Linear independence of the values of $q$-hypergeometric series

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In the present note we are interested in linear independence of the values of a certain class of $q$-hypergeometric series and its generalizations. We give a brief history on this topic in the first section, then state our results in the second and the third sections. Our results here are in [1], a joint work with K. Väänänen.

1. A brief history

Let us call here $q$-hypergeometric series the series of the form

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{q^{-s(z)}}{\prod_{k=0}^{n-1} P(q^{-k})} z^n,$$

where $q$ is a complex number with absolute value greater than one, $s$ is a positive integer, and $P(x)$ is a polynomial with complex coefficients satisfying $P(0) \neq 0$ and $P(q^{-n}) \neq 0$ ($n = 0, 1, 2, ...$). Note that $f(z)$ represents an entire function. By defining $R(x) = x^s P(1/x)$, the series (1.1) can be expressed as

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{\prod_{k=0}^{n-1} R(q^k)}.$$

Then, under the assumption that $\deg P \leq s$ (or equivalently, $R(x)$ is a polynomial), $f(z)$ satisfies the $q$-difference equation

$$(1.2) \quad \{R(D/q) - z\} f(z) = R(1/q), \quad Df(z) := f(qz).$$

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The cases \( R(x) = qx \) and \( R(x) = qx - 1 \) correspond to the Tschakaloff function \( T_q(z) \) and the \( q \)-exponential function \( E_q(z) \), respectively.

The study of the arithmetical nature of the values of the function \( T_q(z) \) goes back to Tschakaloff [10] in 1921. He proved the linear independence over the rational number field \( \mathbb{Q} \) of the numbers 1, \( T_q(\alpha_j) \) \((j = 1, \ldots, m)\) under a certain condition on \( q \in \mathbb{Q} \), where \( \alpha_j \) are nonzero rational numbers satisfying \( \alpha_i/\alpha_j \neq q^n \) \((n \in \mathbb{Z})\) for any \( i \neq j \), while Skolem [8] proved a similar result involving the derivatives of the function. The former result was refined in a quantitative form by Bundschuh and Shiokawa [4], and the later result by Katsurada [5]. Note that both results are valid for \( q \in \mathbb{K} \) and numbers \( \alpha_j \in \mathbb{K} \) with certain conditions, here and in what follows \( \mathbb{K} \) denotes \( \mathbb{Q} \) or an imaginary quadratic number field. Then Stihl [9] generalized the result of Bundschuh and Shiokawa to \( f(z) \) having \( P(x) \in \mathbb{K}[x] \) with \( \deg P < s \), and proved the linear independence over \( \mathbb{K} \) of the numbers

\[
1, \ f(q^k\alpha_j) \quad (j = 1, \ldots, m; k = 0, 1, \ldots, s - 1)
\]

in quantitative form under a certain condition on \( q \in \mathbb{K} \), where \( \alpha_j \) are nonzero elements of \( \mathbb{K} \) satisfying the same conditions as above. Since the functional equation (1.2) for \( f(z) \) with \( \deg P \leq s \) has the order \( s \) with respect to the \( q \)-difference operator \( D \), this result is best possible in qualitative nature. Further, Katsurada [6] put the derivatives of the function in Stihl's result to get the linear independence over \( \mathbb{K} \) of the numbers

\[
(1.3) \quad 1, \ f^{(i)}(q^k\alpha_j) \quad (i = 0, 1, \ldots, \ell; j = 1, \ldots, m; k = 0, 1, \ldots, s - 1)
\]

in quantitative form under the same conditions as Stihl's on \( q \) and \( \alpha_j \)'s, where \( \ell \) is a nonnegative integer.

We now come to the general case in which the degree of \( P(x) \) is not necessarily less than \( s \). In this direction Lototsky [7] in 1943 proved an irrationality result on \( E_q(\alpha) \) with \( q \in \mathbb{Z} \) at a rational point \( \alpha \) different from \( q^n \) \((n \in \mathbb{N})\). A quantitative refinement of this result with \( q \in \mathbb{K} \) was obtained by Bundschuh [3]. After the work of Stihl [9], on noting that \( \{R(q^k)\} \) is a linear recurrent sequence, Bézivin [2] introduced a class of entire series as follows. Let \( \{A(n)\} \) be a linear recurrent sequence of the form

\[
(1.4) \quad A(n) = \lambda_1 \theta_1^n + \cdots + \lambda_h \theta_h^n \quad (n = 0, 1, 2, \ldots),
\]
where \( \theta_i \) are nonzero algebraic integers and \( \lambda_i \) are nonzero algebraic numbers. Assume that \( A(n) \) belong to \( K^\times \), and that
\[
|\theta_1| > |\theta_2| \geq \cdots \geq |\theta_h| \geq 1 \quad \text{and} \quad 1 = \theta_h < |\theta_{h-1}| \text{ if } |\theta_h| = 1.
\]
Then we define an entire function \( \Phi(z) \) by
\[
\Phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \prod_{k=0}^{A(k)}.
\]
Denote by \( \tilde{G} \) the multiplicative group generated by \( \theta_1, \ldots, \theta_h \), Bézivin [2] proved the linear independence over \( K \) of the numbers
\[
1, \Phi^{(i)}(\alpha_j) \quad (i = 0, 1, \ldots, \ell; j = 1, \ldots, m),
\]
where \( \alpha_j \) are nonzero elements of \( K \) such that \( \alpha_i/\alpha_j \not\in \tilde{G} \) for any \( i \neq j \), and in addition that \( \lambda_h \alpha_j \not\in \tilde{G} \) (\( j = 1, \ldots, m \)) if \( \theta_h = 1 \). This result implies that, for \( f(z) \) with \( \deg P \leq s \) and an integer \( q \) in \( K \), the numbers \( (1.3) \) without powers of \( q \) are linearly independent over \( K \).

2. Generalizations of Bézivin's result

We can relax the condition \( (1.5) \) in Bézivin's result to get the following result.

**Theorem 1.** Let \( \theta_1, \ldots, \theta_h \) be nonzero algebraic integers such that
\[
|\theta_1| > 1, \quad |\theta_1| > |\theta_2| \geq \cdots \geq |\theta_h|,
\]
and that \( |\theta_h| < |\theta_{h-1}| \) if \( |\theta_h| < 1 \) and \( \theta_h = 1 < |\theta_{h-1}| \) if \( |\theta_h| = 1 \). Let \( \{A(n)\} \) be the recurrent sequence \( (1.4) \) with nonzero algebraic numbers \( \lambda_1, \ldots, \lambda_h \), and assume that \( A(n) \) belong to \( K^\times \) for all \( n \). Let \( \alpha_1, \ldots, \alpha_m \) be elements of \( K^\times \) satisfying \( \alpha_i/\alpha_j \not\in \tilde{G} \) for any \( i \neq j \). If \( \theta_h = 1 \), assume in addition that \( \lambda_h \alpha_j^{-1} \not\in \tilde{G} \) (\( j = 1, \ldots, m \)). Then the numbers \( (1.7) \) are linearly independent over \( K \).

We give an example of this theorem. Let \( \{F_n\} \) be the Fibonacci sequence defined by \( F_0 = F_1 = 1 \) and \( F_{n+2} = F_{n+1} + F_n \) \( (n = 0, 1, 2, \ldots) \), which is expressed as
\[
F_n = \lambda_1 \alpha^n + \lambda_2 \beta^n \quad (n = 0, 1, 2, \ldots),
\]
where $\alpha = (1 + \sqrt{5})/2, \beta = (1 - \sqrt{5})/2, \lambda_1 = \alpha/\sqrt{5}, \lambda_2 = -\beta/\sqrt{5}$. Since $\beta = -\alpha^{-1}$, the multiplicative group generated by $\alpha^\nu$ and $\beta^\nu$ with a positive integer $\nu$ is $\langle -1 \rangle \times \langle \alpha^\nu \rangle$ or $\langle \alpha^\nu \rangle$ according as $\nu$ is odd or even. Hence the numbers

$$1, \sum_{n=i}^{\infty} \frac{n(n-1) \cdots (n-i+1) \alpha_j^{n-i}}{F_0 F_{\nu} \cdots F_{n\nu}} (i = 0, 1, \ldots, \ell; j = 1, \ldots, m)$$

are linearly independent over $\mathbb{Q}$, if $\nu$ is odd and $\alpha_j$ are nonzero rational numbers having distinct absolute values, or if $\nu$ is even and $\alpha_j$ are nonzero distinct rational numbers.

For the next result let $\theta_i, \lambda_i \in K$ in the above, and assume that $\tilde{G}$ is a free abelian group. We take a free abelian group $\hat{G}$ of finite rank satisfying $\tilde{G} \subseteq \hat{G} \subset \mathbb{Q}^\times$. Let $r$ be the rank of $\hat{G}$, and $\Theta_1, \ldots, \Theta_r$ be a set of generators of $\hat{G}$. By using these generators we can express $\theta_i$ as

$$\theta_i = \Theta_1^{\nu_1(1)} \cdots \Theta_r^{\nu_r(r)} (i = 1, \ldots, h).$$

Define

$$\hat{S} = \{ \Theta_1^{\nu_1} \cdots \Theta_r^{\nu_r} | 0 \leq \nu_j < s_j, j = 1, \ldots, r \},$$

where

$$s_j = \max(0, e(1, j), \ldots, e(h, j)) - \min(0, e(1, j), \ldots, e(h, j)) (j = 1, \ldots, r).$$

Note that $s_j \geq 1$ for all $j$. Then we have the following result.

**Theorem 2.** Let the notations and the assumptions be as above. Let $\alpha_1, \ldots, \alpha_m$ be nonzero elements of $K$ satisfying $\alpha_i/\alpha_j \notin \hat{G}$ for any $i \neq j$. If $\theta_h = 1$, assume in addition that $\lambda_h \alpha_j^{-1} \notin \hat{G}$ $(j = 1, \ldots, m)$. Then the numbers

$$1, \Phi^{(i)}(\lambda \alpha_j) (i = 0, 1, \ldots, \ell; j = 1, \ldots, m; \lambda \in \hat{S})$$

are linearly independent over $K$.

### 3. $q$-hypergeometric series

We can apply Theorem 2 for considering the values of a series generalizing the series (1.1). Let $q_1, \ldots, q_r$ be $r$ nonzero multiplicatively independent integers in $K$. 
with $|q_i| > 1$ for all $i$, and $\mathcal{G}$ be the multiplicative group generated by them. Let $P(x_1, \ldots, x_r)$ be an element of $K[x_1, \ldots, x_r]$ satisfying

\begin{equation}
(3.1) \quad P(0, \ldots, 0) \neq 0, \quad P(q_1^{-n}, \ldots, q_r^{-n}) \neq 0 \quad (n = 0, 1, 2, \ldots).
\end{equation}

Then, for positive integers $t_1, \ldots, t_r$, we define

\begin{equation}
(3.2) \quad \phi(z) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{r} q_i^{-t_i(2)}}{\prod_{k=0}^{n-1} P(q_1^{-k}, \ldots, q_r^{-k})} z^n.
\end{equation}

This series is a particular case of the series (1.6), and reduces to the series (1.1) when $r = 1$. We first restrict ourselves to the case $\deg_{x_i} P \leq t_i \ (i = 1, \ldots, r)$.

**Theorem 3.** Let $q_i$ be as above, and $\phi(z)$ be the series (3.2) with $\deg_{x_i} P \leq t_i \ (i = 1, \ldots, r)$. Let $\alpha_1, \ldots, \alpha_m$ be nonzero elements of $K$ such that $\alpha_i/\alpha_j \notin \mathcal{G}$ for any $i \neq j$, and assume in addition that $p_{t_1, \ldots, t_r} \alpha_i^{-1} \notin \mathcal{G} \ (i = 1, \ldots, m)$ if $p_{t_1, \ldots, t_r} \neq 0$, where $p_{t_1, \ldots, t_r}$ is the coefficient of $x_1^{t_1} \cdots x_r^{t_r}$ in $P(x_1, \ldots, x_r)$. Then the numbers

\begin{equation}
(3.3) \quad 1, \phi^{(0)}(\lambda \alpha_j) \quad (i = 0, 1, \ldots, \ell; j = 1, \ldots, m; \lambda \in S_1)
\end{equation}

are linearly independent over $K$, where

$$S_1 = \{q_1^{k_1} \cdots q_r^{k_r} \mid 0 \leq k_i < t_i \ (i = 1, \ldots, r)\}.$$ 

To give a result without the condition $\deg_{x_i} P \leq t_i \ (i = 1, \ldots, r)$ we assume that $P(x_1, \ldots, x_r)$ is a product of polynomials $P_i(x_i) \in K[x_i]$.

**Theorem 4.** Let $\phi(z)$ be the series (3.2) with $P(x_1, \ldots, x_r) = P_1(x_1) \cdots P_r(x_r)$, where $P_i(x_i) \in K[x_i]$ and the condition (3.1) is satisfied. Let $\alpha_1, \ldots, \alpha_m$ be nonzero elements of $K$ such that $\alpha_i/\alpha_j \notin \mathcal{G}$ for any $i \neq j$, and assume in addition that $p_{t_1, \ldots, t_r} \alpha_j^{-1} \notin \mathcal{G} \ (i = 1, \ldots, m)$ if $p_{t_1, \ldots, t_r} \neq 0$, where $p_{t_1, \ldots, t_r}$ is the coefficient of $x_1^{t_1} \cdots x_r^{t_r}$ in $P_i(x_i)$. Then the numbers (3.3) with $S_2$ instead of $S_1$ are linearly independent over $K$, where

$$S_2 = \{q_1^{k_1} \cdots q_r^{k_r} \mid 0 \leq k_i < s_i \ (i = 1, \ldots, r)\}, \quad s_i = \max(t_i, \deg P_i).$$
The following is a direct consequence of Theorem 4, which generalizes Kat-

Corollary. Let \( q \) be an integer in \( K \) with \( |q| > 1 \). Let \( f(z) \) be the series (1.1) with \( P(z) \in K[z] \) satisfying \( P(0) \neq 0, P(q^{-n}) \neq 0 \) (\( n = 0, 1, 2, \ldots \)). Let \( \alpha_1, \ldots, \alpha_m \) be nonzero elements of \( K \) such that \( \alpha_i/\alpha_j \neq q^n \) (\( n \in \mathbb{Z} \)) for any \( i \neq j \). Assume in addition that \( p_s\alpha_j^{-1} \neq q^n \) (\( n \in \mathbb{Z}, j = 1, \ldots, m \)) if \( p_s \neq 0 \), where \( p_s \) is the coefficient of \( x^s \) in \( P(x) \). Then the numbers (1.3) are linearly independent over \( K \).

References


[10] L. Tschakaloff, Arithmetische Eigenschaften der unendlichen Reihe \( \sum_{\nu=0}^{\infty} x^\nu a^{-\frac{1}{2}\nu(\nu+1)} \) I, Math. Ann. 80 (1921) 62–74; II, ibid. 84 (1921), 100–114.