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Two-fold ground states of the Pauli-Fierz Hamiltonian including spin

Fumio Hiroshima

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Abstract

The Pauli-Fierz Hamiltonian describes an interaction between a low energy electron and photons. Existence of ground states has been established. The purpose of this talk is to show that its ground states is exactly two-fold in a weak coupling region.

1 The Pauli-Fierz Hamiltonian

This is a joint work\(^1\) with Herbert Spohn\(^2\). The Hamiltonian in question is the Pauli-Fierz Hamiltonian in nonrelativisitc QED with spin, which will be denoted by \(H\) acting on the Hilbert space

\[
\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}.
\]

Here \(L^2(\mathbb{R}^3; \mathbb{C}^2)\) denotes the Hilbert space for the electron with spin \(\sigma\), where \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\) denotes the Pauli spin 1/2 matrices,

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

\(\mathcal{F}\) is the symmetric Fock space for the photons given by \(\mathcal{F} = \bigoplus_{n=0}^{\infty} (L^2(\mathbb{R}^3 \times \{1, 2\}))^n_{\text{sym}}\).

Here \((\cdots)^n_{\text{sym}}\) denotes the \(n\)-fold symmetric tensor product of \((\cdots)\) with \((\cdots)^0_{\text{sym}} = \mathbb{C}\).

The photons live in \(\mathbb{R}^3\) and have helicity \(\pm 1\). The Fock vacuum is denoted by \(\Omega\). The photon field is represented in \(\mathcal{F}\) by the two-component Bose field \(a(k, j), j = 1, 2\), with commutation relations

\[
[a(k, j), a^*(k', j')] = \delta_{jj'}\delta(k - k'),
\]
\[ [a(k, j), a(k', j')] = 0, \quad [a^*(k, j), a^*(k', j')] = 0. \]

The energy of the photons is given by
\[ H_f = \sum_{j=1,2} \int \omega(k) a^*(k, j) a(k, j) dk, \]
i.e., \( H_f \) restricted to \((L^2(\mathbb{R}^3 \times \{1, 2\}))^{\text{symm}}_n\) is the multiplication by \( \sum_{j=1}^n \omega(k_j) \), and the momentum of the photons is
\[ P_f = \sum_{j=1,2} \int k a^*(k, j) a(k, j) dk. \]

Throughout units are such that \( \hbar = 1, c = 1 \). Physically \( \omega(k) = |k| \). The case is somewhat singular and we assume that \( \omega \) is continuous, rotation invariant, and that
\begin{enumerate}
  \item \( \inf_{k \in \mathbb{R}^3} \omega(k) \geq \omega_0 > 0 \),
  \item \( \omega(k_1) + \omega(k_2) \geq \omega(k_1 + k_2) \),
  \item \( \lim_{|k| \to \infty} \omega(k) = \infty \).
\end{enumerate}
A typical example is
\[ \omega(k) = \sqrt{|k|^2 + m_{\text{ph}}^2}, \quad m_{\text{ph}} > 0. \]

For a recent result of the massless case see [3]. The quantized transverse vector potential is defined through
\[ A_\varphi(x) = \sum_{j=1,2} \int \frac{\tilde{\varphi}(k)}{\sqrt{2\omega(k)}} e_j(k) \left( a^*(k, j) e^{-ikx} + a(k, j) e^{ikx} \right) dk. \]
Here \( e_1 \) and \( e_2 \) are polarization vectors which together with \( \hat{k} = k/|k| \) form a standard basis in \( \mathbb{R}^3 \). \( \varphi : \mathbb{R}^3 \to \mathbb{R} \) is a form factor which ensures an ultraviolet cutoff. It is assumed to be \( \varphi(Rx) = \varphi(x) \) for an arbitrary rotation \( R \), continuous, bounded with some decay at infinity, and normalized as \( \int \varphi(x) dx = 1 \). We will work with the Fourier transform \( \tilde{\varphi}(k) = (2\pi)^{-3/2} \int \varphi(x) e^{-ikx} dx \). It satisfies
\begin{enumerate}
  \item \( \tilde{\varphi}(Rk) = \tilde{\varphi}(k) \),
  \item \( \overline{\tilde{\varphi}} = \tilde{\varphi} \)
\end{enumerate}
for notational simplicity, \( \tilde{\varphi}(0) = (2\pi)^{-3/2} \), and \( \tilde{\varphi}(k) \) satisfies
\[ \int \left( \omega(k)^{-2} + \omega(k)^{-1} + 1 + \omega(k) \right) |\tilde{\varphi}(k)|^2 dk < \infty. \]

The quantized magnetic field is correspondingly
\[ B_\varphi(x) = i \sum_{j=1,2} \int \frac{\tilde{\varphi}(k)}{\sqrt{2\omega(k)}} (k \times e_j(k)) \left( a^*(k, j) e^{-ikz} - a(k, j) e^{ikx} \right) dk. \]

With these preparation the Pauli-Fierz Hamiltonian, including spin, is defined by
\[ H = \frac{1}{2} (-i \nabla_x \otimes 1 - eA_\varphi(x))^2 + 1 \otimes H_f - \frac{e}{2} \sigma \otimes B_\varphi(x). \] (1.1)
Since obvious from the context we will drop the tensor notation \( \otimes \).
2 Invariances

2.1 Total momentum

Let us define the total momentum by $P_{\text{total}} = -i\nabla_x + P_f$. We see that

$$[P_{\text{total}}, H] = 0.$$  \hspace{1cm} (2.1)

(2.1) immediately implies that $H$ has no ground state. Instead of $H$ we consider the Hamiltonian with a fixed total momentum as follows. By (2.1), we see that (1.1) is decomposable with respect to the spectrum of $P_{\text{total}},$

$$H = \int_{\mathbb{R}^3} H_p dp,$$

where

$$H_p = \frac{1}{2} (p - P_f - eA_\varphi)^2 - \frac{e}{2} \sigma B_\varphi + H_f,$$  \hspace{1cm} (2.2)

acting on $\mathbb{C}^2 \otimes \mathcal{F}$. Here $A_\varphi = A_\varphi(0)$ and $B_\varphi = B_\varphi(0)$. The total momentum $p \in \mathbb{R}^3$ is regarded as a parameter. Recently an adiabatic perturbation of the Hamiltonian (2.2) has been studied in [16]. We define

$$H_{p0} = \frac{1}{2} (p - P_f)^2 + H_f$$

and $H_{ip} = H_p - H_{p0}$. We have $\|H_{ip} \psi\| \leq c_*(e) \|(H_{p0} + 1) \psi\|$, where

$$c_*(e) = c_* \left\{ |e| \left\{ \int \left( \frac{1}{\omega(k)^2} + \omega(k) \right) |\hat{\varphi}(k)|^2 dk \right\}^{1/2} + e^2 \int \left( \frac{1}{\omega(k)^2} + 1 \right) |\hat{\varphi}(k)|^2 dk \right\}$$

with some constant $c_*$. Then $|e| < e_*$ with a certain $e_* > 0$ implies $c_*(e) < 1$. In particular $H_p$ is self-adjoint on $D(H_f) \cap D(P_f^2)$ for all $p \in \mathbb{R}^3$ and bounded from below, for $|e| < e_*$. The ground state energy of $H_p$ is

$$E(p) = \inf \sigma(H_p) = \inf_{\psi \in D(H_p), \|\psi\| = 1} (\psi, H_p \psi).$$

If $E(p)$ is an eigenvalue, the corresponding spectral projection is denoted by $P_p$. $\text{Tr}P_p$ is identical with the multiplicity of ground states. The bottom of the continuous spectrum is denoted by $E_c(p)$. Under our assumptions one knows that

$$E_c(p) = \inf_{k \in \mathbb{R}^3} (E(p - k) + \omega(k)).$$

See [4, 5, 17]. Thus it is natural to set

$$\Delta(p) = E_c(p) - E(p) = \inf_{k \in \mathbb{R}^3} (E(p - k) + \omega(k) - E(p)).$$
2.2 Total angular momentum

Let \( \vec{n} \in \mathbb{R}^3 \) be a unit vector. It follows that, for \( \theta \in \mathbb{R} \),

\[
e^{i(\theta/2)\vec{n}\cdot\theta}\sigma_\mu e^{-i(\theta/2)\vec{n}\cdot\theta} = (R\sigma)_\mu, \quad \mu = 1, 2, 3,
\]

where \( R = (R_{\mu\nu})_{1 \leq \mu, \nu \leq 3} = R(\vec{n}, \theta) \in \text{SO}(3) \) presents the rotation around \( \vec{n} \) through an angle \( \theta \), and \( (R\sigma)_\mu = \sum_{\mu=1,2,3} R_{\mu\nu}\sigma_\nu \). We define the field angular momentum relative to the origin by

\[
J_f = \sum_{j=1,2} \int (k \times (-i\nabla_k))a^*(k,j)a(k,j)dk
\]

and the helicity by

\[
S_f = i \int \hat{k} \{ a^*(k, 2)a(k, 1) - a^*(k, 1)a(k, 2) \} \, dk.
\]

Let \( a^t(f, j) = \int a^t(k, j)f(k)dk \). It holds that

\[
[a(f, 1), S_f] = -ia(\hat{k}f, 2), \quad [a(f, 2), S_f] = ia(\hat{k}f, 1),
\]

\[
[a^*(f, 1), S_f] = -ia^*(\hat{k}f, 2), \quad [a^*(f, 2), S_f] = ia^*(\hat{k}f, 1).
\]

One sees that

\[
e^{i\theta\vec{n}\cdot(J_f+S_f)}H_f e^{-i\theta\vec{n}\cdot(J_f+S_f)} = H_f,
\]

\[
e^{i\theta\vec{n}\cdot(J_f+S_f)}P_f e^{-i\theta\vec{n}\cdot(J_f+S_f)} = RP_f,
\]

\[
e^{i\theta\vec{n}\cdot(J_f+S_f)}A_\varphi e^{-i\theta\vec{n}\cdot(J_f+S_f)} = RA_\varphi.
\]

Define the total angular momentum by

\[
J_{\text{total}} = J_f + S_f + \frac{1}{2}\sigma.
\]

It follows that

\[
e^{i\theta\hat{p}\cdot J_{\text{total}}}H_p e^{-i\theta\hat{p}\cdot J_{\text{total}}} = \frac{1}{2} \left( (R\sigma) \cdot (Rp - RP_f - eRA_\varphi) \right)^2 + H_f = H_p.
\]

In particular \( E(p) = E(Rp) \). Moreover taking \( \vec{n} = \hat{p} = p/|p| \) we have

\[
e^{i\theta\hat{p}\cdot J_{\text{total}}}H_p e^{-i\theta\hat{p}\cdot J_{\text{total}}} = H_p.
\]

Formally we may say that \( H_p \) has a "field angular momentum+helicity+SU(2)" symmetry. It is easily seen that \( \sigma(\hat{p} \cdot (J_f + S_f)) = Z \) and \( \sigma(\hat{p} \cdot \sigma) = \{-1, 1\} \). Thus

\[
\sigma(\hat{p} \cdot J_{\text{total}}) = Z + \frac{1}{2},
\]
which is independent of $p$. Thus $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$ and $H_p$ are decomposable as

$$
\mathbb{C}^2 \otimes \mathcal{F} = \bigoplus_{z \in \mathbb{Z} \pm \frac{1}{2}} \mathcal{H}(z),
$$

and

$$
H_p = \bigoplus_{z \in \mathbb{Z} \pm \frac{1}{2}} H_p(z).
$$

As our main result we state

**Theorem 2.1** Suppose $|e| < e_0$ with some constant $e_0$ given in (3.3), and $\Delta(p) > 0$. Then $H_p$ has two orthogonal ground states, $\psi_{\pm}$, with $\psi_{\pm} \in \mathcal{H}(\pm 1/2)$.

We emphasize that all our estimates on the allowed ranges for $p$ and $e$ do not depend on $m_{\text{ph}}$ if we take $\omega(k) = \sqrt{|k|^2 + m_{\text{ph}}^2}$.

### 3 A proof of Theorem 2.1

In what follows $\psi_p = \begin{pmatrix} \psi_{p+} \\ \psi_{p-} \end{pmatrix}$ denotes an arbitrary ground state of $H_p$. The number operator is defined by

$$
N_f = \sum_{j=1,2} \int a^*(k,j)a(k,j)dk.
$$

The following lemma is shown in [15]

**Lemma 3.1** Suppose $\Delta(p) > 0$. Then

$$
(\psi_p, N_f \psi_p) \leq 2e^2 \int \frac{|k|^2/4 + 6E(p)}{(E(p-k) + \omega(k) - E(p))^2} \frac{|\hat{\varphi}(k)|^2}{\omega(k)}dk ||\psi_p||^2.
$$

We set

$$
\theta(p) = 2 \int \frac{|k|^2/4 + 6E(p)}{(E(p-k) + \omega(k) - E(p))^2} \frac{|\hat{\varphi}(k)|^2}{\omega(k)}dk.
$$

Let $P_\Omega$ be the projection onto $\{\mathbb{C}\Omega\}$.

**Lemma 3.2** Suppose that $\Delta(p) > 0$ and $e^2 < 1/\theta(p)$. Then $(\psi_p, P_\Omega \psi_p) > 0$.

**Proof:** Since $P_\Omega + N_f \geq 1$, we have

$$
(\psi_p, P_\Omega \psi_p) \geq ||\psi_p||^2 - ||N_f^{1/2} \psi_p||^2 > (1 - e^2 \theta(p)) ||\psi_p||^2.
$$

Thus the lemma follows. \qed
Let \( \varphi_+ = \begin{pmatrix} \Omega \\ 0 \end{pmatrix} \) and \( \varphi_- = \begin{pmatrix} 0 \\ \Omega \end{pmatrix} \), which are the ground states of \( H_{p_0} \) with \( p = (0, 0, 1) \) and \( \varphi_\pm \in \mathcal{H}(\pm 1/2) \). Let us denote by \( P \) the projection onto \( \{ c_1 \varphi_1 + c_2 \varphi_2, c_1, c_2 \in \mathbb{C} \} \).

Let \( \{ \phi_i \} \) be a base of the space spanned by ground states of \( H_p \) and \( \{ \psi_j \} \) that of the complement.

**Lemma 3.3** Suppose \( e^2 < 1/(3\theta(p)) \). Then \( \text{Tr} P \leq 2 \).

**Proof:** For \( \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \), since \( (\psi, P\psi) = |(\Omega, \psi_+)|^2 + |(\Omega, \psi_-)|^2 = (\psi, (1 \otimes P_\Omega)\psi) \), we have \( (\psi, (P + 1 \otimes N_f)\psi) = (\psi, 1 \otimes (P_\Omega + N_f)\psi) \geq ||\psi||^2 \). Hence \( P + N_f \geq 1 \). Then

\[
\text{Tr}(P_p(1 - P)) = \sum_{\phi \in \{ \phi_i \} \oplus \{ \psi_j \}} (\phi, P_p(1 - P)\phi) = \sum_{\phi \in \{ \phi_i \}} (\phi, (1 - P)\phi) \\
\leq \sum_{\phi \in \{ \phi_i \}} (\phi, N_f\phi) = \sum_{\phi \in \{ \phi_i \}} (\phi, P_pN_f\phi) = \text{Tr}(P_pN_f).
\]

Thus \( \text{Tr}(P_p(1 - P)) \leq \text{Tr}(P_pN_f) \). It follows that

\[
\text{Tr}(P_pP) = \sum_{\phi \in \{ \phi_i \} \oplus \{ \psi_j \}} (\phi, P_p\phi) = \sum_{\phi \in \{ \phi_i \}} (\phi, P_p\phi) \leq 2.
\]

Thus \( \text{Tr}(P_pP) \leq 2 \). Moreover we have \( \text{Tr}(P_pN_f) \leq e^2\theta(p)\text{Tr} P_p \), since

\[
\text{Tr}(P_pN_f) = \sum_{\phi \in \{ \phi_i \} \oplus \{ \psi_j \}} (\phi, P_pN_f\phi) = \sum_{\phi \in \{ \phi_i \}} (\phi, N_f\phi) \\
\leq e^2\theta(p) \sum_{\phi \in \{ \phi_i \}} (\phi, \phi) = e^2\theta(p)\text{Tr} P_p.
\]

Then \( \text{Tr} P_p - \text{Tr}(P_pP) = \text{Tr} P_p(1 - P) \leq \text{Tr}(P_pN_f) \leq e^2\theta(p)\text{Tr} P_p \). Hence it follows that \( (1 - e^2\theta(p))\text{Tr} P_p \leq \text{Tr}(P_pP) \leq 2 \). We have

\[
\text{Tr} P_p \leq \frac{2}{1 - e^2\theta(p)} < 3.
\]

Thus the lemma follows. \( \square \)

We say that \( \psi \in \mathcal{F} \) is real, if \( \psi^{(n)}(k_1, j_1, \cdots, k_n, j_n) \) is a real-valued function on \( L^2(\mathbb{R}^{3n} \times \{1, 2\}^n) \) for all \( n \geq 0 \). The set of real \( \psi \) is denoted by \( \mathcal{F}_{\text{real}} \). We define the set of reality-preserving operators \( \mathcal{O}_{\text{real}}(\mathcal{F}) \) as follows:

\[
\mathcal{O}_{\text{real}}(\mathcal{F}) = \{ A | A : \mathcal{F}_{\text{real}} \cap D(A) \rightarrow \mathcal{F}_{\text{real}} \}.
\]
It is seen that $H_{I}$ and $P_{I}$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Since, for all $k \in \mathbb{R}$ and $z \in \mathbb{R}^{3}$,

$$(H_{p0} + z)^{k} \psi^{(n)}(k_{1}, j_{1}, \cdots, k_{n}, j_{n})$$

$$= \left( \frac{1}{2} \left( p - \sum_{i=1}^{n} k_{i} \right)^{2} + \sum_{i=1}^{n} \omega(k_{i}) + z \right)^{k} \psi^{(n)}(k_{1}, j_{1}, \cdots, k_{n}, j_{n}),$$

$(H_{p0} + z)^{k}$ is also in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Moreover $A_{\varphi}$ and $iB_{\varphi}$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$.

**Lemma 3.4** Suppose $|e| < e_{*}$. Let $x \in \mathbb{C}^{2}$. Then there exists $a(t) \in \mathbb{R}$ independent of $x$ such that for $t \geq 0$

$$(x \otimes \Omega, e^{-t(H_{p} - E(p))}x \otimes \Omega)_{\mathcal{H}} = a(t)(x, x)_{\mathbb{C}^{2}}. \quad (3.1)$$

**Proof:** Note that $\|H_{IP}(1 + H_{p0})^{-1}\| < 1$ for $|e| < e_{*}$. Then, by spectral theory, one has

$$e^{-t(H_{p} - E(p))} = \lim_{n \to \infty} \left( 1 + \frac{t}{n}(H_{p} - E(p)) \right)^{-n}$$

$$= \lim_{n \to \infty} \lim_{k \to \infty} \left\{ (1 + \frac{t}{n}H_{op})^{-1/2} \left( \sum_{k=0}^{m} (-\frac{t}{n}H_{IP})^{k} \right) (1 + \frac{t}{n}H_{op})^{-1/2} \right\}^{n}.$$

Here

$$\overline{H}_{IP} = H_{IIp} + i\sigma \cdot \tilde{B},$$

$$\tilde{B} = (1 + \frac{t}{n}H_{op})^{-1/2} (iB_{\varphi}) (1 + \frac{t}{n}H_{op})^{-1/2},$$

$$H_{IIp} = -e(p - P_{I}) \cdot A_{\varphi} + \frac{e^{2}}{2}A_{\varphi}^{2}.$$
where $a_m$ and $b_m$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Hence there exist $a_{nm} \in \mathcal{O}_{\text{real}}(\mathcal{F})$ and $b_{nm} \in \mathcal{O}_{\text{real}}(\mathcal{F})$ such that
\[
\left\{ \left( 1 + \frac{t}{n} H_{\Phi} \right)^{-1/2} \left( \sum_{k=0}^{m} \left( -\frac{t}{n} \overline{H_{\mathrm{I}p}} \right)^k \right) \left( 1 + \frac{t}{n} H_{\Phi} \right)^{-1/2} \right\}^n = a_{nm} + i \sigma \cdot b_{nm}.
\]
Finally
\[
(x \otimes \Omega, e^{-t(H_{\mathrm{p}} - E(p))} x \otimes \Omega) = \lim_{n \to \infty} \lim_{k \to \infty} (x, x)(\Omega, a_{nm} \Omega) + i \lim_{n \to \infty} \lim_{k \to \infty} (x, \sigma x)(\Omega, b_{nm} \Omega).
\]
Since the left-hand side is real, the second term of the right-hand side vanishes and $a(t) = \lim_{n \to \infty} \lim_{k \to \infty} (\Omega, a_{nm} \Omega)$ exists, which establishes the desired result. \hfill \square

**Lemma 3.5** Suppose $|e| < e_*$ and $|e| < 1/\sqrt{\theta(p)}$. Then there exists $a > 0$ such that
\[
PP_p P = a P.
\]

**Proof:** Note that $P_p = s - \lim_{t \to -\infty} e^{-t(H_{\mathrm{p}} - E(p))}$. Thus by Lemma 3.4,
\[
(x \otimes \Omega, P_p x \otimes \Omega) = \lim_{t \to -\infty} (x \otimes \Omega, e^{-t(H_{\mathrm{p}} - E(p))} x \otimes \Omega) = \lim_{t \to -\infty} a(t)(x, x)
\]
for all $x \in \mathbb{C}^2$. Since by Lemma 3.2, $(x \otimes \Omega, P_p x \otimes \Omega) \neq 0$ for some $x \in \mathbb{C}^2$, $\lim_{t \to -\infty} a(t)$ exists and it does not vanish. For arbitrary $\phi_1, \phi_2 \in \mathcal{H}$, the polarization identity leads to $(\phi_1, PP_p P \phi_2) = a(\phi_1, P \phi_2)$. The lemma follows. \hfill \square

**Lemma 3.6** Suppose $|e| < e_*$ and $|e| < 1/\sqrt{\theta(p)}$. Then $\text{Tr} P_p \geq 2$.

**Proof:** Suppose $\text{Tr} P_p = 1$. Let $P = |\varphi_+\rangle \langle \varphi_+| + |\varphi_-\rangle \langle \varphi_-|$ and $P_p = |\psi_p\rangle \langle \psi_p|$. Lemma 3.5 yields that
\[
PP_p P = (|\varphi_+\rangle \langle \varphi_+| + |\varphi_-\rangle \langle \varphi_-|)|\psi_p\rangle \langle \psi_p|( |\varphi_+\rangle \langle \varphi_+| + |\varphi_-\rangle \langle \varphi_-|)
\]
\[
= |(\varphi_+, \psi_p)|^2 |\varphi_+\rangle \langle \varphi_+| + |(\varphi_-, \psi_p)|^2 |\varphi_-\rangle \langle \varphi_-|
\]
\[
+ (\varphi_+, \psi_p)(\psi_p, \varphi_-)|\varphi_+\rangle \langle \varphi_-| + (\varphi_-, \psi_p)(\psi_p, \varphi_+) |\varphi_-\rangle \langle \varphi_+|
\]
\[
= a(|\varphi_+\rangle \langle \varphi_+| + |\varphi_-\rangle \langle \varphi_-|). \quad (3.2)
\]
It follows that $(\varphi_+, \psi_p)(\psi_p, \varphi_-) = 0$. Let us assume $(\psi_p, \varphi_-) = 0$. It implies in terms of (3.2) that $|(\varphi_+, \psi_p)|^2 |\varphi_+\rangle \langle \varphi_+| = a(|\varphi_+\rangle \langle \varphi_+| + |\varphi_-\rangle \langle \varphi_-|)$. This contradicts $(\varphi_+, \psi_p) \neq 0$ and $a \neq 0$. Thus the lemma follows. \hfill \square
We define
\[ e_0 = \inf \{ |e| \, |e| < 1/\sqrt{3\theta(p)}, |e| < e_* \}. \]  
\hspace{1em} (3.3)

A proof of Theorem 2.1

By Lemma 3.6, TrP_p \geq 2, and by Lemma 3.3, TrP_p \leq 2. Hence TrP_p = 2 follows. Without loss of generalization we may assume that p = (0,0,1). Then \( \varphi_\pm \in \mathcal{H}(\pm 1/2) \).

Let \( \psi_\pm \) be ground states of \( H_p \) such that \( \psi_+ \in \mathcal{H}(z) \) and \( \psi_- \in \mathcal{H}(z') \) with some \( z, z' \in \mathbb{Z} + 1/2 \). Since \( P\varphi P = a\varphi \) we have \( (\varphi_\pm, P\varphi_\pm) = a > 0 \). Let \( Q_\pm \) be the projections to \( \mathcal{H}(\pm 1/2) \). Then \( Q_+ P \varphi_+ \neq 0 \) and \( Q_- P \varphi_- \neq 0 \). The alternative \( Q_+ \psi_+ \neq 0 \) or \( Q_- \psi_- \neq 0 \) holds, or the alternative \( Q_- \psi_+ \neq 0 \) or \( Q_+ \psi_- \neq 0 \) holds. We may set \( Q_+ \psi_+ \neq 0 \). Then \( \psi_+ \in \mathcal{H}(+1/2) \) and \( \psi_- \in \mathcal{H}(-1/2) \). The theorem follows. \( \square \)

4 Confining potentials

In this section we set \( \omega(k) = |k| \) and
\[ H = \frac{1}{2}(-i\nabla - eA_\varphi(x))^2 + H_f \frac{e}{2} \sigma B_\varphi(x) + V. \]

Let \( V \) be relatively bounded with respect to \( -\Delta/2 \) with a relative bound strictly smaller than one. It has been established in [10, 11] that \( H \) is self-adjoint on \( D(-\Delta) \cap D(H_f) \) and bounded from below, for arbitrary \( e \). A confining potential \( V \) breaks the total momentum invariance,
\[ [P_{\text{total}}, H] \neq 0. \]  
\hspace{1em} (4.1)

Existence of ground states of \( H \) is expected by (4.1). Actually by many authors it has been established that \( H \) has ground states, e.g., [1, 6, 7, 8, 14, 13], and in a spinless case, the ground state is unique [9].

Let \( H_0 = H_{el} + H_f \) and \( H_{el} = \frac{1}{2}p^2 + V \). We set \( E = \inf \sigma(H) \), \( E_{el} = \inf \sigma(H_{el}) \) and \( \Sigma_{el} = \inf \sigma_{\text{ess}}(H_{el}) \).

We define a class of external potentials.

**Definition 4.1** (1) We say \( V = Z + W \in V_{\text{exp}} \) if the following (i)-(iv) hold, (i) \( Z \in L_{\text{loc}}^1(\mathbb{R}^3) \), (ii) \( Z > -\infty \), (iii) \( W < 0 \), (iv) \( W \in L^p(\mathbb{R}^3) \) for some \( p > 3/2 \).

(2) We say \( V \in V(m) \), \( m \geq 1 \), if (i) \( V \in V_{\text{exp}} \), (ii) \( Z(x) \geq \gamma |x|^{2m} \), outside a compact set for some positive constant \( \gamma \).

(3) We say \( V \in V(0) \), \( m \geq 1 \), if (i) \( V \in V_{\text{exp}} \), (ii) \( \lim_{|x| \to \infty} Z(x) > \inf \sigma(H) \).
We assume that $V$ satisfies that (1) $\|Vf\| \leq a\|f^2/2\| + b\|f\|$ with some $a < 1$ and some $b \geq 0$, (2) $V \in V(m)$ with some $m \geq 0$, (3) $V(x) = V(-x)$, (4) $\Sigma_{e1} - E_{e1} > 0$ and the ground state $\phi_0$ of $H_{e1}$ is unique and real.

(1) guarantees self-adjointness of $H$, (2) derives a boundedness of $\|x|\psi_0\|$ for ground states $\psi_0$ of $H$, and (3) will be needed to estimate a lower bound of the multiplicity of ground states of $H$. (4) ensures that $H$ has ground states and $H_0$ has twofold ground states. Actually $H_0$ has the two ground states, $\phi_+ = \left(\phi_0 \otimes \Omega, 0\right)$ and $\phi_- = \left(0, \phi_0 \otimes \Omega\right)$.

Let $P_{\phi_0}$ denote the projection onto $\{C\phi_0\}$. Define

$$P = P_{\phi_0} \otimes P_{\Omega}, \quad Q = P_{\phi_0}^\perp \otimes P_{\Omega}.$$ 

Furthermore $P_e$ denotes the projection onto the space spanned by ground states of $H$. Let $\psi$ be arbitrary ground state of $H$. It is proven in [1] that

$$\|N_{1/2}\psi\|^2 \leq \theta_1(e)\|x|\psi\|^2,$$

and in [2, 12] that

$$\|x^k\psi\|^2 \leq \theta_2(e)\|\psi\|^2.$$ 

Then together with (4.2) and (4.3), we have

$$\|N_{1/2}\psi\|^2 \leq \theta_1(e)\theta_2(e)\|\psi\|^2.$$ 

Suppose $\Sigma_{e1} - E > 0$. Then there exists $\theta_3(e)$ such that

$$\|Q\psi\|^2 \leq \theta_3(e)\|\psi\|^2.$$ 

Note that $\lim_{|e| \to 0} \theta_j(e) = 0$.

**Lemma 4.2** Suppose $\theta_1(e)\theta_2(e) + \theta_3(e) < 1$. Then $(\psi_0, P\psi_0) > 0$.

**Proof:** It follows from (4.4), (4.5) and $P \geq 1 - N_t - Q$. \hfill \Box

**Lemma 4.3** Suppose $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then $\text{Tr} P_e \leq 2$.

**Proof:** It can be proven in the similar way as Lemma 3.3. \hfill \Box
Next we estimate $\text{Tr} P_{e}$ from below using the realness argument used in the previous section. Let $F$ denote the Fourier transformation on $L^{2}(\mathbb{R}^{3})$. We define the unitary operator $\mathcal{O}$ on $\mathcal{H}$ by $\mathcal{O} = (F \otimes 1)e^{ix \otimes P_{t}}$. Then $\mathcal{O}$ maps $D(-\Delta) \cap D(H_{t})$ onto $D(|x|^{2}) \cap D(H_{t})$ with

$$\tilde{H} = \mathcal{O}H\mathcal{O}^{-1} = \frac{1}{2}(x - P_{t} - eA(0))^{2} + \tilde{V} + H_{t} - \frac{e}{2}\sigma \cdot B(0).$$

Here $\tilde{V}$ is defined by

$$\tilde{V}f = FVF^{-1}f = \hat{V} * f$$

where $*$ denotes the convolution. By the assumption $V(x) = V(-x)$ we see that $\tilde{V}$ is a reality preserving operator. Let

$$\tilde{H}_{0} = \frac{1}{2}(x - P_{t})^{2} + H_{t} + \tilde{V}.$$

**Lemma 4.4** We have $(\tilde{H}_{0} - z)^{-n} \in \mathcal{O}_{\text{real}}(L^{2}(\mathbb{R}^{3}; \mathcal{F}))$ for all $z \in \mathbb{R}$ with $z \not\in \sigma(\tilde{H}_{0})$ and $n \in \mathbb{R}$.

**Proof:** We have

$$(\tilde{H}_{0} - z)^{-n} = \frac{1}{\Gamma(n)} \int_{0}^{\infty} t^{-1+n} e^{-t\tilde{H}_{0}} e^{tz} dt,$$

where $\Gamma(\cdot)$ denotes the Gamma function. It is enough to prove $e^{-t\tilde{H}_{0}} \in \mathcal{O}_{\text{real}}(L^{2}(\mathbb{R}^{3}; \mathcal{F}))$. Since by the Trotter product formula,

$$e^{-t\tilde{H}_{0}} = \lim_{n \to \infty} \left( e^{-\frac{t}{n}(P_{t} - x)^{2}/2} F^{-1} e^{-\frac{t}{n}V} F \right)^{n},$$

$$F^{-1} e^{-sV} F \in \mathcal{O}_{\text{real}}(L^{2}(\mathbb{R}^{3}; \mathcal{F})),$$

and

$$e^{-s(P_{t} - x)^{2}} \in \mathcal{O}_{\text{real}}(L^{2}(\mathbb{R}^{3}; \mathcal{F})), $$

it follows that $e^{-t\tilde{H}_{0}} \in \mathcal{O}_{\text{real}}(L^{2}(\mathbb{R}^{3}; \mathcal{F}))$. The lemma follows. \qed

From this lemma it follows that $(\tilde{H}_{0} - z)^{-1}, (\tilde{H}_{0} - z)^{-1/2} \in \mathcal{O}_{\text{real}}(L^{2}(\mathbb{R}^{3}; \mathcal{F}))$. We decompose $\tilde{H} = \tilde{H} - E$ as $\tilde{H} = \tilde{H}_{0} + \tilde{H}_{1}$, where

$$\tilde{H}_{1} = -\frac{e}{2}(x - P_{t})A_{\varphi}(0) - \frac{e}{2}A_{\varphi}(0)(x - P_{t}) + \frac{e^{2}}{2}A_{\varphi}^{2}(0) - \frac{e}{2}\sigma B_{\varphi}(0) - E.$$

**Lemma 4.5** There exists $e_{c} > 0$ such that for all $|e| < e_{c}$, $\text{Tr} P_{e} \geq 2$. 


Proof: First we prove $PP_eP = aP$ with some $a > 0$ in the similar way as Lemma 3.4 with $H_p$ and $H_{ip}$ replaced by $\overline{H}$ and $\overline{H}_1$, respectively. Then the lemma follows from the proof of Lemma 3.6.\(\square\)

Theorem 4.6 Suppose $\Sigma_{el} - E > 0$, $|e| < e_c$ and $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then $\text{Tr} P_e = 2$.

Proof: It follows from Lemmas 4.3 and 4.5.\(\square\)

Suppose that $V$ is rotation invariant. Let

$$J_{\text{total}} = x \times (-i\nabla_x) + J_t + S_t + \frac{1}{2}\sigma.$$

Then we have for $\theta \in \mathbb{R}$, $\vec{n} \in \mathbb{R}^3$ with $|\vec{n}| = 1$,

$$e^{i\theta\vec{n} \cdot J_{\text{total}}} H e^{-i\theta\vec{n} \cdot J_{\text{total}}} = H.$$

Since $\sigma(\vec{n} \cdot J_{\text{total}}) = z+1/2$ for each $\vec{n}$, $\mathcal{H}$ and $H$ are decomposable as $\mathcal{H} = \bigoplus_{z \in \mathbb{Z}+1/2} \mathcal{H}(z)$, and $H = \bigoplus_{z \in \mathbb{Z}+1/2} H(z)$. In the same way as the proof of Theorem 2.1 one can prove the following corollary.

Corollary 4.7 Suppose that $V$ is translation invariant, and $\Sigma_{el} - E > 0$, $|e| < e_c$ and $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then $H$ has two orthogonal ground states, $\psi_\pm$, with $\psi_\pm \in \mathcal{H}(\pm 1/2)$.

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