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Transcendence of certain reciprocal sums of linear recurrences

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1 Introduction

Let $C$ be a field of characteristic 0 and $d$ an integer greater than 1. We consider the function $f(z)$ defined by

$$f(z) = \sum_{k \geq 0} \frac{a^k z^{d^k}}{H(z^{d^k})}, \quad (1)$$

where $H(z) \in C[z]$ with $H(0) = 1$ and $\deg H(z) \geq 1$, and $a \in C$ with $a \neq 0$. Then the function $f(z)$ satisfies the functional equation

$$af(z^d) = f(z) - \frac{z}{H(z)}. \quad (2)$$

It is known that $f(z)$ represents a rational function in the following four cases:

(i) If $d = 2$, $a = 4$, and $H(z) = (1 + z)^2$, then

$$f(z) = \sum_{k \geq 0} \frac{4^k z^{2^k}}{(1 + z^{2^k})^2} = \frac{z}{(1-z)^2}.$$

(ii) If $d = 2$, $a = -2$, and $H(z) = 1 - z + z^2$, then

$$f(z) = \sum_{k \geq 0} \frac{(-2)^k z^{2^k}}{1 - z^{2^k} + z^{2^{k+1}}} = \frac{z}{1 + z + z^2}.$$

(iii) If $d = 2$, $a = 2$, and $H(z) = 1 + z$, then

$$f(z) = \sum_{k \geq 0} \frac{2^k z^{2^k}}{1 + z^{2^k}} = \frac{z}{1 - z}.$$
(iv) If \( d = 2, a = 1, \) and \( H(z) = 1 - z^2, \) then
\[
f(z) = \sum_{k \geq 0} \frac{z^{2^k}}{1 - z^{2^{k+1}}} = \frac{z}{1 - z}.
\]

It is natural to ask whether there exist rational functions of the form (1) other than these four cases. The purpose of this paper is to answer this question.

**Theorem 1.1.** Let \( f(z) \) be the function defined by (1). Suppose that \( \deg H \leq 3. \) Then \( f(z) \) is a transcendental function over \( C(z) \) except in the four cases stated above.

In the case of \( a \neq 1, \) we can dispense with the assumption \( \deg H \leq 3. \)

**Theorem 1.2.** Let \( f(z) \) be the function defined by (1). Suppose that \( a \neq 1. \) Then \( f(z) \) is a transcendental function over \( C(z) \) except in the three cases stated above.

We shall apply Theorem 1.1 to establish the transcendence of new type of reciprocal sums of binary linear recurrences.

Let \( \{F_n\}_{n \geq 0} \) be the sequence of the Fibonacci numbers defined by
\[
F_0 = 0, \ F_1 = 1, \ F_{n+2} = F_{n+1} + F_n \quad (n \geq 0),
\]
and \( \{L_n\}_{n \geq 0} \) be the sequence of the Lucas numbers defined by
\[
L_0 = 2, \ L_1 = 1, \ L_{n+2} = L_{n+1} + L_n \quad (n \geq 0).
\]

Lucas [6] proved that
\[
\theta_1 = \sum_{k \geq 0} \frac{1}{F_{2^k}} = \frac{7 - \sqrt{5}}{2}.
\]

Erdős and Graham [5] asked for arithmetic character of the related sums
\[
\theta_2 = \sum_{k \geq 0} \frac{1}{L_{2^k}}, \quad \theta_3 = \sum_{k \geq 0} \frac{1}{F_{2^k+1}}.
\]

Transcendence of \( \theta_2 \) and that of \( \theta_3 \) were proved by Bundschuh and Pethő [2] and by Becker and Töpfer [1], respectively.

Let \( \{R_n\}_{n \geq 0} \) be a sequence of integers satisfying the binary linear recurrence relation
\[
R_{n+2} = A_1R_{n+1} + A_2R_n \quad (n \geq 0), \quad (3)
\]
where \( A_1 \neq 0, A_2 \) are integers, \( \Delta = A_1^2 + 4A_2 > 0 \) is not a perfect square, and \( R_0, R_1 \) are integers not both zero. We can express \( \{ R_n \}_{n \geq 0} \) as follows:

\[
R_n = g_1 \alpha^n + g_2 \beta^n \quad (n \geq 0),
\]

where \( g_1 = (R_1 - \beta R_0)/(\alpha - \beta), g_2 = (\alpha R_0 - R_1)/(\alpha - \beta) \), and \( \alpha, \beta \) are the roots of

\[
X^2 - A_1 X - A_2 = 0.
\]

Then we define \( R_l \) for any \( l \in \mathbb{Z} \) by

\[
R_l = g_1 \alpha^l + g_2 \beta^l.
\]

Becker and Töpfer [1] proved a more general theorem.

**Theorem A (Becker and Töpfer [1])**. Let \( \{ R_n \}_{n \geq 0} \) be a sequence of integers satisfying (3), \( \{ a_k \}_{k \geq 0} \) be a periodic sequence of algebraic numbers which is not identically zero, and \( d, c, \) and \( l \) be integers with \( d \geq 2 \) and \( c \geq 1 \). Then the number

\[
\theta = \sum_{k \geq 0}' \frac{a_k}{R_{cd^k+l}},
\]

where the sum \( \sum_{k \geq 0}' \) is taken over those \( k \) with \( cd^k + l \geq 0 \) and \( R_{cd^k+l} + b \neq 0 \), is algebraic if and only if \( \{ a_k \}_{k \geq 0} \) is a constant sequence, \( d = 2, |A_2| = 1 \), and \( R_l = 0 \).

Their result was much more improved by Nishioka, Tanaka, and Toshimitu [10]. Indeed they established the algebraic independence of the numbers

\[
\sum_{k \geq 0}' \frac{a_k}{(R_{cd^k+l})^m} \quad (d \geq 2, m \geq 1, l \in \mathbb{Z})
\]
even under a weaker condition on \( \{ R_n \}_{n \geq 0} \).

Duverney [3] showed that

\[
\sum_{k \geq 1} \frac{4^k}{L_{2^k} + 2} = 4, \quad \sum_{k \geq 1} \frac{(-2)^k}{L_{2^k} - 1} = -\frac{1}{2}.
\]

These numbers are special cases of the following reciprocal sums

\[
\phi = \sum_{k \geq 0}' \frac{a_k}{R_{cd^k+l} + b'}
\]

where the sum \( \sum_{k \geq 0}' \) is taken over those \( k \) with \( cd^k + l \geq 0 \) and \( R_{cd^k+l} + b \neq 0 \), \( \{ a_k \}_{k \geq 0} \) is a linear recurrence of algebraic numbers which is not identically zero, and \( b, c, d, \) and \( l \) are integers with \( c \geq 1 \) and \( d \geq 2 \). Using Theorem 1.1 and applying a method developed in [9], we can show that these numbers are transcendental except some few cases including the numbers given by (4).
Theorem 1.3. Let \( \{R_n\}_{n \geq 0} \) be a sequence of integers satisfying (3). Then the number \( \phi \) defined by (5) is transcendental except in the following three cases:

(i) \( |A_2| = 1, d = 2, b = 0, R_t = 0, \) and \( \{a_k\}_{k \geq 0} \) is a constant sequence.

(ii) \( |A_2| = 1, d = 2, A_1 R_t = 2 R_{t+1}, R_t = b, \) and \( a_k = c 4^k \) (\( k \geq 0 \)) for some nonzero \( c \in \mathbb{Q} \).

(iii) \( |A_2| = 1, d = 2, A_1 R_t = 2 R_{t+1}, R_t = -2b, \) and \( a_k = c(-2)^k \) (\( k \geq 0 \)) for some nonzero \( c \in \mathbb{Q} \).

Remark 1.1. Becker and Töpfer's result stated above can be deduced from Theorem 1.3.

2 Proof of Theorems

2.1 Proof of Theorem 1.1

The function \( f(z) \) is transcendental over \( C(z) \) if \( f(z) \notin C(z) \) (cf. [7]). Suppose on the contrary that \( f(z) = P(z)/Q(z) \) with \( P(z), Q(z) \in C[z] \) prime to each other. As \( f(0) = 0 \), we have \( P(0) = 0 \) and \( Q(0) \neq 0 \), so that we may assume \( Q(0) = 1 \). By (2) we have

\[
a \frac{P(z^d)}{Q(z^d)} = \frac{P(z)}{Q(z)} - \frac{z}{H(z)},
\]

and so

\[
a P(z^d)Q(z)H(z) = P(z)Q(z^d)H(z) - zQ(z)Q(z^d). \tag{6}
\]

As \( P(z^d)/Q(z^d) \) is irreducible, \( Q(z^d) \) divides \( Q(z)H(z) \). Therefore there exist \( A(z) \in C[z] \) such that

\[
A(z)Q(z^d) = Q(z)H(z). \tag{7}
\]

As \( P(0) = 0 \), we put \( P(z) = zR(z) \). We have from (6)

\[
Q(z)^2 = A(z)\{R(z)Q(z^d) - az^{d-1}R(z^d)Q(z)\}. \tag{8}
\]

In what follows, let \( h, p, q, \) and \( r \) be the degrees of \( H, P, Q, \) and \( R, \) respectively. Then we have by (7) and (8)

\[
\deg A = h - (d - 1)q \leq 2q. \tag{9}
\]
We shall prove

\[ 1 \leq 1 + r = p \leq q. \tag{10} \]

As \( P(0) = 0 \), we have \( p \geq 1 \). If \( p > q \), we get \( \deg PH > \deg zQ \), since \( 1 \leq h \) (\( \leq 3 \)) by (2) and (7). Then (6) yield \( dp + q + h = dq + p + h \), a contradiction and (10) follows.

The proof will be done in three cases; Case I. \( p < q \), Case II. \( p = q \) and \( a \neq 1 \), Case III. \( p = q \) and \( a = 1 \).

Case I. Let \( p < q \). We have \( q \geq 2 \) by (10) and \( 2q = \deg A + r + dq \) by (8). This with (10) implies \( \deg A = 0 = r \), and \( d = 2 \). Hence \( A(z) = 1 \) and \( R(z) = 1 \), since \( A(0) = 1 \) by (7) and \( R(0) = 1 \) by (8). Then we have by (8)

\[ Q(z)^2 = Q(z^2) - azQ(z). \tag{11} \]

Writing \( Q(z) = a_q z^q + a_{q-s} z^{q-s} + \cdots \), where \( a_q \neq 0 \), \( a_{q-s} \neq 0 \) (\( 1 \leq s \leq q \)), we have from (11)

\[
\begin{align*}
a_q z^{2q} + 2a_q a_{q-s} z^{2q-s} + \cdots &= a_q z^{2q} + a_{q-s} z^{2q-2s} + \cdots - az(a_q z^q + a_{q-s} z^{q-s} + \cdots).
\end{align*}
\]

We see that \( a_q = 1 \). First we consider the case of \( q \geq 3 \). If \( 1 \leq s \leq q - 2 \), then \( 2q - s > 2q - 2s \) and \( 2q - s > q + 1 \), so we have \( a_{q-s} = 0 \), which is a contradiction. Therefore we have \( s = q - 1 \) or \( s = q \). Thus we have \( Q(z) = z^q + a_1 z + 1 \), where \( a_1 \neq 0 \) if \( s = q - 1 \), \( = 0 \) if \( s = q \). We have from (11)

\[
\begin{align*}
z^{2q} + 2a_1 z^{q+1} + 2z^q + a_1^2 z^2 + 2a_1 z + 1 &= z^{2q} + a_1 z^2 + 1 - az(z^q + a_1 z + 1).
\end{align*}
\]

Noting that \( q \geq 3 \) and comparing the coefficients of \( z^q \) in the both sides, we have a contradiction.

Therefore we have \( q = 2 \), and so \( Q(z) = z^2 + a_1 z + 1 \). It follows from (11) that

\[
\begin{align*}
z^4 + 2a_1 z^3 + 2z^2 + a_1^2 z^2 + 2a_1 z + 1 &= z^4 + a_1 z^2 + 1 - az(z^2 + a_1 z + 1).
\end{align*}
\]

Comparing the coefficients of the both sides, we have

\[
2a_1 = -a, \quad 2 + a_1^2 = a_1 - aa_1.
\]

Hence we have \( (a, a_1) = (4, -2) \) or \( (-2, 1) \), and so we get

\[
f(z) = \sum_{k \geq 0} \frac{4^k z^{2k}}{(1 + 2^k z^k)^2} = \frac{z}{(1 - z)^2}.
\]
which are the rational functions given in the case (i) and (ii), respectively.

Case II. Let $p = q$ and $a \neq 1$. We have from (8) $2q = \deg A + r + dq$. This with (10) implies $\deg A = 0 = r, q = 1$, and $d = 2$. Hence $A(z) = 1$ and $R(z) = 1$, since $A(0) = 1$ by (7) and $R(0) = 1$ by (8). Writing $Q(z) = 1 - bz$ with $b \neq 0$, we have from (8)

$$1 - 2bz + b^2z^2 = 1 - bz^2 - az(1 - bz).$$

Comparing the coefficients of both sides, we have

$$b^2 = -b + ab, \quad 2b = a.$$ 

Hence we have $a = 2, b = 1$, and so we get

$$f(z) = \sum_{k \geq 0} \frac{2^k z^{2^k}}{1 + z^{2^k}} = \frac{z}{1 - z}.$$ 

which is the rational function given in the case (iii).

Case III. Let $p = q$ and $a = 1$. From (8) we have

$$Q(z)^2 = A(z)\{R(z)Q(z^d) - z^{d-1}R(z^d)Q(z)\}.$$ 

Lemma 2.1. We can express $Q(z)$ as

$$Q(z) = \Pi_{i=1}^{d-1} (1 - \gamma_i^{-1}z)^{n_i} Q_1(z),$$

where $\gamma_i (1 \leq i \leq d - 1)$ are the $(d - 1)$-th roots of unity, $n_i \geq 1 (1 \leq i \leq d - 1)$, and $Q_1(z) \in C[z]$ such that $Q_1(\gamma_i) \neq 0$ for any $i$. Furthermore

$$A(z) = \Pi_{i=1}^{d-1} (1 - \gamma_i^{-1}z)^{n_i} A_1(z),$$

where $A_1(z) \in C[z]$ such that $A_1(\gamma_i) \neq 0$ for any $i$. In particular,

$$d - 1 \leq \deg A \quad \text{and} \quad d - 1 \leq q.$$ 

Proof. Letting $z = \gamma_i$ in (12) we have $Q(\gamma_i) = 0$ for any $i$. We may put

$$Q(z) = \Pi_{i=1}^{d-1} (1 - \gamma_i^{-1}z)^{n_i} Q_1(z),$$
where $n_i \geq 1 \ (1 \leq i \leq d - 1)$ and $Q_1(z) \in C[z]$ such that $Q_1(\gamma_i) \neq 0$ for any $i$. From (12) we have

$$
\Pi_{i=1}^{d-1}(1 - \gamma_i^{-1}z)^{n_i} Q_1(z)^2 = A(z)\{R(z)Q_1(z^d)\Pi_{i=1}^{d-1}\varphi(\gamma_i^{-1}z)^{n_i} - z^{d-1}R(z^d)Q_1(z)\},
$$

where $\varphi(z) = (1 - z^d)/(1 - z)$. Letting $z = \gamma_j$ for fixed $j$, we have

$$
0 = A(\gamma_j)R(\gamma_j)Q_1(\gamma_j)\Pi_{i=1}^{d-1}\varphi(\gamma_i^{-1}\gamma_j)^{n_i}(-:1).
$$

We note that $\varphi(\gamma_i^{-1}\gamma_j) = 1$ if $i \neq j$ and $\varphi(\gamma_i^{-1}\gamma_j) = d$ if $i = j$. So $\Pi_{i=1}^{d-1}\varphi(\gamma_i^{-1}\gamma_j)^{n_i} - 1 = d^{n_j} - 1 \neq 0$. Since $R(\gamma_j)Q_1(\gamma_j) \neq 0$, we obtain $A(\gamma_j) = 0$ for any $j$. Therefore we may put

$$
A(z) = \Pi_{i=1}^{d-1}(1 - \gamma_i^{-1}z)^{n_i}A_1(z),
$$

where $A_1(z) \in C[z]$ such that $A_1(\gamma_i) \neq 0$ for any $i$. The proof of the lemma is completed.

Now we return to the proof in Case III. It follows from (9) and (13) that

$$
1 \leq \max\{d - 1, \frac{h}{d+1}\} \leq q \leq \frac{h}{d-1} - 1.
$$

(14)

In particular, we have

$$
2 \leq d(d-1) \leq h.
$$

(15)

In the case of $h = 2$, we have $d = 2$ by (15) and so $q = 1$ by (14). We have $R(z) = 1$ by (10) and $Q(z) = 1 - z$ by Lemma 2.1, which implies $A(z) = 1 - z$ by (12), and so $H(z) = 1 - z^2$ by (7). This gives the functional equation

$$
f(z) = \sum_{k \geq 0} \frac{z^{2k}}{1 - z^{2k+1}} = \frac{z}{1 - z},
$$

which is the rational function given in the case (iv).

If $h = 3$, we have $d = 2$ by (15), and hence $q = 1$ or 2 by (14). Assume that $q = 1$. Then we have $Q(z) = 1 - z, R(z) = 1$, and $A(z) = 1 - z$ by (12), which contradicts (9). If $q = 2$, we have $Q(z) = (1 - z)(1 - bz)$, where $b \neq 1$, $A(z) = 1 - z$, and $R(z) = 1 - cz$, which implies $(1 - bz)^2 = (1 - cz)(1 + z)(1 - bz^2) - z(1 - cz^2)(1 - bz)$. Letting $z = 1$ we have $b = c$ since $b \neq 1$, which is impossible since $P, R$ are coprime. The proof of Theorem 1.1 is completed.

### 2.2 Proof of Theorem 1.2

The proof is the same as those of Case I and II in Theorem 1.1, since the condition $\deg H \leq 3$ is not used there.
References


