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On special values of the j-function and related modular functions

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1. Introduction

Classically, the theory of complex multiplication asserts that the value of the usual elliptic modular function j at an imaginary quadratic number τ is an algebraic integer and generates a ring class field of the corresponding imaginary quadratic number field. Of course, these singular moduli also play an important role in modern number theory (see the well-known papers by Gross-Zagier, Borcherds and others and most recently by Zagier [3] where traces of singular moduli are studied).

Here we would like to report on recent joint work with J.H. Bruinier and K. Ono in which we consider a weighted average sum of the values of j (and of a sequence of related modular functions) over the points of the divisor of an arbitrary non-zero meromorphic modular function f. We explicitly relate these average sums to the exponents in the q-product expansion of f. For details (and other related results) we refer the reader to [1].

2. The j-function and Heegner points

We let $\mathcal{H} := \{z \in \mathbf{C} \,|\, \Im z > 0\}$ be the complex upper half-plane and put $q := e^{2\pi i z}$ for $z \in \mathcal{H}$. All modular functions that occur are with respect to the full modular group $\Gamma_1 := SL_2(\mathbf{Z})$.

Recall that by definition

$$j(z) := \frac{E_4^3(z)}{\Delta(z)}$$

where E_4 resp. Δ are the normalized Eisenstein series of weight 4 resp. the discriminant function of weight 12.

Then we have the following *facts*:

i) j is a modular function of weight zero, holomorphic on $\mathcal{H}$, with a simple pole at infinity;
ii) $j = \frac{1}{q} + 744 + 196884q + \cdots$ has a q-expansion with integral coefficients;
iii) j gives a bijection between $\Gamma_1 \backslash \mathcal{H}$ and $\mathbf{C}$;
iv) Any meromorphic modular function f of weight zero is a rational function in j. If f has Fourier coefficients in a field K, then $f \in K(j)$.

Recall that a point $\tau \in \mathcal{H}$ is called a Heegner point if τ satisfies a quadratic equation $a\tau^2 + b\tau + c = 0$ with $a, b, c \in \mathbf{Z}$, $(a, b, c) = 1$ and $b^2 - 4ac = D < 0$.

These points τ are very important in classical and modern number theory. For example, one knows that the "singular modulus" $j(\tau)$ is an algebraic integer and generates a ring class field of $\mathbf{Q}(\sqrt{D})$ ("theory of complex multiplication").

By the famous Gross-Zagier formula (1983), the heights of Heegner points (viewed as points on modular curves, where in the above definition one has to add appropriate congruence conditions) relate –very roughly speaking– to derivatives of L-series of modular curves at $s = 1$.

Furthermore, Borcherds (1995) describes modular functions whose divisors are supported at the cusps and Heegner points in terms of infinite product expansions where the exponents involve Fourier coefficients of modular functions of half-integral weight.

Finally, Zagier [3] relates traces of singular moduli to Fourier coefficients of half-integral weight modular functions.

A maybe naive but at first sight reasonable question is the following: Are there other "natural" points $\tau \in \mathcal{H}$ such that $j(\tau)$ is algebraic and has interesting properties?

After a little thought, of course, one remembers the famous result of Schneider which asserts that if $\tau \in \mathcal{H}$ is algebraic of degree > 2, then $j(\tau)$ is transcendental. Thus regarding the above question, it seems to be completely unclear what numbers τ one would have to look for.

However, there is the following

Lemma 1. *Let f be a non-zero meromorphic modular function of integral weight k. Write $f = \sum_{n \geq h} a(n) q^n$ and suppose that $a(h) = 1$. Assume that $a(n) \in K$ for all n where K is a field. Let $\tau \in \mathcal{H}$ be a zero or pole of f. Then $j(\tau)$ is algebraic over K.*

For the *proof* one simply notes that $\frac{f^{12}}{\Delta^k}$ is a meromorphic modular function of weight zero with Fourier coeffcients in K, hence is a K-rational function in j. Thus the result follows.

Now a natural question seems to be the following: What can be said about arithmetic properties of the values $j(\tau)$ for τ as above?

3. Values of j at points of the divisors of modular functions

More generally as above, for a positive integer m let $j_m(z)$ be the unique meromorphic modular function of weight zero that is holomorphic on $\mathcal{H}$ and with q-expansion

$$j_m(z) = q^{-m} + \mathcal{O}(q).$$

It is clear that these functions exist and are uniquely determined. They also occur in [3]. Each j_m is a monic polynomial of degree m in j with integral coefficients (e.g. $j_1 = j - 744$, $j_2 = j^2 - 1488j + 159768, \cdots$).

We put

$$j_0(z) := 1.$$

Clearly the assertion of Lemma 1 is also true with j replaced by j_m.

In order to state our result, we need the following

Lemma 2 (Eholzer-Skoruppa [2] a.o.) *Let $f = \sum_{n \geq h} a(n)q^n$ be a function meromorphic at $q = 0$, with $a(h) = 1$. Then there are uniquely determined complex numbers $c(n)$ $(n \in \mathbf{N})$ such that*

$$f = q^h \prod_{n \geq 1} (1 - q^n)^{c(n)} \qquad (|q| < \epsilon).$$

Moreover, if $\theta f = q\frac{d}{dq}f$, then

$$\frac{\theta f}{f} = h - \sum_{n \geq 1} (\sum_{d|n} dc(d))q^n.$$

Theorem [1]. *Let f be a non-zero meromorphic modular function of integral weight k. Write $f = \sum_{n \geq h} a(n)q^n$ and assume that $a(h) = 1$. Then for all $n \geq 1$ one has*

$$\sum_{d|n} dc(d) = 2k\sigma_1(n) + \sum_{\tau \in \Gamma_1 \backslash \mathcal{H}} \frac{1}{e_\tau} \operatorname{ord}_\tau f \cdot j_n(\tau)$$

where e_τ is equal to 3 resp. 2 resp. 1 according as τ is equivalent to $\rho := e^{2\pi i/3}$, to i or neither.

For the *proof* one appropriately modifies the usual proof of the valence formula for the number of zeros and poles of f in the standard fundamental domain for Γ_1, roughly speaking by inserting the function $j_n(z)$ under the integral. For details we refer to [1].

Let us indicate several corollaries of the Theorem. The following is immediate, by Lemma 2.

Corollary 1 [1]. *If all the $a(n)$ are contained in a field K, then $\sum_{\tau \in \Gamma_1 \backslash \mathcal{H}} \frac{1}{e_\tau} \operatorname{ord}_\tau f \cdot j_n(\tau)$ is contained in K, for all n.*

If one takes for f certain modular functions of weight zero constructed by Borcherds, one obtains for the above sums the traces of singular moduli.

Corollary 2 [1]. *Let $\partial f := 12\theta f - kE_2 f$ (where $E_2 := 1 - 24\sum_{n \geq 1} \sigma_1(n)q^n$) be the usual derivation on modular forms raising the weight by 2. Then*

$$\partial f = -12f \cdot \sum_{n \geq 0} (\sum_{\tau \in \Gamma_1 \backslash \mathcal{H}} \frac{1}{e_\tau} \operatorname{ord}_\tau f \cdot j_n(\tau))q^n.$$

The *proof* follows easily by combining the Theorem and Lemma 2, bearing in mind the valence formula for f and the definition $j_0 := 1$.

One can refine Lemma 1 to get integrality statements, too. For example, if f is a modular form and all the $a(n)$ are contained in a ring R, then one easily sees that $j_n(\tau)$ is integral over R if τ is a zero of f, for all n. One can then also prove congruences. For example

Corollary 3 [1]. *Let E_k be the normalized Eisenstein series of weight k (constant Fourier coefficient equal to 1). Let $p \geq 5$ be a prime with $(p-1)|k$. Then for all n one has*

$$\sum_{\tau \in \Gamma_1 \backslash \mathcal{H}} \frac{1}{e_\tau} \operatorname{ord}_\tau E_k \cdot j_n(\tau) \equiv -2k\sigma_1(n) \pmod{p}.$$

The *proof* is immediate from the Theorem and the v. Staudt-Clausen congruence.

References

[1] J.H. Bruinier, W. Kohnen and K. Ono: The arithmetic of the values of modular functions and the divisors of modular forms. Preprint 2001

[2] W. Eholzer and N.-P. Skoruppa: Product expansions of conformal characters. Phys. Lett. B 388 (1996), 82-89

[3] D. Zagier: Traces of singular moduli. Preprint 2001

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