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Two-fold ground states of the Pauli-Fierz Hamiltonian including spin

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Abstract

The Pauli-Fierz Hamiltonian describes an interaction between a low energy electron and photons. Existence of ground states has been established. The purpose of this talk is to show that its ground states is exactly two-fold in a weak coupling region.

1 The Pauli-Fierz Hamiltonian

This is a joint work\(^1\) with Herbert Spohn\(^2\). The Hamiltonian in question is the Pauli-Fierz Hamiltonian in nonrelativistic QED with spin, which will be denoted by \(H\) acting on the Hilbert space

\[
\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}.
\]

Here \(L^2(\mathbb{R}^3; \mathbb{C}^2)\) denotes the Hilbert space for the electron with spin \(\sigma\), where \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\) denotes the Pauli spin 1/2 matrices,

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

\(\mathcal{F}\) is the symmetric Fock space for the photons given by \(\mathcal{F} = \bigoplus_{n=0}^{\infty} (L^2(\mathbb{R}^3 \times \{1, 2\}))_{\text{sym}}^n\).

Here \((\cdots)_{\text{sym}}^n\) denotes the \(n\)-fold symmetric tensor product of \((\cdots)_{\text{sym}}^0 = \mathbb{C}\).

The photons live in \(\mathbb{R}^3\) and have helicity \(\pm 1\). The Fock vacuum is denoted by \(\Omega\). The photon field is represented in \(\mathcal{F}\) by the two-component Bose field \(a(k, j), j = 1, 2\), with commutation relations

\[
[a(k, j), a^*(k', j')] = \delta_{jj'}\delta(k - k'),
\]

\(^1\) [12].

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\[ [a(k,j), a(k',j')] = 0, \quad [a^*(k,j), a^*(k',j')] = 0. \]

The energy of the photons is given by
\[
H_f = \sum_{j=1,2} \int \omega(k) a^*(k,j) a(k,j) dk,
\]
i.e., \( H_f \) restricted to \((L^2(\mathbb{R}^3 \times \{1,2\}))_{\text{symm}}^n\) is the multiplication by \( \sum_{j=1}^n \omega(k_j) \), and the momentum of the photons is
\[
P_f = \sum_{j=1,2} \int k a^*(k,j) a(k,j) dk.
\]
Throughout units are such that \( \hbar = 1, \ c = 1 \). Physically \( \omega(k) = |k| \). The case is somewhat singular and we assume that \( \omega \) is continuous, rotation invariant, and that
(1) \( \inf_{k \in \mathbb{R}^3} \omega(k) \geq \omega_0 > 0 \),
(2) \( \omega(k_1) + \omega(k_2) \geq \omega(k_1 + k_2) \),
(3) \( \lim_{|k| \to \infty} \omega(k) = \infty \).
A typical example is
\[ \omega(k) = \sqrt{|k|^2 + m_{\text{ph}}^2}, \quad m_{\text{ph}} > 0. \]
For a recent result of the massless case see [3]. The quantized transverse vector potential is defined through
\[
A_{\varphi}(x) = \sum_{j=1,2} \int \frac{\hat{\varphi}(k)}{\sqrt{2\omega(k)}} e_j(k) \left( a^*(k,j) e^{-ikx} + a(k,j) e^{ikx} \right) dk.
\]
Here \( e_1 \) and \( e_2 \) are polarization vectors which together with \( \hat{k} = k/|k| \) form a standard basis in \( \mathbb{R}^3 \). \( \varphi : \mathbb{R}^3 \to \mathbb{R} \) is a form factor which ensures an ultraviolet cutoff. It is assumed to be \( \varphi(Rx) = \varphi(x) \) for an arbitrary rotation \( R \), continuous, bounded with some decay at infinity, and normalized as \( \int \varphi(x) dx = 1 \). We will work with the Fourier transform \( \hat{\varphi}(k) = (2\pi)^{-3/2} \int \varphi(x) e^{-ikx} dx \). It satisfies
(1) \( \hat{\varphi}(Rk) = \hat{\varphi}(k) \),
(2) \( \overline{\hat{\varphi}} = \hat{\varphi} \) for notational simplicity,
(3) \( \hat{\varphi}(0) = (2\pi)^{-3/2} \), and
(4) the decay
\[
\int \left( \omega(k)^{-2} + \omega(k)^{-1} + 1 + \omega(k) \right) |\hat{\varphi}(k)|^2 dk < \infty.
\]
The quantized magnetic field is correspondingly
\[
B_{\varphi}(x) = i \sum_{j=1,2} \int \frac{\hat{\varphi}(k)}{\sqrt{2\omega(k)}} (k \times e_j(k)) \left( a^*(k,j) e^{-ikx} - a(k,j) e^{ikx} \right) dk.
\]
With these preparation the Pauli-Fierz Hamiltonian, including spin, is defined by
\[
H = \frac{1}{2} (-i \nabla_x \otimes 1 - e A_{\varphi}(x))^2 + 1 \otimes H_f - \frac{e}{2} \sigma \otimes B_{\varphi}(x).
\]
(1.1)
Since obvious from the context we will drop the tensor notation \( \otimes \).
2 Invariances

2.1 Total momentum

Let us define the total momentum by $P_{\text{total}} = -i \nabla_z + P_t$. We see that

$$[P_{\text{total}}, H] = 0. \quad (2.1)$$

(2.1) immediately implies that $H$ has no ground state. Instead of $H$ we consider the Hamiltonian with a fixed total momentum as follows. By (2.1), we see that (1.1) is decomposable with respect to the spectrum of $P_{\text{total}},$

$$H = \int_{\mathbb{R}^3} H_p dp,$$

where

$$H_p = \frac{1}{2} (p - P_t - e A_\varphi)^2 - \frac{e}{2} \sigma B_\varphi + H_f, \quad (2.2)$$

acting on $\mathbb{C}^2 \otimes \mathcal{F}$. Here $A_\varphi = A_\varphi(0)$ and $B_\varphi = B_\varphi(0)$. The total momentum $p \in \mathbb{R}^3$ is regarded as a parameter. Recently an adiabatic perturbation of the Hamiltonian (2.2) has been studied in [16]. We define

$$H_{p0} = \frac{1}{2} (p - P_t)^2 + H_f,$$

and $H_{tp} = H_p - H_{p0}$. We have $\|H_{tp}\psi\| \leq c_\ast(e)\|(H_{p0} + 1)\psi\|$, where

$$c_\ast(e) = c_\ast \left\{ |e| \left( \left( \frac{1}{\omega(k)^2} + \omega(k) \right)|\hat{\varphi}(k)|^2 dk \right)^{1/2} + e^2 \left( \left( \frac{1}{\omega(k)^2} + 1 \right)|\hat{\varphi}(k)|^2 dk \right) \right\}$$

with some constant $c_\ast$. Then $|e| < e_\ast$ with a certain $e_\ast > 0$ implies $c_\ast(e) < 1$. In particular $H_p$ is self-adjoint on $D(H_f) \cap D(P_f^2)$ for all $p \in \mathbb{R}^3$ and bounded from below, for $|e| < e_\ast$. The ground state energy of $H_p$ is

$$E(p) = \inf \sigma(H_p) = \inf_{\psi \in D(H_p), \|\psi\| = 1} (\psi, H_p \psi).$$

If $E(p)$ is an eigenvalue, the corresponding spectral projection is denoted by $P_p$. $\text{Tr} P_p$ is identical with the multiplicity of ground states. The bottom of the continuous spectrum is denoted by $E_c(p)$. Under our assumptions one knows that

$$E_c(p) = \inf_{k \in \mathbb{R}^3} (E(p - k) + \omega(k)).$$

See [4, 5, 17]. Thus it is natural to set

$$\Delta(p) = E_c(p) - E(p) = \inf_{k \in \mathbb{R}^3} (E(p - k) + \omega(k) - E(p)).$$
2.2 Total angular momentum

Let $\vec{n} \in \mathbb{R}^3$ be a unit vector. It follows that, for $\theta \in \mathbb{R}$,

$$e^{i(\theta/2)\vec{n} \cdot \theta} \sigma_{\mu} e^{-i(\theta/2)\vec{n} \cdot \theta} = (R\sigma)_{\mu}, \quad \mu = 1, 2, 3,$$

where $R = (R_{\mu\nu})_{1\leq\mu,\nu\leq 3} = R(\vec{n}, \theta) \in \text{SO}(3)$ presents the rotation around $\vec{n}$ through an angle $\theta$, and $(R\sigma)_{\mu} = \sum_{\mu=1,2,3} R_{\mu\nu} \sigma_{\nu}$. We define the field angular momentum relative to the origin by

$$J_f = \sum_{j=1,2} \int (k \times (-i \nabla_k)) a^*(k,j)a(k,j)dk$$

and the helicity by

$$S_f = i \int \hat{k} \{a^*(k,2)a(k,1) - a^*(k,1)a(k,2)\} dk.$$  

Let $a^t(f,j) = \int a(k,j)f(k)dk$. It holds that

$$[a(f,1), S_t] = -ia(-k f,2), \quad [a(f,2), S_t] = ia(k f,1),$$

$$[a^*(f,1), S_t] = -ia^*(-k f,2), \quad [a^*(f,2), S_t] = ia^*(k f,1).$$

One sees that

$$e^{i\theta(\hat{p} \cdot (J_t + S_t))} H_f e^{-i\theta(\hat{p} \cdot (J_t + S_t))} = H_f,$$

$$e^{i\theta(\hat{p} \cdot (J_t + S_t))} P_t e^{-i\theta(\hat{p} \cdot (J_t + S_t))} = RP_t,$$

$$e^{i\theta(\hat{p} \cdot (J_t + S_t))} A_\varphi e^{-i\theta(\hat{p} \cdot (J_t + S_t))} = RA_\varphi.$$

Define the total angular momentum by

$$J_{\text{total}} = J_f + S_f + \frac{1}{2} \sigma.$$  

It follows that

$$e^{i\theta(\hat{p} \cdot J_{\text{total}})} H_{p} e^{-i\theta(\hat{p} \cdot J_{\text{total}})} = \frac{1}{2} \{ (R\sigma) \cdot (Rp - RP_t - eRA_\varphi) \}^2 + H_f = H_p.$$  

In particular $E(p) = E(Rp)$. Moreover taking $\vec{n} = \hat{p} = p/|p|$ we have

$$e^{i\theta(\hat{p} \cdot J_{\text{total}})} H_{p} e^{-i\theta(\hat{p} \cdot J_{\text{total}})} = H_p.$$  

Formally we may say that $H_p$ has a "field angular momentum+helicity+SU(2)" symmetry. It is easily seen that $\sigma(\hat{p} \cdot (J_t + S_t)) = \mathbb{Z}$ and $\sigma(\hat{p} \cdot \sigma) = \{-1,1\}$. Thus

$$\sigma(\hat{p} \cdot J_{\text{total}}) = \mathbb{Z} + \frac{1}{2},$$
which is independent of $p$. Thus $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$ and $H_p$ are decomposable as

$$\mathbb{C}^2 \otimes \mathcal{F} = \bigoplus_{z \in \mathbb{Z} + \frac{1}{2}} \mathcal{H}(z),$$

and

$$H_p = \bigoplus_{z \in \mathbb{Z} + \frac{1}{2}} H_p(z).$$

As our main result we state

**Theorem 2.1** Suppose $|e| < e_0$ with some constant $e_0$ given in (3.3), and $\Delta(p) > 0$. Then $H_p$ has two orthogonal ground states, $\psi_\pm$, with $\psi_\pm \in \mathcal{H}(\pm 1/2)$.

We emphasize that all our estimates on the allowed ranges for $p$ and $e$ do not depend on $m_{\text{ph}}$ if we take $\omega(k) = \sqrt{|k|^2 + m_{\text{ph}}^2}$.

### 3 A proof of Theorem 2.1

In what follows $\psi_p = \begin{pmatrix} \psi_{p+} \\ \psi_{p-} \end{pmatrix}$ denotes an arbitrary ground state of $H_p$. The number operator is defined by

$$N_f = \sum_{j=1,2} \int a^*(k,j)a(k,j)dk.$$  

The following lemma is shown in [15]

**Lemma 3.1** Suppose $\Delta(p) > 0$. Then

$$(\psi_p, N_f \psi_p) \leq 2e^2 \int \frac{|k|^2/4 + 6E(p)}{(E(p-k) + \omega(k) - E(p))^2} \frac{|	ilde{\varphi}(k)|^2}{\omega(k)}dk ||\psi_p||^2.$$  

We set

$$\theta(p) = 2 \int \frac{|k|^2/4 + 6E(p)}{(E(p-k) + \omega(k) - E(p))^2} \frac{|	ilde{\varphi}(k)|^2}{\omega(k)}dk.$$  

Let $P_\Omega$ be the projection onto $\{\mathbb{C}\Omega\}$.

**Lemma 3.2** Suppose that $\Delta(p) > 0$ and $e^2 < 1/\theta(p)$. Then $(\psi_p, P_\Omega \psi_p) > 0$.

**Proof:** Since $P_\Omega + N_f \geq 1$, we have

$$(\psi_p, P_\Omega \psi_p) \geq ||\psi_p||^2 - ||N_f^{1/2} \psi_p||^2 > (1 - e^2\theta(p))||\psi_p||^2.$$  

Thus the lemma follows. □
Let $\varphi_{+} = \left( \begin{array}{l} \Omega \\ 0 \end{array} \right)$ and $\varphi_{-} = \left( \begin{array}{l} 0 \\ \Omega \end{array} \right)$, which are the ground states of $H_{p0}$ with $p = (0, 0, 1)$ and $\varphi_{\pm} \in \mathcal{H}(\pm 1/2)$. Let us denote by $P$ the projection onto $\{c_1\varphi_1 + c_2\varphi_2, c_1, c_2 \in \mathbb{C}\}$.

Let $\{\phi_i\}$ be a base of the space spanned by ground states of $H_p$ and $\{\psi_j\}$ that of the complement.

**Lemma 3.3** Suppose $e^2 < 1/(3\theta(p))$. Then $\text{Tr} P_p \leq 2$.

**Proof:** For $\psi = \left( \begin{array}{l} \psi_+ \\ \psi_- \end{array} \right)$, since $(\psi, P\psi) = |(\Omega, \psi_+)|^2 + |(\Omega, \psi_-)|^2 = (\psi, (1 \otimes P_{\Omega})\psi)$, we have $(\psi, (P + 1 \otimes N_f)\psi) = (\psi, 1 \otimes (P_{\Omega} + N_f)\psi) \geq ||\psi||^2$. Hence $P + N_f \geq 1$. Then

$$\text{Tr}(P_p(1 - P)) = \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_p(1 - P)\phi) = \sum_{\phi \in \{\phi_i\}} (\phi, (1 - P)\phi) \leq \sum_{\phi \in \{\phi_i\}} (\phi, N_f\phi) = \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_pN_f\phi) = \text{Tr}(P_pN_f).$$

Thus $\text{Tr}(P_p(1 - P)) \leq \text{Tr}(P_pN_f)$. It follows that

$$\text{Tr}(P_pP) = \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_pP\phi) = \sum_{\phi \in \{\phi_i\}} (\phi, P_p\phi) \leq 2.$$ 

Thus $\text{Tr}(P_pP) \leq 2$. Moreover we have $\text{Tr}(P_pN_f) \leq e^2\theta(p)\text{Tr} P_p$, since

$$\text{Tr}(P_pN_f) = \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_pN_f\phi) = \sum_{\phi \in \{\phi_i\}} (\phi, N_f\phi) \leq e^2\theta(p) \sum_{\phi \in \{\phi_i\}} (\phi, \phi) = e^2\theta(p)\text{Tr} P_p.$$

Then $\text{Tr} P_p - \text{Tr}(P_pP) = \text{Tr}(P_p(1 - P)) \leq \text{Tr}(P_pN_f) \leq e^2\theta(p)\text{Tr} P_p$. Hence it follows that $(1 - e^2\theta(p))\text{Tr} P_p \leq \text{Tr}(P_pP) \leq 2$. We have

$$\text{Tr} P_p \leq \frac{2}{1 - e^2\theta(p)} < 3.$$ 

Thus the lemma follows. \hfill \Box

We say that $\psi \in \mathcal{F}$ is real, if $\psi^{(n)}(k_1, j_1, \cdots, k_n, j_n)$ is a real-valued function on $L^2(\mathbb{R}^{3n} \times \{1, 2\}^n)$ for all $n \geq 0$. The set of real $\psi$ is denoted by $\mathcal{F}_{\text{real}}$. We define the set of reality-preserving operators $\mathcal{O}_{\text{real}}(\mathcal{F})$ as follows:

$$\mathcal{O}_{\text{real}}(\mathcal{F}) = \{A|A : \mathcal{F}_{\text{real}} \cap D(A) \longrightarrow \mathcal{F}_{\text{real}}\}.$$
It is seen that $H_1$ and $P_1$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Since, for all $k \in \mathbb{R}$ and $z \in \mathbb{R}^3$,

$$(H_{p0} + z)^k \psi^{(n)}(k_1, j_1, \cdots, k_n, j_n)$$

$$= \left( \frac{1}{2} \left( p - \sum_{i=1}^{n} k_i \right)^2 + \sum_{i=1}^{n} \omega(k_i) + z \right)^k \psi^{(n)}(k_1, j_1, \cdots, k_n, j_n),$$

$(H_{p0} + z)^k$ is also in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Moreover $A_\varphi$ and $iB_\varphi$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$.

**Lemma 3.4** Suppose $|e| < e_*$. Let $x \in \mathbb{C}^2$. Then there exists $a(t) \in \mathbb{R}$ independent of $x$ such that for $t \geq 0$

$$(x \otimes \Omega, e^{-t(H_p - E(p))} x \otimes \Omega)_{\mathcal{H}} = a(t)(x, x)_{\mathbb{C}^2}. \quad (3.1)$$

**Proof:** Note that $\|H_\varphi(1 + H_{p0})^{-1}\| < 1$ for $|e| < e_*$. Then, by spectral theory, one has

$$e^{-t(H_p - E(p))} = \lim_{n \to \infty} \left( 1 + \frac{t}{n} (H_p - E(p)) \right)^{-n}$$

$$= \lim_{n \to \infty} \lim_{k \to \infty} \left\{ \left( 1 + \frac{t}{n} H_{0p} \right)^{-1/2} \left( \sum_{k=0}^{m} \left( -\frac{t}{n} \overline{H_{\varphi}} \right)^k \right) \left( 1 + \frac{t}{n} H_{0p} \right)^{-1/2} \right\}^n.$$  

Here

$$\overline{H_{\varphi}} = \overline{H_{\varphi}} + i\sigma \cdot \tilde{B},$$

$$\tilde{B} = \left( 1 + \frac{t}{n} H_{0p} \right)^{-1/2} (iB_\varphi) \left( 1 + \frac{t}{n} H_{0p} \right)^{-1/2},$$

$$\overline{H_{\varphi}} = \left( 1 + \frac{t}{n} H_{0p} \right)^{-1/2} (H_{\varphi} - E(p)) \left( 1 + \frac{t}{n} H_{0p} \right)^{-1/2},$$

$$H_{\varphi} = -e(p - P_1) \cdot A_\varphi + \frac{e^2}{2} A_\varphi^2.$$  

It is seen that

$$\overline{H_{\varphi}}^2 = \overline{H_{\varphi}} \overline{H_{\varphi}} - \tilde{B} \cdot \overline{B} + i\sigma \cdot (\overline{H_{\varphi}} \tilde{B} + \tilde{B} \overline{H_{\varphi}} - \tilde{B} \wedge \tilde{B}) = M + i\sigma \cdot L.$$  

Here both of $M = \overline{H_{\varphi}} \overline{H_{\varphi}} - \tilde{B} \cdot \overline{B}$ and $L = \overline{H_{\varphi}} \tilde{B} + \tilde{B} \overline{H_{\varphi}} - \tilde{B} \wedge \tilde{B}$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Moreover

$$\overline{H_{\varphi}}^3 = \overline{H_{\varphi}} M - \tilde{B} L + i\sigma \cdot (\tilde{B} M + \overline{H_{\varphi}} L - \tilde{B} \wedge L),$$

where both of $\overline{H_{\varphi}} M - \tilde{B} L$ and $\tilde{B} M + \overline{H_{\varphi}} L - \tilde{B} \wedge L$ are also in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Thus, repeating above procedure, one obtains

$$\sum_{k=0}^{m} \left( -\frac{t}{n} \overline{H_{\varphi}} \right)^k = a_m + i\sigma \cdot b_m,$$
where \( a_m \) and \( b_m \) are in \( O_{\text{real}}(\mathcal{F}) \). Hence there exist \( a_{nm} \in O_{\text{rea}1}(%0000(\mathcal{F}) \) such that

\[
\left\{ \left( 1 + \frac{t}{n} H_{\Phi} \right)^{-1/2} \left( \sum_{k=0}^{m} \left( \frac{t}{n} H_{\Phi} \right)^{k} \right) \left( 1 + \frac{t}{n} H_{\Phi} \right)^{-1/2} \right\}^{n} = a_{nm} + i \sigma \cdot b_{nm}.
\]

Finally

\[
(x \otimes \Omega, e^{-t(H_{p}-E(p))}x \otimes \Omega) = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} (x \otimes \Omega, a_{nm} \Omega) + i \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} (x \otimes \Omega, b_{nm} \Omega).
\]

Since the left-hand side is real, the second term of the right-hand side vanishes and \( a(t) = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} (\Omega, a_{nm} \Omega) \) exists, which establishes the desired result.

\[\square\]

**Lemma 3.5** Suppose \(|e| < e_* \) and \(|e| < 1/\sqrt{\theta(p)}\). Then there exists \( a > 0 \) such that

\[PP_P P = aP.\]

**Proof:** Note that \( P_p = s - \lim_{t \rightarrow \infty} e^{-t(H_{p}-E(p))} \). Thus by Lemma 3.4,

\[
(x \otimes \Omega, P_p x \otimes \Omega) = \lim_{t \rightarrow \infty} (x \otimes \Omega, e^{-t(H_{p}-E(p))} x \otimes \Omega)
\]

for all \( x \in \mathbb{C}^2 \). Since by Lemma 3.2, \( (x \otimes \Omega, P_p x \otimes \Omega) \neq 0 \) for some \( x \in \mathbb{C}^2 \), \( \lim_{t \rightarrow \infty} a(t) \) exists and it does not vanish. For arbitrary \( \phi_1, \phi_2 \in \mathcal{H} \), the polarization identity leads to \( (\phi_1, PP_P P \phi_2) = a(\phi_1, P \phi_2) \). The lemma follows. \[\square\]

**Lemma 3.6** Suppose \(|e| < e_* \) and \(|e| < 1/\sqrt{\theta(p)}\). Then \( \text{Tr} P_P \geq 2 \).

**Proof:** Suppose \( \text{Tr} P_P = 1 \). Let \( P = |\varphi_+\rangle \langle \varphi_+| + |\varphi_-\rangle \langle \varphi_-| \) and \( P_p = |\psi_p\rangle \langle \psi_p| \). Lemma 3.5 yields that

\[
PP_P P = (|\varphi_+\rangle \langle \varphi_+| + |\varphi_-\rangle \langle \varphi_-|)P_P(|\psi_p\rangle \langle \psi_p|(|\varphi_+\rangle \langle \varphi_+| + |\varphi_-\rangle \langle \varphi_-|)\\
= |(\varphi_+, \psi_p)\rangle \langle \varphi_+| + |(\varphi_-, \psi_p)\rangle \langle \varphi_-|\\
+(\varphi_+, \psi_p)\langle \psi_p, \varphi_-|(|\varphi_+\rangle \langle \varphi_-| + (\varphi_-, \psi_p)\langle \psi_p, \varphi_+| |\varphi_-\rangle \langle \varphi_+|
= a(|\varphi_+\rangle \langle \varphi_+| + |\varphi_-\rangle \langle \varphi_-|).
\]

It follows that \( (\varphi_+, \psi_p)\langle \psi_p, \varphi_-| = 0 \). Let us assume \( (\psi_p, \varphi_-) = 0 \). It implies in terms of (3.2) that \( |(\varphi_+, \psi_p)\rangle \langle \varphi_+| = a(|\varphi_+\rangle \langle \varphi_+| + |\varphi_-\rangle \langle \varphi_-|) \). This contradicts \( (\varphi_+, \psi_p) \neq 0 \) and \( a \neq 0 \). Thus the lemma follows. \[\square\]
We define 
\[ e_0 = \inf \left\{ |e| \left| |e| < 1/\sqrt{3\theta(p)}, |e| < e_* \right. \right\}. \] (3.3)

A proof of Theorem 2.1

By Lemma 3.6, $\text{Tr} P_p \geq 2$, and by Lemma 3.3, $\text{Tr} P_p \leq 2$. Hence $\text{Tr} P_p = 2$ follows. Without loss of generalization we may assume that $p = (0,0,1)$. Then $\varphi_{\pm} \in \mathcal{H}(\pm 1/2)$.

Let $\psi_{\pm}$ be ground states of $H_p$ such that $\psi_+ \in \mathcal{H}(z)$ and $\psi_- \in \mathcal{H}(z')$ with some $z, z' \in \mathbb{Z} + 1/2$. Since $PP_p P = aP$ we have $(\varphi_{\pm}, P_p \varphi_{\pm}) = a > 0$. Let $Q\pm$ be the projections to $H(\pm 1/2)$. Then $Q_+ P_p \varphi_+ \neq 0$ and $Q_- P_p \varphi_- \neq 0$.

The alternative $Q_+ \psi_+ \neq 0$ or $Q_- \psi_- \neq 0$ holds, or the alternative $Q_- \psi_+ \neq 0$ or $Q_- \psi_- \neq 0$ holds. We may set $Q_+ \psi_+ \neq 0$. Then $\psi_+ \in \mathcal{H}(+1/2)$ and $\psi_- \in \mathcal{H}(-1/2)$. The theorem follows. \(\square\)

4 Confining potentials

In this section we set $\omega(k) = |k|$ and

\[ H = \frac{1}{2}(-i\nabla - eA_\varphi(x))^2 + H_f - \frac{e}{2} \sigma B_\varphi(x) + V. \]

Let $V$ be relatively bounded with respect to $-\Delta/2$ with a relative bound strictly smaller than one. It has been established in [10, 11] that $H$ is self-adjoint on $D(-\Delta) \cap D(H_f)$ and bounded from below, for arbitrary $e$. A confining potential $V$ breaks the total momentum invariance,

\[ [P_{\text{total}}, H] \neq 0. \] (4.1)

Existence of ground states of $H$ is expected by (4.1). Actually by many authors it has been established that $H$ has ground states, e.g., [1, 6, 7, 8, 14, 13], and in a spinless case, the ground state is unique [9].

Let $H_0 = H_{\text{el}} + H_f$ and $H_{\text{el}} = \frac{1}{2}p^2 + V$. We set $E = \inf \sigma(H)$, $E_{\text{el}} = \inf \sigma(H_{\text{el}})$ and $\Sigma_{\text{el}} = \inf \sigma_{\text{ess}}(H_{\text{el}})$.

We define a class of external potentials.

**Definition 4.1**

1. We say $V = Z + W \in V_{\text{exp}}$ if the following (i)-(iv) hold, (i) $Z \in L_{\text{loc}}^1(\mathbb{R}^3)$, (ii) $Z > -\infty$, (iii) $W < 0$, (iv) $W \in L^p(\mathbb{R}^3)$ for some $p > 3/2$.

2. We say $V \in V(m)$, $m \geq 1$, if (i) $V \in V_{\text{exp}}$, (ii) $Z(x) \geq \gamma|x|^{2m}$, outside a compact set for some positive constant $\gamma$.

3. We say $V \in V(0)$, $m \geq 1$, if (i) $V \in V_{\text{exp}}$, (ii) $\lim \inf_{|x| \to \infty} Z(x) > \inf \sigma(H)$. 

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We assume that $V$ satisfies that (1) $||Vf|| \leq a(||(p^2/2)f|| + b||f||$ with some $a < 1$ and some $b \geq 0$, (2) $V \in V(m)$ with some $m \geq 0$, (3) $V(x) = V(-x)$, (4) $\Sigma_{\text{el}} - E_{\text{el}} > 0$ and the ground state $\phi_0$ of $H_{\text{el}}$ is unique and real.

(1) guarantees self-adjointness of $H$, (2) derives a boundedness of $||x|\psi_0||$ for ground states $\psi_0$ of $H$, and (3) will be needed to estimate a lower bound of the multiplicity of ground states of $H$. (4) ensures that $H$ has ground states and $H_0$ has twofold ground states. Actually $H_0$ has the two ground states, $\phi_+ = \left( \phi_0 \otimes \Omega, 0 \right)$ and $\phi_- = \left( 0, \phi_0 \otimes \Omega \right)$.

Let $P_{\phi_0}$ denote the projection onto $\{\mathbb{C}\phi_0\}$. Define

$$P = P_{\phi_0} \otimes P_{\Omega}, \quad Q = P_{\phi_0}^{\perp} \otimes P_{\Omega}.$$ 

Furthermore $P_e$ denotes the projection onto the space spanned by ground states of $H$. Let $\psi$ be arbitrary ground state of $H$. It is proven in [1] that

$$||N_{t}^{1/2}\psi||^2 \leq \theta_1(e)||x|\psi||^2, \quad (4.2)$$

and in [2, 12] that

$$||x|^k\psi||^2 \leq \theta_2(e)||\psi||^2. \quad (4.3)$$

Then together with (4.2) and (4.3), we have

$$||N_{t}^{1/2}\psi||^2 \leq \theta_1(e)\theta_2(e)||\psi||^2. \quad (4.4)$$

Suppose $\Sigma_{\text{el}} - E > 0$. Then there exists $\theta_3(e)$ such that

$$||Q\psi||^2 \leq \theta_3(e)||\psi||^2. \quad (4.5)$$

Note that $\lim_{|e|\rightarrow 0} \theta_j(e) = 0$.

**Lemma 4.2** Suppose $\theta_1(e)\theta_2(e) + \theta_3(e) < 1$. Then $(\psi_0, P\psi_0) > 0$.

**Proof:** It follows from (4.4), (4.5) and $P \geq 1 - N_t - Q$. \hfill $\square$

**Lemma 4.3** Suppose $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then $\text{Tr}P_e \leq 2$.

**Proof:** It can be proven in the similar way as Lemma 3.3. \hfill $\square$
Next we estimate $\text{Tr}P_e$ from below using the realness argument used in the previous section. Let $F$ denote the Fourier transformation on $L^2(\mathbb{R}^3)$. We define the unitary operator $O$ on $\mathcal{H}$ by $O = (F \otimes 1)e^{ixP_t}$. Then $O$ maps $D(-\Delta) \cap D(H_t)$ onto $D(|x|^2) \cap D(H_t)$ with

$$\tilde{H} = OHO^{-1} = \frac{1}{2}(x - P_t - eA(0))^2 + \tilde{V} + H_t - \frac{e}{2}A \cdot B(0).$$

Here $\tilde{V}$ is defined by

$$\tilde{V}f = FVF^{-1}f = \hat{V} \ast f$$

where $\ast$ denotes the convolution. By the assumption $V(x) = V(-x)$ we see that $\tilde{V}$ is a reality preserving operator. Let

$$\tilde{H}_0 = \frac{1}{2}(x - P_t)^2 + H_t + \tilde{V}.$$ 

**Lemma 4.4** We have $(\tilde{H}_0 - z)^{-n} \in \mathcal{O}_{\text{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$ for all $z \in \mathbb{R}$ with $z \not\in \sigma(\tilde{H}_0)$ and $n \in \mathbb{R}$.

**Proof:** We have

$$(\tilde{H}_0 - z)^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{-1+n}e^{-t\tilde{H}_0}e^{tz}dt,$$

where $\Gamma(\cdot)$ denotes the Gamma function. It is enough to prove $e^{-t\tilde{H}_0} \in \mathcal{O}_{\text{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$. Since by the Trotter product formula,

$$e^{-t\tilde{H}_0} = \lim_{n \to \infty} \left( e^{-(t/n)(P_t-x)^2/2}F^{-1} e^{-(t/n)V}F \right)^n,$$

and

$$e^{-s(P_t-x)^2} \in \mathcal{O}_{\text{real}}(L^2(\mathbb{R}^3; \mathcal{F})),$$

it follows that $e^{-t\tilde{H}_0} \in \mathcal{O}_{\text{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$. The lemma follows. $\square$

From this lemma it follows that $(\tilde{H}_0 - z)^{-1}, (\tilde{H}_0 - z)^{-1/2} \in \mathcal{O}_{\text{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$. We decompose $\overline{H} = \tilde{H} - E$ as $\overline{H} = \tilde{H}_0 + \tilde{H}_I$, where

$$\tilde{H}_I = -\frac{e}{2}(x - P_t)A_{\varphi}(0) - \frac{e}{2}A_{\varphi}(0)(x - P_t) + \frac{e^2}{2}A_{\varphi}^2(0) - \frac{e}{2}A \cdot B_{\varphi}(0) - E.$$ 

**Lemma 4.5** There exists $e_c > 0$ such that for all $|e| < e_c$, $\text{Tr}P_e \geq 2$. 
**Proof:** First we prove $PP_e P = aP$ with some $a > 0$ in the similar way as Lemma 3.4 with $H_p$ and $H_{Ip}$ replaced by $H$ and $H_1$, respectively. Then the lemma follows from the proof of Lemma 3.6. □

**Theorem 4.6** Suppose $\Sigma_{el} - E > 0$, $|e| < e_c$ and $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then $\text{Tr} P_e = 2$.

**Proof:** It follows from Lemmas 4.3 and 4.5. □

Suppose that $V$ is rotation invariant. Let

$$J_{\text{total}} = x \times (-i\nabla_x) + J_t + S_t + \frac{1}{2} \sigma.$$  

Then we have for $\theta \in \mathbb{R}, \vec{n} \in \mathbb{R}^3$ with $|\vec{n}| = 1$,

$$e^{i\theta \vec{n} \cdot J_{\text{total}}} H e^{-i\theta \vec{n} \cdot J_{\text{total}}} = H.$$  

Since $\sigma(\vec{n} \cdot J_{\text{total}}) = z + 1/2$ for each $\vec{n}$, $\mathcal{H}$ and $H$ are decomposable as $\mathcal{H} = \bigoplus_{z \in Z + \frac{1}{2}} \mathcal{H}(z)$, and $H = \bigoplus_{z \in Z + \frac{1}{2}} H(z)$. In the same way as the proof of Theorem 2.1 one can prove the following corollary.

**Corollary 4.7** Suppose that $V$ is translation invariant, and $\Sigma_{el} - E > 0$, $|e| < e_c$ and $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then $H$ has two orthogonal ground states, $\psi_{\pm}$, with $\psi_{\pm} \in \mathcal{H}(\pm 1/2)$.

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**References**


