = M$. This result can be written generally as:

$$\triangle(\vec{J}) \ = \ \vec{J} \otimes \mathbf{1} + \mathbf{1} \otimes \vec{J}$$

which implies that the vector addition of angular momenta defines a commutative coproduct in a Hopf algebra (cf. Abe (1980)). Accordingly a commutative Hopf algebra structure is implicit in q-physics, recognized as such only in recent times.

For the first time, Koozekanani and Biedenharn (1974) initiated the study of *non trivial* zeros of the 6-j coefficient. They tabulated the zero-valued 6-j coefficients for the arguments of any one of the six angular momenta being $\leq 37/2$. Subsequnetly, Varshaloivch et.al (1988) gave a listing of the zero-valued 3-j coefficients. Bowick (1976) reduced the sizes of these tables by taking the Regge (1958, 1959) symmetries of these coefficients into account and tabulating only the Regge inequivalent zeros.

3. Hahn polynomial and the 3-j coefficient: The 3-j or Clebsch-Gordan coefficient is defined (Van der Waerden, 1932) as:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta(m_1+m_2+m_3,0)\ (-1)^{j_1-j_2-m_3} \prod_{i=1}^{3} [(j_i-m_i)!(j_i+m_i)!]^{1/2}$$

$$\times \triangle\,(j_1,j_2,j_3)\ \sum_t (-1)^t \left[t! \prod_{k=1}^{2}(t-\alpha_k)! \prod_{l=1}^{3}(\beta_l - t)!\right]^{-1},$$

where $\delta(m,n)$ is the Kronecker delta function, being 1 for $m=n$ and 0 for $m \neq n$, m,n being integers;

$$max(0,\alpha_1,\alpha_2) \leq t \leq min(\beta_1,\beta_2,\beta_3),$$

$$\alpha_1 = j_1 - j_3 + m_2, \quad \alpha_2 = j_2 - j_3 - m_1,$$

$$\beta_1 = j_1 - m_1, \quad \beta_2 = j_2 + m_2, \quad \text{and} \quad \beta_3 = j_1 + j_2 - j_3,$$

$$\Delta(x,y,z) = \left[\frac{(-x+y+z)!(x-y+z)!(x+y-z)!}{(x+y+z+1)!}\right]^{1/2}.$$

Srinivasa Rao (1978) showed that the 72 symmetries of the 3-j coefficient can be easily understood in terms of the set of six hypergeometric functions of unit argument:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta(m_1+m_2+m_3,0) \prod_{i,k=1}^{3} [R_{ik}!/(J+1)!]^{1/2} \; (-1)^{\sigma(pqr)}$$

$$\times \left[\Gamma(1-A,1-B,1-C,D,E\right]^{-1} {}_3F_2(A,B,C;D,E;1),$$

where

$$A = -R_{2p}, \quad B = -R_{3q}, \quad C = -R_{1r},$$

$$D = 1 + R_{3r} - R_{2p}, \qquad E = 1 + R_{2r} - R_{3q},$$

$$\Gamma(x,y,\cdots) = \Gamma(x)\Gamma(y)\cdots; \quad \text{and} \quad J = j_1 + j_2 + j_3,$$

$$\sigma(pqr) = \begin{cases} R_{3p} - R_{2q} & \text{for even permutations of } (pqr)=(123) \\ R_{3p} - R_{2q} + J & \text{for odd permutations of } (pqr)=(123). \end{cases}$$

The Hahn polynomial is defined in terms of the ${}_3F_2(1)$, by Karlin and MacGregor (1961), as:

$$Q_n(x) \equiv Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix} -n,-x, & n+\alpha+\beta+1 \\ \alpha+1, & -N+1 \end{matrix} ;1\right)$$

for $Rl\ \alpha > -1$, $Rl\ \beta > -1$ and positive integral N, with $Q_n(x)$ satisfying the orthogonality relations :

$$\sum_{x=0}^{n-1} Q_n(x)Q_m(x)\rho(x) = \frac{1}{\pi_n}\delta(m,n),$$

$$\sum_{n=0}^{n-1} Q_n(x)Q_n(x)\pi_n = \frac{1}{\rho(x)}\delta(x,y), \qquad \text{(Dual)}$$

where $\delta(x,y)$ is the Kronecker delta function and the weight functions are:

$$\rho(x) = \rho(x;\alpha,\beta,N) = \frac{\begin{pmatrix} \alpha+x \\ N \end{pmatrix}\begin{pmatrix} \beta+N-x-1 \\ N-x-1 \end{pmatrix}}{\begin{pmatrix} N+\alpha+\beta \\ N-1 \end{pmatrix}},$$

$$\pi_n = \pi_n(\alpha,\beta,N) = \frac{\begin{pmatrix} N-1 \\ n \end{pmatrix}}{\begin{pmatrix} N+\alpha+\beta+n \\ n \end{pmatrix}} \frac{(2n+\alpha+\beta+1)}{(\alpha+\beta+1)}$$

$$\times \quad \frac{\Gamma\left(\beta+1, \quad n+\alpha+1, \quad n+\alpha+\beta+1\right)}{\Gamma\left(\alpha+1, \quad \alpha+\beta+1, \quad n+\alpha+\beta+1, \quad (n+1)\right)},$$

$\binom{n}{r}$ being the binomial coefficient.

The Van der Waerden form is not suitable for relating the 3-j coefficient to the Hahn polynomial. This can be overcome by the use of the Thomae transformation:

$${}_3F_2\left(\begin{matrix}\alpha,\beta,-n\\ \gamma,\delta\end{matrix};1\right)=\frac{\Gamma(\gamma,\gamma-\alpha+n)}{\Gamma(\gamma+n,\gamma-\alpha)}{}_3F_2\left(\begin{matrix}\alpha,\delta-\beta,-n\\ 1+\alpha-\gamma-n,\delta\end{matrix};1\right)$$

on the ${}_3F_2(1)$ for the 3-j coefficient, with $\alpha=C,\ \beta=A,\ -n=B,\ \gamma=D,\ \delta=E$, to establish the relation:

$$Q_n(x)=(-1)^{2j_2+m+n+x}\frac{(j_3-j_2+m_1)!}{(2j_2)!}\left[\frac{(j_2-n)!n!(2j_3+n+1)!}{(2(j_3-j_2)+n)!}\right]^{1/2}$$

$$\times\left[\frac{(j_3-j_2-m_1+n)!(2j_2-x)!x!}{(j_3-j_2+m_1+n)!(j_3-j_2+m_1+x)!(j_3+j_2-m_1-x)!}\right]^{1/2}$$

$$\times\begin{pmatrix} j_3-j_2+n & j_2 & j_3\\ m_1 & x-j_2 & j_2-m_1-x\end{pmatrix}$$

Therefore, the zeros of the 3-j coefficient are nothing but the zeros of the Hahn polynomial.

4. Racah polynomial and the 6-j coefficient: The 6-j coefficient can be rearranged to give rise to a set of three ${}_4F_3(1)$s, by Srinivasa Rao et.al. (1975), as:

$$\begin{Bmatrix} a & b & e\\ d & c & f\end{Bmatrix}=N(-1)^{\beta_k}\sum_s(-1)^s\,(\beta_k-s+1)!$$

$$\times\left[\prod_{i=1}^{4}(\beta_k-\alpha_i-s)!\ \prod_{j=1}^{3}(s-\beta_k+\beta_j)!\right]^{-1}$$

$$=N(-1)^{E+1}\Gamma(1-E)\left[\Gamma(1-A,1-B,1-C,1-D,F,G)\right]^{-1}$$

$$\times{}_4F_3(A,B,C,D;E,F,G;1)$$

where

$$A=e-a-b,\ B=e-c-d,\ C=f-a-c,\ D=f-b-d,$$
$$E=-a-b-c-d-1,\ F=e+f-a-d+1,\ G=e+f-b-c+1;$$

$$A=a-b-e,\ B=d-c-e,\ C=a-c-f,\ D=d-b-f,$$
$$E=-b-c-e-f-1,\ F=a+d-b-c+1,\ G=a+d-e-f+1;$$

$$A=b-a-e,\ B=c-d-e,\ C=c-a-f,\ D=b-d-f,$$
$$E=-a-d-e-f-1,\ F=b+c-a-d+1,\ G=b+c-e-f+1.$$

Equivalently, the 6-j coefficient can also be rearranged to give rise to a set of four ${}_4F_3(1)$s (Srinivasa Rao et.al. (1977)):

$$\left\{\begin{array}{ccc} a & b & e \\ d & c & f \end{array}\right\} = N(-1)^{\alpha_l} \quad \sum_s (-1)^s(\alpha_l + s + 1)!$$

$$\times \left[\prod_{i=1}^{4}(s + \alpha_l - \alpha_i)! \ \prod_{j=1}^{3}(\beta_j - \alpha_l - s)!\right]^{-1}$$

$$= N(-1)^{A'-2}\Gamma(A')\,[\Gamma(1 - B', 1 - C', 1 - D', E', F', G')]^{-1}$$

$$\times {}_4F_3(A', B', C', D'; E', F', G'; 1),$$

where

$$A' = a + b + e + 2,\ B' = a - c - f,\ C' = b - d - f,\ D' = e - c - d,$$
$$E' = a + b - c - d + 1,\ F' = a + e - d - f + 1,\ G' = b + e - c - f + 1;$$

$$A' = c + d + e + 2,\ B' = c - a - f,\ C' = d - b - f,\ D' = e - a - d,$$
$$E' = c + d - a - b + 1,\ F' = c + e - b - f + 1,\ G' = d + e - a - f + 1;$$

$$A' = a + c + f + 2,\ B' = c - d - e,\ C' = a - b - e,\ D = f - b - d,$$
$$E' = a + c - b - d + 1,\ F' = a + f - d - e + 1,\ G' = c + f - b - e + 1;$$

$$A' = b + d + f + 2,\ B' = b - a - e,\ C' = d - c - e,\ D' = f - a - c,$$
$$E' = b + d - a - c + 1,\ F' = b + f - c - e + 1,\ G' = d + f - a - e + 1.$$

These two sets of ${}_4F_3(1)$s for the 6-j coefficient have been shown by Srinivasa Rao et.al. (1985) to be related to each other through the *reversal* of hypergeometric series, which clearly establishes why the sets of three and four ${}_4F_3(1)$s are *necessary and sufficient* to account for the 144 symmetries of this coefficient.

Though Racah established the discrete orthogonality property satisfied by the 6-j coefficient, he was unaware that it is indeed a manifestation of its connection to an orthogonal polynomial. Askey and Wilson (1985) who discovered this, christened the polynomial as the Racah polynomial. In terms of the Racah polynomial the 6-j coefficient can be written as:

$$\left\{\begin{array}{ccc} a & b & c+d-x \\ d & c & b+d-n \end{array}\right\} = (-1)^{a+b+c+d-n}\Delta(b, d, b + d - n)\Delta(a, e, b + d - n) \quad \times$$

$$\times\Delta(a, b, c + d - x)\Delta(c, d, c + d - x)\Gamma(a + b + c + d - n + z)$$

$$\times\,[\Gamma(1 + n, 1 + n + a - b + c - d, 1 + x, 1 + 2d - x, 1 + N - x, 1 + a + b - c - d + x)]^{-1}$$

$$\times \quad P_n(t^2; a', b', c', d'),$$

where

$$t = x - c - d - 1/2 \quad a' = -c - d - 1/2, \quad b' = -a - b - 1/2,$$

$$c' = a - b + 1/2, \quad d' = c - d + 1/2,$$

$$P_n(t^2; a', b', c', d') = \Gamma(a' + b' + n, a' + c' + n, a' + d' + n)$$

$$\times\left[\Gamma(a'+b',a'+c',a'+d')\right]^{-1}$$

$$\times{}_4F_3\left(\begin{array}{c}-n,\ a'+b'+c'+d'+n-1,\ a'-t,\ a'+t\\ a'+b',\ a'+c',\ a'+d'\end{array};1\right).$$

Thus, the *non-trivial* zeros of the 6-j coefficient, are nothing but the polynomial zeros of the 6-j coefficient.

5. Polynomial zeros of the 3-j and the 6-j coefficients: The identification of the 3-j and the 6-j coefficient to the Hahn and the Racah polynomials, provides us with a means for the classification of their zeros by the degree of the polynomial. The trivial zeros of these coefficients are those which arise due to:

(i) violation of triangle inequalities, or, violation of additive properties of vector quantum numbers – Note: i.e. if $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$, then the triad j_1, j_2, j_3 is said to satisfy the triangle inequality;

(ii) violation of additive property of scalar (projection) quantum numbers (only in the case of the 3-j coefficient);

(iii) symmetry properties of the coefficients.

Koozekanani and Biedenharn (1974), noted that *Non-trivial* zeros arise due to the summation part *accidentally* becoming zero and not due to (i), (ii) or (iii) stated above. We prefer to call these structural zeros as *polynomial zeros* since the 3-j and 6-j coefficients have been related to the Hahn and Racah polynominals, respectively.

Sato and Kaguei (1972) rearranged the 3-j coefficient to obtain the following form:

$$\left(\begin{array}{ccc} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{array}\right)=\boxed{\begin{array}{ccc}\mathrm{a}&\mathrm{b}&\mathrm{c}\\ \mathrm{d}&\mathrm{e}&\mathrm{f}\\ \mathrm{g}&\mathrm{h}&\mathrm{i}\end{array}}=\delta(m_1+m_2+m_3,0)(-)^{b+c-i}$$

$$\times\quad\left[\frac{b^{(b-g)}d^{(d-i)}f^{(f-a)}h^{(h-c)}}{e!(a+b+c+1)!}\right]^{1/2}\quad(a.i-c.g)^{(e)}$$

where $(a.i - c.g)^{(e)}$ is a symbolic binomial expansion:

$$(a.i\ -c.g)^{(e)}=\sum_{k=0}^{e}(-1)^k\left(\begin{array}{c}e\\ k\end{array}\right)c^{(e-k)}g^{(e-k)}a^{(k)}i^{(k)},$$

with

$$p^{(\sigma)}=\frac{p!}{(p-\sigma)!}=p(p-1)(p-2)\cdots(p-\sigma+1),$$

being the *lowering factorial* and $a, b, \cdots, i$ being the nine integer parameters that can be formed out of the j_i's and the m_i's ($i = 1, 2, 3$) to give rise to the 3×3 Regge (1959) square symbol for the 3-j coefficient:

$$||R_{ik}||\equiv\boxed{\begin{array}{ccc}\mathrm{a}&\mathrm{b}&\mathrm{c}\\ \mathrm{d}&\mathrm{e}&\mathrm{f}\\ \mathrm{g}&\mathrm{h}&\mathrm{i}\end{array}}\equiv\left\|\begin{array}{ccc}-j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3\\ j_1-m_1 & j_2-m_2 & j_3-m_3\\ j_1+m_1 & j_2+m_2 & j_3+m_3\end{array}\right\|.$$

Srinivasa Rao and Venkatesh (1977) observe that for σ =1, $p^{(1)} = p!/(p-1)! = p$ is an *exact* power and immediately this establishes that the binomial expansion is also exact in this case. Therefore, when $e = 1$, the Sato-Kaguei expression above reveals polynomial zeros of degree 1, when

$$a.i = c.g \iff (-j_1 + j_2 + j_3)(j_3 + m_3) = (j_1 + j_2 - j_3)(j_1 + m_1).$$

In the case of the 6-j coefficient, Sato (1955) used the definitions:

$$p^{(\sigma)} = \frac{p!}{(p-\sigma)!} \qquad \text{and} \qquad p^{(-\sigma)} = \frac{(p+\sigma)!}{p!}$$

to obtain a formal binomial expansion for the 6-j coefficient. Notice that while $p^{(1)} = p$, unfortunately $p^{(-1)} = p + 1$ according to Sato's definitions. Therefore, we redefine the generalized powers as:

$$p^{(\sigma)} = \frac{p!}{(p-\sigma)!} \qquad \text{and} \qquad p^{(-\sigma)} = \frac{1}{p^{(\sigma)}}$$

and this enables Srinivasa Rao and Venkatesh (1977) to obtain the formal binomial expansion for the 6-j coefficient as:

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-)^{a+b+c+d}\ \Delta(abe)\Delta(cde)\Delta(acf)\Delta(bdf)$$

$$\times(-1)^{\beta_0}\Gamma(\beta_0+2)\left[\Gamma(n+1, C_p+1, C_q+1, C_r+1, D_u+n+1, D_v+n+1)\right]^{-1}$$
$$\times((D_u+n)(D_v+n) - C_pC_qC_r(\beta_0+1)^{(-1)})^{(n)}$$

where $n = \beta_0 - \alpha_0$, β_0 being any one of $\beta_1, \beta_2, \beta_3$; α_0 being any one of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$; $C_i = \beta_0 - \alpha_i$, $(i = p, q, r)$, $D_j = \beta_j - \beta_0$, $(j = u, v)$, and the indices p, q, r and u, v stand for those α's and β's other than $\beta_0 = min(\beta_1, \beta_2\beta_3)$ and $\alpha_0 = max(\alpha_1, \alpha_2, \alpha_3\alpha_4)$, with

$$\alpha_1 = a+b+e,\ \alpha_2 = c+d+e,\ \alpha_3 = a+c+f,\ \alpha_4 = b+d+f,$$
$$\beta_1 = a+b+c+d,\ \beta_2 = a+d+e+f,\ \beta_3 = b+c+e+f.$$

As in the case of the 3-j coefficient, with our redefined generalized powers, for $n = 1$, the binomial expansion for the 6-j coefficient is *exact*, and therefore the condition:

$$(\beta_0+1)(D_u+n)(D_v+n) = C_pC_qC_r$$

reveals all the polynomial zeros of degree 1 of the 6-j coefficient.

Explicitly, these identifications enable us to write down closed form expressions for degree 1 polynomial zeros of both the 3-j and the 6-j coefficients as

$$1 - \delta(X, Y)\ \delta(n, 1)$$

where for the 3-j coefficient:

$$X = R_{mr}\ R_{kp}, \qquad Y = R_{mp}\ R_{lq}, \qquad n = R_{lq},$$

(lmk) and (pqr) correspond to permutations of (123). And for the 6-j coefficient:

$$X = (\beta_u - \alpha_0)(\beta_v - \alpha_0)(\beta_0 + 1), \qquad Y = (\beta_0 - \alpha_p)(\beta_0 - \alpha_q)(\beta_0 - \alpha_r),$$

where β_u, β_v correspond to the pair of $(\beta_1, \beta_2, \beta_3)$ other than β_0 and $\alpha_p, \alpha_q, \alpha_r$ correspond to three of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ other than α_0.

Finally, notice also that in terms of the set of ${}_3F_2(1)$s, the polynomial zeros of degree 1 of the 3-j coefficient, the condition to be statisfied by the parameters is:

$$1 + \frac{ABC}{DE} = 0 \qquad \text{or} \qquad ABC = -DE,$$

with one of the numerator parameters A, B or C being -1. Thus, the general form of this equation is:

$$x_1 x_2 = u_1 u_2,$$

which is called a homogeneous Multiplicative Diophantine Equation of degree 2. Similarly, in terms of the sets of ${}_4F_3(1)$s for the 6-j coefficient, the polynomial zeros of degree 1 are obtained when

$$1 + \frac{ABCD}{EFG} = 0 \qquad \text{or} \qquad ABCD = -EFG$$

with A, B, C or D being -1. The general form of this conditional equation is:

$$x_1 x_2 x_3 = u_1 u_2 u_3$$

which is a homogeneous Multiplicative Diophantine Equation of degree 3. Therefore, the polynomial zeros of degree 1 of the 3-j and the 6-j angular momentum coefficients are intimately connected with the solutions of Multiplicative Diophantine Equations of degree 2 and degree 3, respectively.

E.T. Bell (1933) classified the Multiplicative Diophatine Equations and obtained their solutions in his classic paper entitled *Reciprocal Arrays and Diophantine Analysis*. S-rinivasa Rao et.al (1993) showed that the concept of reciprocal arrays can be dispensed with in the Proof of the main theorem and what is more this enabled us to provide a simpler and elegant induction proof of the theorems. This formed the basis for providing a complete solution to the problem of the degree 1 zeros of the 3-j and 6-j coefficients by Srinivasa Rao, Rajeswari and King (1988), who also provided an alternative induction proof for the main theorem - which states that n^2 independent parameters, arranged in the form of a $n \times n$ square array, with n greatest common divisor conditions applying to the diagonal elments of the array, are necessary and sufficient to obtain *all* the solutions for the homogeneous Multiplicative Diophantine equation of degree n.

6. Zeros of the 9-j coefficient: The 9-j (or the $LS - jj$ transformation) coefficient expressed as a sum over the product of three 6-j coefficients is the one which is most often used for numerical computations. Wu (1972) pointed out that it is not a ${}_7F_6(1)$ and later Wu (1973) claimed that he had found a new generalized hypergeometric function in three variables: $\Phi^{(3)}(\alpha_{kl}; \beta_i, \gamma_m; w_k)$ for the 9-j coefficient. The triple sum series of

Jucys and Bandzaitis (1977) is the simplest known algebraic form for the 9-j c
Srinivasa Rao and Rajeswari (1989) showed that this formula does not exh
the well known 72 symmetries of the 9-j coefficient and it has been ident
with a formal triple hypergeometric series due to Lauricella (1893), Saran
Srivastava (1964):

$$F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b''):(c);(c');(c'')\\(e)::(f);(f');(f''):(g);(g');(g'')\end{array};x,y,z\right]$$

$$=\sum_{m,n,p}\frac{((a),m+n+p)}{((e),m+n+p)}\frac{((b),m+n)((b'),n+p)((b''),p+m)}{((f),m+n)((f'),n+p)((f''),p+m)}$$

$$\times\frac{((c),m)((c'),n)((c''),p)}{((g),m)((g'),n)((g''),p)}\ \frac{x^m y^n z^p}{m!n!p!}$$

where (a) stands for a sequence of parameters $(a_1, a_2, \cdots)$ and

$$((a),m+n+p)\equiv((a))_{m+n+p}=(a_1)_{m+n+p}(a_2)_{m+n+p}\cdots$$

$$((b),m+n)\equiv((b))_{m+n}=(b_1)_{m+n}(b_2)_{m+n}\cdots$$

$$((c),m)\equiv((c))_m=(c_1)_m(c_2)_m\cdots.$$

The 9-j coefficient was shown by Srinivasa Rao and Rajeswari (1989) to be:

$$\left\{\begin{array}{ccc}a&b&c\\d&e&f\\g&h&i\end{array}\right\}=(-1)^{x5}\frac{(dag)(beh)(igh)}{(def)(bac)(icf)}$$

$$\times\frac{\Gamma(1+x1,1+x2,1+x3)}{\Gamma(1+x4,1+x5)}\frac{\Gamma(1+y1,1+y2)}{\Gamma(1+y3,1+y4,1+y5)}$$

$$\times\frac{\Gamma(1+z1,1+z2)}{\Gamma(1+z3,1+z4,1+z5)}\frac{\Gamma(1+p1}{1+p2,1+p3)}$$

$$\times F^{(3)}\left[\begin{array}{lr}-::-;-;-\ :& 1+x2,1+x3,-x4,-x5;\\-::1+p2;-p1;1+p3\ :& -x1;\end{array}\right.$$

$$\left.\begin{array}{cc}1+y1,1+y2,-y4,-y5;& 1+z2,-z3,-z4,-z5;\\1+y3;&-z1;\end{array}1,1,1\right],$$

where

$$\begin{array}{lll}
x1=2f, & y1=-b+e+h, & z1=2a,\\
x2=d+e-f, & y2=g+h-1, & z2=-a+b+c,\\
x3=c-f+i, & y3=2h+1, & z3=a+d+g+1,\\
x4=-d+e+f, & y4=b+e-h, & z4=a-b+c,\\
x5=c+f-1, & y5=g-h+i, & z5=a-b+c,\\
p1=a+d-h+i, & p2=-b+d-f+h, & p3=-a+b-f+i,
\end{array}$$

$$0\le x\le min(x4,x5)=XF, 0\le y\le min(y4,y5)=YF, 0\le z\le min(z4,$$

$$(abc) = \triangle(abc)\frac{(a+b+c+1)!}{(-a+b+c)!}.$$

The realization that the triple sum series is a triple hypergeometric series, evaluated at unit argument for all the variables, enables Rajeswari and Srinivasa Rao (1989) to *define for the first time* polynomial zeros for this 9-j coefficient, derive a formal binomial expansion for it, and obtain the closed form expression to get *all* the polynomial zeros of the 9-j coefficient as:

$$1 - \delta^{\alpha1,XF,YF,ZF}_{\beta1,\ 1,\ 0,\ 0} - \delta^{\alpha2,XF,YF,ZF}_{\beta2,\ 0,\ 1,\ 0} - \delta^{\alpha3,XF,YF,ZF}_{\beta3,\ 0,\ 0,\ 1},$$

where

$$\begin{array}{ll} \alpha1 = (1+x2)(1+x3)x4\ x5, & \beta1 = (1+p2)(1+p3)x1, \\ \alpha2 = (1+y1)(1+y2)y4\ y5, & \beta2 = (1+y3)\ p1\ (1+p2), \\ \alpha3 = (1+z2)\ z3\ z4\ z5, & \beta3 = z1\ p1\ (1+p3), \end{array}$$

and the notation[1] for the product of four Kronecker delta functions is:

$$\delta^{p,q,r,s}_{a,b,c,d} = \delta(a,p)\delta(b,q)\delta(c,r)\delta(d,s).$$

The first few 'generic' zeros of the 9-j coefficient are given below:

a	b	c	d	e	f	g	h	i	$\sigma = a + \cdots + i$
0.5	1	1.5	1	1.5	1.5	1.5	2.5	2	13
0.5	1	1.5	1	2	3	1.5	3	3.5	17
0.5	1	1.5	1.5	0.5	2	2	1.5	1.5	12
0.5	1	1.5	1.5	2	1.5	2	3	2	15
0.5	1	1.5	1.5	2.5	3	2	3.5	3.5	19

A detailed study revealed – Rajeswari and Srinivasa Rao (1989) – that equations like $\alpha1 = \beta1$ represent Multiplicative Diophantine Equations of degree 3 (since either $x4$ or $x5$ or both have to be 1) for polynomial zeros of degree 1 to occur. 24 cases arose and of these 12 did not yield any degree 1 zeros because of inherent inconsistencies and of the remaining 12, only four (two corresponding to $XF = 1$ and two to $YF = 1$) are full nine parameter solutions, the other eight being fewer (than nine) parameter solutions having one of the angular momenta itself as a free parameter.

This comprehensive work on the polynomial zeros of degree 1 of the 3-j, 6-j and 9-j coefficients via Multiplicative Diophantine Equaitons revealed **all** the zeros and set at rest work of Bremner (1986), Brudno (1985, 1987), Bremner and Brudo (1986), Brudno and Louck (1985) and others which were based on fewer parameteric solutions which giving rise to partial lists of these zeros only. For a detailed account of this problem and its resolution, the interested reader is referred to the monograph of Srinivasa Rao and Rajeswari (1993).

[1]Note: the definitions of the α and β parameters of sections 5 and 6 are different.

7. Degree Vs. Order of the zeros of 3-j coefficients: The polynomial zeros of the 3-j coefficients were defined in terms of the number of terms of the hypergeometric series minus one which is the *degree* of the coefficient. A detailed study of the destribution of the zeros of the 3-j coefficient with respect to the degree n for $J = j_1 + j_2 + j_3 \leq 240$, by Jacques Raynal (1985) revealed that most of the zeros of high degree had small magnetic (m_i, $i = 1, 2, 3$) quantum numbers. This led Raynal et.al (1993) to define the **order** m to classify the zeros of the 3-j coefficient. Changing the notation and suing a, b, c for j_1, j_2, j_3, it is to be noted that when $J = a + b + c$ is odd,

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} = 0. \qquad (\mathrm{Parity}3 - j\mathrm{coefficient})$$

On the contrary if $J = a + b + c$ is even, the 3-j coefficient cannot vanish and it has the value:

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{J/2} \left[\frac{(J-2a)!(J-2b)!(J-2c)!}{(J+1)!} \right]^{1/2} \frac{(J/2)!}{(J/2-a)!(J/2-b)!(J/2-c)!},$$

a result which can be deduced from Dixon's theorem for the ${}_3F_2(1)$. From recurrence relations for the 3-j coefficient, Raynal (1979) deduced three more sets of $3j$-coefficients which never vanish and these are:

$$\begin{pmatrix} a & b & c \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = (-1)^{J/2+1} \left[\frac{(J-2a)!(J-2b)!(J-2c)!}{(J+1)!(2a+1)(2c+1)} \right]^{1/2}$$
$$\times \frac{2(J/2)!}{(J/2-a-\frac{1}{2})!(J/2-b)!(J/2-c-\frac{1}{2})!},$$

for even J, and the following for J odd:

$$\begin{pmatrix} a & b & c \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = (-1)^{J/2+3/2} \left[\frac{(J-2a)!(J-2b)!(J-2c)!}{(J+1)!(2a+1)(2c+1)} \right]^{1/2}$$
$$\times \frac{2(J/2+\frac{1}{2})!}{(J/2-a)!(J/2-b-\frac{1}{2})!(J/2-c)!},$$

$$\begin{pmatrix} a & b & c \\ 0 & 1 & -1 \end{pmatrix} = (-1)^{J/2+1/2} \left[\frac{(J-2a)!(J-2b)!(J-2c)!}{(J+1)!b(b+1)c(c+1)} \right]^{1/2}$$
$$\times \frac{2(J/2)!}{(J/2-a-\frac{1}{2})!(J/2-b-\frac{1}{2})!(J/2-c-\frac{1}{2})!}.$$

These non-zero 3-j-coefficients will be called as zeros of **order** 0.

Contiguous 3-j coefficients, satisfy the recurrence relations (Raynal (1979)):

$$-S(a,b,c,\alpha,\beta,\gamma) \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} - T(a,b,\alpha,\beta) \begin{pmatrix} a & b & c \\ \alpha-1 & \beta+1 & \gamma \end{pmatrix}$$
$$= S(a,b,c,-\alpha,-\beta,\gamma) \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} + T(a,b,-\alpha,-\beta) \begin{pmatrix} a & b & c \\ \alpha+1 & \beta-1 & \gamma \end{pmatrix}$$

$$
\begin{aligned}
&= -S(b,c,a,\beta,\gamma,\alpha)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} - T(b,c,\beta,\gamma)\begin{pmatrix} a & b & c \\ \alpha & \beta-1 & \gamma+1 \end{pmatrix} \\
&= S(b,c,a,-\beta,-\gamma,\alpha)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} + T(b,c,-\beta,-\gamma)\begin{pmatrix} a & b & c \\ \alpha & \beta+1 & \gamma-1 \end{pmatrix} \\
&= -S(c,a,b,\gamma,\alpha,\beta)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} - T(c,a,\gamma,\alpha)\begin{pmatrix} a & b & c \\ \alpha+1 & \beta & \gamma-1 \end{pmatrix} \\
&= S(c,a,b,-\gamma,-\alpha,\beta)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} + T(c,a,-\gamma,-\alpha)\begin{pmatrix} a & b & c \\ \alpha-1 & \beta & \gamma+1 \end{pmatrix},
\end{aligned}
$$

where

$$S(a,b,c,\alpha,\beta,\gamma) = \tfrac{1}{2}\big(a(a+1)+b(b+1)-c(c+1)\big)+\alpha\beta+\tfrac{1}{3}(\alpha-\beta),$$
$$T(a,b,\alpha,\beta) = \big((a+\alpha)(a-\alpha+1)(b-\beta)(b+\beta+1)\big)^{1/2}.$$

(i) Setting $\alpha=\beta=\gamma=0$ in the recurrence relations and using a symmetry for the 3-j coefficient, we get for even J :

$$\begin{pmatrix} a & b & c \\ 0 & 1 & -1 \end{pmatrix} = \frac{a(a+1)-b(b+1)-c(c+1)}{2\{b(b+1)c(c+1)\}^{\frac{1}{2}}}\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} \quad (J \text{ even}).$$

(ii) Setting $(\alpha,\beta,\gamma)=(0,1,-1)$ in the above recurrence relations and using the symmetries of 3-j coefficients, we get for odd values of J:

$$\begin{pmatrix} a & b & c \\ 0 & 2 & -2 \end{pmatrix} = \frac{a(a+1)-b(b+1)-c(c+1)+2}{\{(b-1)(b+2)(c-1)(c+2)\}^{\frac{1}{2}}}\begin{pmatrix} a & b & c \\ 0 & 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} a & b & c \\ 1 & 1 & -2 \end{pmatrix} = \frac{(b-a)(a+b+1)}{\{a(a+1)(c-1)(c+2)\}^{\frac{1}{2}}}\begin{pmatrix} a & b & c \\ 0 & 1 & -1 \end{pmatrix}. \quad (*)$$

For (*) a zero can be found only for $a=b$ and it is a trivial zero, since J is odd. Setting $(\alpha,\beta,\gamma)=(0,1/2,-1/2)$ (and using symmetries and relabellings for a, b and c), we get five new relations. They are, for J even:

$$\begin{pmatrix} a & b & c \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{pmatrix} = \frac{a(a+1)-b(b+1)-c(c+1)-(b+\frac{1}{2})(c+\frac{1}{2})+\frac{1}{2}}{\{(b-\frac{1}{2})(b+\frac{3}{2})(c-\frac{1}{2})(c+\frac{3}{2})\}^{\frac{1}{2}}}\begin{pmatrix} a & b & c \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix};$$

$$\begin{pmatrix} a & b & c \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{pmatrix} = \frac{a(a+1)-b(b+1)-c(c+1)+(b+\frac{1}{2})(c+\frac{1}{2})+\frac{1}{2}}{\{(b-\frac{1}{2})(b+\frac{3}{2})(c-\frac{1}{2})(c+\frac{3}{2})\}^{\frac{1}{2}}}\begin{pmatrix} a & b & c \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad (J \text{ odd});$$

$$\begin{pmatrix} a & b & c \\ \frac{1}{2} & 1 & -\frac{3}{2} \end{pmatrix} = \frac{(a+\frac{1}{2})(a+c+1)-b(b+1)}{\{b(b+1)(c-\frac{1}{2})(c+\frac{3}{2})\}^{\frac{1}{2}}}\begin{pmatrix} a & b & c \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}, \quad (J \text{ even});$$

$$\begin{pmatrix} a & b & c \\ \frac{1}{2} & 1 & -\frac{3}{2} \end{pmatrix} = \frac{(a+\frac{1}{2})(a-c)-b(b+1)}{\{b(b+1)(c-\frac{1}{2})(c+\frac{3}{2})\}^{\frac{1}{2}}}\begin{pmatrix} a & b & c \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \quad (J \text{ odd});$$

$$\begin{pmatrix} a & b & c \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} = -\frac{(c+\frac{1}{2})+(-1)^J(a+\frac{1}{2})}{\{b(b+1)\}^{\frac{1}{2}}}\begin{pmatrix} a & b & c \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}, \quad (J \text{ even or odd}).$$

Clearly, the six equations above in section (ii), except (*), give the zeros of the $3j$ coefficients of (recurrence) **order** 1.

The 3-j coefficients of order 2 are obtained with recurrence relations involving 3-j coefficients of recursion order 0 and 1 for the two other members. More generally, the 3-j coefficients of order m are obtained with recurrence relations involving 3-j coefficients of order $m-2$ and $m-1$ for the two other members. In order to characterise the order m in another way, consider the Regge symbol associated with a 3-j symbol. In this Regge symbol, perform a transformation bringing the two rows or two columns with minimum absolute difference (this is the sum of the absolute values of the differences member by member) to the last two rows. Then, one has :

- if α, β and γ are all integers :

$$\begin{aligned} m &= \max\{|\alpha|,|\beta|,|\gamma|\} \qquad \text{if } J \text{ is even,} \\ m &= \max\{|\alpha|,|\beta|,|\gamma|\}-1 \qquad \text{if } J \text{ is odd;} \end{aligned}$$

- if α, β and γ are not all integers :

$$m = \lfloor \max\{|\alpha|,|\beta|,|\gamma|\} \rfloor,$$

  where $\lfloor x \rfloor$ stands for the integer part of x.

Note that this definition gives the order $m = -1$ for the trivial zeros. A complete calssification of order 2 and 3 has been obtained and Raynal et.al. (1993) found that there are 12 types of zeros of order 2 and 17 types of zeros of order 3. The problem of zeros of order 1 has been completely solved and the zeros of order 2 and 3 classified. While the zeros of degree 1,2,3 and 4 were found to be infinite in number, it is **not known** whether the number of zeros of degree $n > 4$ is finite or infinite, If the zeros are arranged with $n > m$, for increasing values of $m > 4$, it appears that there are no zeros of high order!

8. Distribution of polynomial zeros: Relating the polynomial zeros of degree 1 of $3n$-j coefficients, ($n = 1,2,3$), to the solution of homogeneous Multiplicative Diophantine equations by Srinivasa Rao and coworkers has solved this problem completely. Bayer, Louck and Stein (1987) and Louck and Stein (1987) related the solutions of the Pell's quadratic Diophantine equation to that of the polynomial zeros of degree 2, for the 3-j and the 6-j coefficients, using transformations of quadratic forms over the integers. In the case of the 6-j coefficient, their orbit classification of the zero's of Pell's equation, unfortunately does not reveal all the zeros fo degree 2. Srinivasa Rao and Chiu (1989) proposed, therefore, a simple algorithm to generate *all* the polynomial zeros of degree 2 of the 3-j and the 6-j coefficients, using the principle of factorization of an integer and the solution to a quadratic or a cubic equation.

Biedenharn and Louck (1981) pointed out that the distribution of these polynomial zeros is a number theoretic problem. In the field of integers and half-integers, it is possible to count the number of polynomial zeros which occur out of the allowed (non-zero) angular momentum coefficients. We, Raynal et.al. (1993), did a search for the polynomial zeros of degree 1 to 7 for polynomial zeros of the 3-j coefficient and found that the number of zeros of degree 1 and 2 are infinite; though the number of zeros of degree larger than 3 decreases very quickly.

In the table below are given the number of polynomial zeros of degree 1 of the 6-j coefficient corresponding the allowed coefficients (Srinivasa Rao and Rajeswari (1993)) in the first two columns and the primenumber $\pi(x)$ in the interger field x:

No. of allowed 6-j coefficients N	Zeros Z(N) (KSR-VR,1993)	x	primes $\pi(x)$ (Zagier,1977)
10^1	0	10^1	4
10^2	1	10^2	25
10^3	8	10^3	168
10^4	39	10^4	1229
10^5	177	10^5	9592
10^6	728	10^6	78498
10^7	2612	10^7	664579
10^8	8749	10^8	5761455

The comparison made in the table shows that the distribution of the number of polynomial zeros of degree 1 of the 6-j coefficient, in the allowed anugular momentum field, is much smaller than the distribution of primes in the integer field.

An extensive computer search by Raynal and Van der Jeugt (1993) for the zeros of 6-j coefficients, similar to the one by us (Raynal et.al. 1993) for the 3-j coefficient, led to the conclusion that the number of zeros of degree 1,2 and 3 is infinite; no results were obtained for zeros of degree 4; an indication that the degree 3 zeros belonged to a finite family (though no proof that their total number is finite); that the number of zeros of higher degree are relatively smaller as the degree goes higher and that for $J = a+b+c+d+e+f \leq 1200$, in $\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}$, only one zero of degree 9 and none of higher degree than that.

9. Open problems: The study of polynomial zeros of 3-j and 6-j coefficients, started by Koozekanani and Biedenharn, in 1974, has been extended considerably by scores of papers by Beyer, Louck and Stein (1986); Bremner (1986); Brudno (1985, 1987); Brudno and Louck (1985); De Meyer, Vanden Berghe and Van der Jeugt (1983, 1984); Labarthe (1986); Lindner (1984, 1985); Rajeswari and Srinivasa Rao (1989, 1990); Raynal (1978, 1979, 1985); Raynal et.al. (1993); Srinivasa Rao (1978, 1985); Srinivasa Rao and Rajeswari (1985, 1987, 1988, 1989, 1993); Srinivasa Rao et.al. (1975, 1977, 1988, 1992); Van der Jeugt and Raynal (1993).

Racah (1949) recognized that $\left\{\begin{smallmatrix} 5 & 5 & 3 \\ 3 & 3 & 3 \end{smallmatrix}\right\} = 0$ elucidates the embedding of the exceptional Lie algebra G_2 into the SO_7 Lie algebra. Judd (1970) related $\left\{\begin{smallmatrix} 5 & 5 & 2 \\ 2 & 2 & 4 \end{smallmatrix}\right\} = 0$ to the vanishing of the fractional parentage coefficient in the atomic g-shell. Vanden Berghe et.al. in a series of papers demonstrated that tensor operator realizations of the exceptional Lie algebras F_4 and E_6 provide a basis to explain 12 generic or Regge inequivalent polynomial zeros of the 6-j coefficient taking the the suggestion of Koozekanani and Biedenharn (1974) that realizations of exceptional Lie algebras might provide bases for explaining the polynomial zeros of the 6-j coefficient. However, Srinivasa Rao (1985) observed that of the 12 generic entries of Vanden Berghe et.al. 11 are polynomial zeros of degree 1 and only one is a polynomial zero of degree 2. It was also pointed out by Srinivasa Rao et.al. that simple closed form expressions can be given for all the polynomial zeros of degree 1 for not only 6-j, but also 3-j and 9-j coefficients, since formal binomial expansions for these coefficients in terms of generalized powers are indeed *exact* expressions for degree 1.

The problem of degree 1 zeros of the $3n$-j coefficients ($n = 1, 2, 3$) has been completely solved via the study of homogeneous Multiplicative Diophanitine equations – initiated by Bremner (1986), Brudno ()1985, 1987), Bremner and Brudno (1986), Beyer, Louck and Stein (1986), and completed by Srinivasa Rao, Rajeswari and King 1988. The extensive numerical studies of the zeros of 3-j coefficients, especially by Jacques Raynal(1978, 1979, 1985), provided an opportunity to define the *order* of their zeros via the use of recurrence relations satisfied by them. However, many questions remain unanswered. We mention some of these open problems:

- The distribution of polynomial zeros of angular momentum coefficients is basically a number-theoretic problem which has not been studied completely.
- Degree 1 zeros were related to solutions of Multiplicative Diophatine equations and degree 2 zeros to Pell's equation and this raises the question of whether a hierarchy of equations can be associated with integer solutions of known polynomial expressions. Such an association will then enable one to understand the sparse number of polynomial zeros as their degree n increases and it is conjectured that there is a limit n beyond which there are no slutions to the equations and hence no polynomial zeros. The open problem is to establish this degree n, for the 3-j, 6-j and the 9-j coefficients.
- The Hahn and Racah polynomials are hypergeometric functions of unit argument. The study of the zeros of these polynomials, as well as polynomial zeros of multiple hypergeometric series is an interesting open problem.
- The triple hypergeometric series representation for the 9-j coefficient is the simplest known algebraic form for this recoupling coefficient, which satisfies an orthogonality property. There is no three-term recurrence relation associated with the 9-j coefficient. These are the clues available for the construction of an orthogonal polynomial associated with the 9-j coefficient. (Suslov (1983) has shown that the 9-j coefficient is a polynomial in two variables but it is not established that this result is unique!).

- A preliminary study (Srinivasa Rao, 2001) shows that the 12-j coefficient of the second kind is related to a multiple hypergeometric series in four variables, all four at unit argument. The results of a study of this coefficient will be reported elsewhere. He is also thankful to the organizers of this Conference for the invitation to participate abd to present this summary of the research work done.

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