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ON THE SIMULTANEOUS DISTRIBUTION OF THE FRACTIONAL PARTS OF DIFFERENT POWERS OF PRIMES

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1. Introduction

In 1940, I.M.Vinogradov[1] considered the distribution of the fractional parts of the sequence $f\sqrt{p}$, where p runs over prime numbers and f is a positive constant. This celebrated work motivated the interests of many authors to investigate the distribution of p^α modulo 1 by various methods.

In 1991, D.I. Tolev[2] studied the simultaneous distribution of the fractional parts of different powers of primes . Suppose $k \geq 2$ is a fixed integer and $0 < \alpha_k < \cdots < \alpha_1 < 1$ are real numbers, $\Gamma \subset \mathbb{R}^k$ is defined by

$$\Gamma = \Gamma(\xi_1, \eta_1, \cdots, \xi_k, \eta_k) = \{(x_1, \cdots, x_k) : \xi_i < x_i < \eta_i, 1 \leq i \leq k\},$$

where $0 < \xi_i < \eta_i \leq 1, 1 \leq i \leq k$. Let $\mu(\Gamma) = \prod_{i=1}^k (\eta_i - \xi_i)$, and let $S(x;\Gamma)$ denote the number of primes not greater than x and satisfy the condition

$$(\{p^{\alpha_1}\}, \cdots, \{p^{\alpha_k}\}) \in \Gamma,$$

where $\{t\}$ means the fractional part of t. Then Tolev proved that

$$S(x;\Gamma) = \pi(x)\left(\mu(\Gamma) + O(x^{-\frac{\delta}{3}} \log^{k+9} x)\right) \tag{1}$$

with

$$\delta = \min(1 - \alpha_1, \alpha_1 - \alpha_2, \cdots, \alpha_{k-1} - \alpha_k, \alpha_k, 1/4).$$

We first give the outline of Tolev's proof. It suffices to establish the inequality

$$R(Y) \ll Y^{-\delta/3} \log^{k+9} Y \tag{2}$$

for all $Y \in [x^{1-\delta}, x]$, where

$$R(Y) = \sup_{\Gamma} \left| \frac{S(2Y;\Gamma) - S(Y;\Gamma)}{\pi(2Y) - \pi(Y)} - \mu(\Gamma) \right|.$$

The following Lemma 1 can be used to transform the estimation of $R(Y)$ into an exponential sum problem.

Lemma 1. If $Z_n = (Z_{1,n}, \cdots, Z_{k,n})(n = 1, 2, 3, \cdots)$ is a sequence of k-dimensional vectors and its discrepency is defined by

$$D_N = \sup_{\Gamma} \left| \frac{1}{N} \sum_{\substack{n \le N \\ (Z_{1,n}, \cdots, Z_{k,n}) \in \Gamma}} 1 - \mu(\Gamma) \right|.$$

Then for any $H > 0$, we have

$$D_N \ll \frac{1}{H} + \sum_{0 < \|h\| \le H} \frac{1}{r(h)} \left| \frac{1}{N} \sum_{n \le N} e(< h, Z_n >) \right|,$$

where $h = (h_1, \cdots, h_k)$ denotes the k-dimensional integer vector,

$$\| h \| = \max_{1 \le i \le k} |h_i|, r(h) = \prod_{i=1}^{k} \max(|h_i|, 1),$$

$< \cdot, \cdot >$ denotes the Euclidean scalar product in $\mathbb{R}^k$ and $e(x) = e^{2\pi i x}$.

So for every $H > 2$, by Lemma 1 one has

(3)
$$\begin{aligned} R(Y) &\ll H^{-1} + \sum_{0 < \|h\| \le H} \frac{1}{r(h)} \\ &\quad \times \left| \frac{1}{\pi(2Y) - \pi(Y)} \sum_{Y < p \le 2Y} e(h_1 p^{\alpha_1} + \cdots + h_k p^{\alpha_k}) \right| \\ &\ll H^{-1} + Y^{-1/2} \log^{k+2} Y + Y^{-1} \log Y \sum_{0 < \|h\| \le H} \frac{1}{r(h)} |U(h)|, \end{aligned}$$

where

$$U(h) = \sum_{Y < n \le 2Y} \Lambda(n) e(V(t)),$$
$$V(t) = h_1 t^{\alpha_1} + \cdots + h_k t^{\alpha_k},$$

$\Lambda(n)$ is the Mangoldt function.

Now the problem is reduced to estimate the exponential sum $U(h)$. Tolev connected the sum $U(h)$ with the well-known formula

$$\sum_{n\leq x}\Lambda(n)=x-\sum_{|\rho|\leq T}\frac{x^\rho}{\rho}+O(\frac{x\log^2 xT}{T}+\log x).$$

Then he obtained his result with the help of the zero-density estimates.

2. Some new results

Tolev's result can be further improved by different methods.

Let

$$\delta_1=\min(1-\alpha_1,\alpha_1-\alpha_2,\cdots,\alpha_{k-1}-\alpha_k,\alpha_k/3,20/177).$$

We take $H=Y^{\delta_1}/\log Y$ in (3).

For a fixed $h=(h_1,\cdots,h_k)\neq(0,\cdots,0)$ with $|h_i|\leq H(1\leq i\leq k)$, consider the function

$$V(t)=h_1t^{\alpha_1}+\cdots+h_kt^{\alpha_k},$$

where $Y<t\leq 2Y$. Let d be the first integer with $h_j\neq 0$, then

$$V(t)=h_dt^{\alpha_d}+g(t).$$

Since $\delta_1\leq\alpha_d-\alpha_{d+1}$, we have $g(t)=O(|h_d|Y^{\alpha_d}/\log Y)$.

Now we can write

$$U(h)=\sum_{Y<n\leq 2Y}\Lambda(n)e(h_dn^{\alpha_d}+g(n)).$$

So $U(h)$ can be estimated more effectively by using the method of exponential sums directly and Finally we can prove that

$$U(h)\ll Y^{1-\delta_1}\log^{11.5}Y, \tag{4}$$

which yields the following(see next Section)

Theorem 1. We have

$$S(x;\Gamma)=\pi(x)\left(\mu(\Gamma)+O(x^{-\delta_1}\log^{k+11.5}x)\right) \tag{5}$$

$$\delta_1 = \min(1 - \alpha_1, \alpha_1 - \alpha_2, \cdots, \alpha_{k-1} - \alpha_k, \alpha_k/3, 20/177).$$

Example 1. Take $k = 2$. If $80/177 < \alpha_1 < 157/177, 60/177 < \alpha_2 < \alpha_1 - 20/177$, then

$$S(x;\Gamma) = \pi(x)\mu(\Gamma) + O(x^{157/177}\log^{k+12.5} x).$$

Similarly we can prove

Theorem 2. We have

$$S(x;\Gamma) = \pi(x)\left(\mu(\Gamma) + O(x^{-\delta_2}\log^{k+11.5} x)\right) \tag{6}$$

with

$$\delta_2 = \min(\alpha_1 - \alpha_2, \cdots, \alpha_{k-1} - \alpha_k, \alpha_k/3, 40/407).$$

Example 2. Take $k = 2$. If $160/407 < \alpha_1 < 1, 120/407 < \alpha_2 < \alpha_1 - 40/407$, then

$$S(x;\Gamma) = \pi(x)\mu(\Gamma) + O(x^{367/407}\log^{k+12.5} x).$$

Both of the above Theorems improve Tolev's result. If α_1 is very close to 1, then Theorem 2 is better.

It is obvious that Theorem 1 and Theorem 2 are very weak if

$$\delta_0 = \min(\alpha_1 - \alpha_2, \cdots, \alpha_{k-1} - \alpha_k)$$

is very small. We shall use a different approach to study this case. In this approach, we need to estimate exponential sums of the type

$$S_d(M) = \sum_{M<m\le M_1} e(f_d(m)),$$

where

$$f_d(m) = a_1 m^{\gamma_1} + \cdots + a_d m^{\gamma_d},$$

$d \ge 2$ is a fixed integer, $a_1, \cdots, a_d$ are any real numbers such that $a_1 a_2 \cdots a_d \ne 0$, $\gamma_1, \cdots, \gamma_d$ are real non-integer constants, M and M_1 are real numbers such that $5 < M < M_1 \le 2M$.

We shall use the method of van der Corput to estimate $S_d(M)$. For example, we use the second order derivative method. It is possible that for some $t \in (M, M_1]$, $|f_d''(t)|$ is very small. Consider this example:

$$f_2(m) = a_1 m^{\gamma_1} - a_2 m^{\gamma_2}, a_1 > 0, a_2 > 0.$$

Let

$$m_0 = \left(\frac{a_2\gamma_2(\gamma_2 - 1)}{a_1\gamma_1(\gamma_1 - 1)}\right)^{\frac{1}{\gamma_1 - \gamma_2}},$$

and we suppose $m_0 \in (M, M_1]$. Obviously $f''(m_0) = 0$. So we can not use the method of van der Corput in the whole interval $(M, M_1]$ directly (the second order derivative). Suppose $\eta > 0$ is a parameter to be chosen later. We divide the interval $(M, M_1]$ into two parts as follows:

$$I_1 = \{t \in (M, M_1] : |f_d''(t)| \le \eta\},$$
$$I_2 = \{t \in (M, M_1] : |f_d''(t)| > \eta\}.$$

Then

$$S_d(M) = \sum_{m \in I_1} e(f_d(m)) + \sum_{m \in I_2} e(f_d(m)) = S_1 + S_2.$$

S_2 can be estimated by the method of van der Corput directly, S_1 is bounded by the number of integers in I_1. Finally we choose an η such that the two estimates are equal.

Set $R = |a_1|M^{\gamma_1} + \cdots + |a_d|M^{\gamma_d}$. Using the idea above we can prove the following two Lemmas, which have been published in Zhai[3].

Lemma 2. If $R \le \Delta M$, where Δ is a fixed positive constant small enough, then

$$S_d(M) \ll MR^{-1/d}.$$

Lemma 3. If $R \ll M^2$, then

$$S_d(M) \ll R^{1/2} + MR^{-1/(d+1)}.$$

Let $\delta_3 = \min(1/(4k+6), \alpha_k/(4k-2))$, take $H = Y^{\delta_3}$ in (3) and then estimate $U(h)$ by the above two Lemmas. Finally we can get the following

Theorem 3. We have

$$S(x;\Gamma) = \pi(x)\left(\mu(\Gamma) + O(x^{-\delta_3}\log^{k+5.5} x)\right). \tag{7}$$

Example 3. Take $k = 2$. Suppose $6/14 < \alpha_2 < \alpha_1 < 1, \alpha_1 - \alpha_2 < 1/14$.

From Theorem 1 we have

$$S(x;\Gamma) = \pi(x)\mu(\Gamma) + O(x^{1-\delta_4}\log^{12.5} x)$$

with $\delta_4 = \min(1 - \alpha_1, \alpha_1 - \alpha_2)$.

From Theorem 2 we have

$$S(x;\Gamma) = \pi(x)\mu(\Gamma) + O(x^{1-\delta_5}\log^{12.5} x)$$

with $\delta_5 = \alpha_1 - \alpha_2$.

However Theorem 3 yields

$$S(x;\Gamma) = \pi(x)\mu(\Gamma) + O(x^{1-\delta_6}\log^{6.5} x)$$

with $\delta_6 = 1/14$..

3. Proofs of Theorems 1 and 2

From Section 2 we know that in order to prove Theorems 1 and 2, we should estimate exponential sums of the form

$$S(Y;h,\alpha) = \sum_{Y<m\le 2Y} \Lambda(m)e(h_d m^{\alpha} + g(m)),$$

where Y is a large positive real number, $0 < \alpha < 1$, $0 < \delta < 1/3$ is a function of α, h is an integer such that $1 \le h \ll T^{\delta}$, and $g(m)$ is a real function on $[Y, 2Y]$ of the form

$$g(m) = u_1 m^{\gamma_1} + \cdots + u_l m^{\gamma_l}$$

such that $|g^{(j)}(m)| \le \varepsilon h Y^{\alpha-j} (j = 0, 1, 2, \cdots, 6)$ for some fixed integer $l \ge 1$ and $\gamma_1, \cdots, \gamma_l$ real constants. According to Vaughen's identity, $S(Y;h,\alpha)$ can be written as sums of so-called Type I and Type II sums. Both of Type I and Type II sums can be estimated by the method of van der Corput. And finally we can get the following Propositions.

Proposition 3.1. Suppose $340/351 < \alpha < 1$, $\delta = \delta(\alpha) = \min(1-\alpha, 20/177)$, $0 < \Delta \le \delta$. Then, for $h \ll Y^{\delta}$, we have

$$S(Y; h, \alpha) \ll Y^{1-\Delta} \log^{11.5} Y.$$

Proposition 3.2.. Suppose $340/351 < \alpha < 1$, $\delta = 40/407$, $0 < \Delta \le \delta$. Then, for $h \ll Y^{\delta}$, we have

$$S(Y; h, \alpha) \ll Y^{1-\Delta} \log^{11.5} Y.$$

Proposition 3.3. Suppose $0 < \alpha < 4/5$, $\delta = \min((1-\alpha)/3, \alpha/4)$, $0 < \Delta \le \delta$. Then, for $h \ll Y^{\delta}$, we have

$$S(Y; h, \alpha) \ll Y^{1-\Delta} \log^{5.5} Y.$$

Proposition 3.4. Suppose $0 < \alpha < 2/3$, $\delta = \min((1-\alpha)/3, \alpha/2, 1/6)$. Then, for $h \ll Y^{\delta}$, we have

$$\sum_{m \sim M} \Lambda(m) e(hm^{\alpha}) \ll Y^{1-\delta} \log^{4.5} Y.$$

Proof of Theorem 1 : Let $h = (h_1, \cdots, h_k)$ satisfy $0 < \|h\| \le H$ and d be the first integer j with $h_j \ne 0$, then $V(t) = h_d t^{\alpha_d} + g(t)$.

If $\alpha_d > 340/351$, we use Proposition 3.1 to estimate $U(h)$. We take $\Delta = \alpha_d - \alpha_{d+1}$ if $\alpha_d - \alpha_{d+1} \le \min(1-\alpha_d, 20/177)$, and $\Delta = \min(1-\alpha_d, 20/177)$. We get

$$\begin{aligned} U(h) &\ll Y^{1-\min(1-\alpha_d, \alpha_d-\alpha_{d+1}, 20/177)} \log^{11.5} Y \\ &\ll Y^{1-\min(1-\alpha_1, \alpha_d-\alpha_{d+1}, 20/177)} \log^{11.5} Y \\ &\ll Y^{1-\delta_1} \log^{11.5} Y. \end{aligned}$$

Now suppose $\alpha_d \le 340/351$. If $h_{d+1} = \cdots = h_k = 0$, then by Proposition 3.4 we get

$$\begin{aligned} U(h) &\ll Y^{1-\min((1-\alpha_d)/3, \alpha_d/2, 1/6)} \log^{4.5} Y \\ &\ll Y^{1-\min(\alpha_k/2, 191/1593)} \log^{4.5} Y \\ &\ll Y^{1-\delta_1} \log^{11.5} Y. \end{aligned}$$

If there is at least one $h_j \ne 0 (j > d)$, then $d \le k-1$. By Proposition 3.3 we have

$$U(h) \ll Y^{1-\min((1-\alpha_d)/3, \alpha_d-\alpha_{d+1}, \alpha_d/4)} \log^{5.5} Y.$$

If $\alpha_d - \alpha_{d+1} \leq \alpha_d/4$, then

$$\min((1-\alpha_d)/3, \alpha_d - \alpha_{d+1}, \alpha_d/4) = \min((1-\alpha_d)/3, \alpha_d - \alpha_{d+1}).$$

If $\alpha_d - \alpha_{d+1} > \alpha_d/4$, then

$$\alpha_d/4 \geq \alpha_{d+1}/3 \geq \alpha_k/3.$$

So we have

$$\begin{aligned} U(h) \quad &\ll Y^{1-\min((1-\alpha_d)/3, \alpha_d-\alpha_{d+1}, \alpha_d/4)} \log^{5.5} Y \\ &\ll Y^{1-\min((1-\alpha_d)/3, \alpha_d-\alpha_{d+1}, \alpha_k/3)} \log^{5.5} Y \\ &\ll Y^{1-\delta_1} \log^{5.5} Y. \end{aligned}$$

This copletes the proof of (4) and hence Theorem 1.

Using Proposition 3.2 instead of Proposition 3.1 we can Theorem 2.

4. Proof of Theorem 3

Suppose $l \geq 2$ is a fixed integer, $1 > \gamma_1 > \gamma_2 > \cdots > \gamma_l > 0$ are real numbers, Y is a large positive number, $0 < \delta = \delta(\gamma_1) < 1/2$ is a constant depending only on γ_1. Let

$$S(Y; h_1, \cdots, h_l, \gamma_1, \cdots, \gamma_l) = \sum_{Y<n\leq 2Y} \Lambda e\left(\sum_{j=1}^{l} h_j n^{\gamma_j}\right),$$

where h_j are real numbers such that $1 \leq |h_j| \leq Y^{\delta}, j = 1, \cdots, l$. From Section 2 we know that in order to prove Theorem 3 , we should estimate the exponential sum $S(Y; h_1, \cdots, h_l, \gamma_1, \cdots, \gamma_l)$.

By Lemma 2 and Lemma 3 we can prove the following

Proposition 4.1. Let $\delta = \min(\gamma_1/(4l-2), 1/(4l+6))$. Then we have

$$S(Y; h_1, \cdots, h_l, \gamma_1, \cdots, \gamma_l) \ll Y^{1-\delta} \log^{5.5} Y.$$

Proof of Theorem 3. Following the proof of Theorem 1, we only need to estimate $U(h)$ for fixed $h = (h_1, \cdots, h_k) \neq (0, \cdots, 0)$. We take $H = Y^{\delta_3}$ in (3).

Let $n_0(h)$ denote the number of h_j such that $h_j \neq 0$, and let d denote the first integer j with $h_j \neq 0$. If $n_0(h) \geq 2$, then by Proposition 4.1 we have

$$\begin{aligned} U(h) &\ll Y^{1-\min(1/(4n_0(h)+6),\alpha_d/(4n_0(h)-2))} \log^{5.5} Y \\ &\ll Y^{1-\min(1/(4k+6),\alpha_k/(4k-2))} \log^{5.5} Y. \end{aligned}$$

Now suppose $n_0(h) = 1$. If $\alpha_d \geq 340/531$, then by Propositopn 3.2 we have

$$U(h) \ll Y^{1-40/407} \log^{11.5} Y \ll Y^{1-\delta_3} \log^{5.5} Y.$$

If $\alpha_d < 340/531$, then by Proposition 3.4 we get

$$\begin{aligned} U(h) &\ll Y^{(1-\alpha_d)/3,1/6,\alpha_d/2} \log^{4.5} Y \\ &\ll Y^{1-\delta_3} \log^{5.5} Y. \end{aligned}$$

This completes the proof of Theorem 3.

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