Two-fold ground states of the Pauli-Fierz Hamiltonian including spin

Fumio Hiroshima

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Abstract

The Pauli-Fierz Hamiltonian describes an interaction between a low energy electron and photons. Existence of ground states has been established. The purpose of this talk is to show that its ground states is exactly two-fold in a weak coupling region.

1 The Pauli-Fierz Hamiltonian

This is a joint work\textsuperscript{1} with Herbert Spohn\textsuperscript{2}. The Hamiltonian in question is the Pauli-Fierz Hamiltonian in nonrelativisitc QED with spin, which will be denoted by $H$ acting on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}.$$ 

Here $L^2(\mathbb{R}^3; \mathbb{C}^2)$ denotes the Hilbert space for the electron with spin $\sigma$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli spin $1/2$ matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

$\mathcal{F}$ is the symmetric Fock space for the photons given by $\mathcal{F} = \bigoplus_{n=0}^{\infty} (L^2(\mathbb{R}^3 \times \{1, 2\}))^n_{\text{sym}}$. Here $(\cdots)^n_{\text{sym}}$ denotes the $n$-fold symmetric tensor product of $(\cdots)$ with $(\cdots)^0_{\text{sym}} = \mathbb{C}$.

The photons live in $\mathbb{R}^3$ and have helicity $\pm 1$. The Fock vacuum is denoted by $\Omega$. The photon field is represented in $\mathcal{F}$ by the two-component Bose field $a(k, j), j = 1, 2$, with commutation relations

$$[a(k, j), a^*(k', j')] = \delta_{jj'}\delta(k - k'),$$

\textsuperscript{1} [12].

\textsuperscript{2} Zentrum Mathematik, Technische Universität München, D80290, München, Germany.
\[ [a(k, j), a(k', j')] = 0, \quad [a^*(k, j), a^*(k', j')] = 0. \]

The energy of the photons is given by

\[ H_f = \sum_{j=1}^{2} \int \omega(k) a^*(k, j) a(k, j) \, dk, \]

i.e., \( H_f \) restricted to \((L^2(\mathbb{R}^3 \times \{1, 2\}))_{\text{symm}}^n\) is the multiplication by \( \sum_{j=1}^{n} \omega(k_j) \), and the momentum of the photons is

\[ P_f = \sum_{j=1}^{2} \int k a^*(k, j) a(k, j) \, dk. \]

Throughout units are such that \( \hbar = 1 \), \( c = 1 \). Physically \( \omega(k) = |k| \). The case is somewhat singular and we assume that \( \omega \) is continuous, rotation invariant, and that

1. \( \inf_{k \in \mathbb{R}^3} \omega(k) \geq \omega_0 > 0 \),
2. \( \omega(k_1) + \omega(k_2) \geq \omega(k_1 + k_2) \),
3. \( \lim_{|k| \to \infty} \omega(k) = \infty \).

A typical example is

\[ \omega(k) = \sqrt{|k|^2 + m_{\text{ph}}^2}, \quad m_{\text{ph}} > 0. \]

For a recent result of the massless case see [3]. The quantized transverse vector potential is defined through

\[ A_{\varphi}(x) = \sum_{j=1}^{2} \int \frac{\varphi(k)}{\sqrt{2 \omega(k)}} e_j(k) \left( a^*(k, j) e^{-ikx} + a(k, j) e^{ikx} \right) \, dk. \]

Here \( e_1 \) and \( e_2 \) are polarization vectors which together with \( \hat{k} = k/|k| \) form a standard basis in \( \mathbb{R}^3 \). \( \varphi : \mathbb{R}^3 \to \mathbb{R} \) is a form factor which ensures an ultraviolet cutoff. It is assumed to be \( \varphi(Rx) = \varphi(x) \) for an arbitrary rotation \( R \), continuous, bounded with some decay at infinity, and normalized as \( \int \varphi(x) \, dx = 1 \). We will work with the Fourier transform \( \hat{\varphi}(k) = (2\pi)^{-3/2} \int \varphi(x) e^{-ikx} \, dx \). It satisfies

1. \( \hat{\varphi}(Rk) = \hat{\varphi}(k) \),
2. \( \overline{\hat{\varphi}} = \hat{\varphi} \) for notational simplicity,
3. \( \hat{\varphi}(0) = (2\pi)^{-3/2} \), and
4. the decay

\[ \int \left( \omega(k)^{-2} + \omega(k)^{-1} + 1 + \omega(k) \right) |\hat{\varphi}(k)|^2 \, dk < \infty. \]

The quantized magnetic field is correspondingly

\[ B_{\varphi}(x) = i \sum_{j=1}^{2} \int \frac{\varphi(k)}{\sqrt{2 \omega(k)}} (k \times e_j(k)) \left( a^*(k, j) e^{-ikx} - a(k, j) e^{ikx} \right) \, dk. \]

With these preparation the Pauli-Fierz Hamiltonian, including spin, is defined by

\[ H = \frac{1}{2} (-i \nabla_x \otimes 1 - e A_{\varphi}(x))^2 + 1 \otimes H_f - \frac{e}{2} \sigma \otimes B_{\varphi}(x). \quad (1.1) \]

Since obvious from the context we will drop the tensor notation \( \otimes \).
2 Invariances

2.1 Total momentum

Let us define the total momentum by $P_{\text{total}} = -i \nabla_x + P_f$. We see that

$$[P_{\text{total}}, H] = 0. \quad (2.1)$$

(2.1) immediately implies that $H$ has no ground state. Instead of $H$ we consider the Hamiltonian with a fixed total momentum as follows. By (2.1), we see that (1.1) is decomposable with respect to the spectrum of $P_{\text{total}},$

$$H = \int_{\mathbb{R}^3}^\oplus H_p dp,$$

where

$$H_p = \frac{1}{2}(p - P_f - eA_\varphi)^2 - \frac{e}{2}\sigma B_\varphi + H_f, \quad (2.2)$$

acting on $c^2 \otimes \mathcal{F}$. Here $A_\varphi = A_\varphi(0)$ and $B_\varphi = B_\varphi(0)$. The total momentum $p \in \mathbb{R}^3$ is regarded as a parameter. Recently an adiabatic perturbation of the Hamiltonian (2.2) has been studied in [16]. We define

$$H_{p0} = \frac{1}{2}(p - P_f)^2 + H_f,$$

and $H_{ip} = H_p - H_{p0}$. We have $\|H_{ip}\psi\| \leq c_*(e)\|(H_{p0} + 1)\psi\|$, where

$$c_*(e) = c_* \left\{ |e| \left\{ \int \left( \frac{1}{\omega(k)^2} + \omega(k) \right) |\hat{\varphi}(k)|^2 dk \right\}^{1/2} + e^2 \int \left( \frac{1}{\omega(k)^2} + 1 \right) |\hat{\varphi}(k)|^2 dk \right\}$$

with some constant $c_*$. Then $|e| < e_*$ with a certain $e_* > 0$ implies $c_*(e) < 1$. In particular $H_p$ is self-adjoint on $D(H_f) \cap D(P_f^2)$ for all $p \in \mathbb{R}^3$ and bounded from below, for $|e| < e_*$. The ground state energy of $H_p$ is

$$E(p) = \inf \sigma(H_p) = \inf_{\psi \in D(H_p), \|\psi\|=1} (\psi, H_p\psi).$$

If $E(p)$ is an eigenvalue, the corresponding spectral projection is denoted by $P_p$. $\text{Tr} P_p$ is identical with the multiplicity of ground states. The bottom of the continuous spectrum is denoted by $E_c(p)$. Under our assumptions one knows that

$$E_c(p) = \inf_{k \in \mathbb{R}^3} (E(p - k) + \omega(k)).$$

See [4, 5, 17]. Thus it is natural to set

$$\Delta(p) = E_c(p) - E(p) = \inf_{k \in \mathbb{R}^3} (E(p - k) + \omega(k) - E(p)).$$
2.2 Total angular momentum

Let $\vec{n} \in \mathbb{R}^3$ be a unit vector. It follows that, for $\theta \in \mathbb{R},$

$$e^{i(\theta/2)\vec{n} \cdot \theta} \sigma_{\mu} e^{-i(\theta/2)\vec{n} \cdot \theta} = (R \sigma)_{\mu}, \quad \mu = 1, 2, 3,$$

where $R = (R_{\mu \nu})_{1 \leq \mu, \nu \leq 3} = R(\vec{n}, \theta) \in \text{SO}(3)$ presents the rotation around $\vec{n}$ through an angle $\theta,$ and $(R \sigma)_{\mu} = \sum_{\mu=1,2,3} R_{\mu \nu} \sigma_{\nu}.$ We define the field angular momentum relative to the origin by

$$J_f = \sum_{j=1,2} \int (k \times (-i \nabla_k)) a^*(k, j) a(k, j) dk$$

and the helicity by

$$S_f = i \int \hat{k} \{a^*(k, 2)a(k, 1) - a^*(k, 1)a(k, 2)\} dk.$$

Let $a^t(j, f) = \int a^t(k, j) f(k) dk.$ It holds that

$$[a(f, 1), S_f] = -ia(\hat{k} f, 2), \quad [a(f, 2), S_f] = ia(\hat{k} f, 1),$$

$$[a^*(f, 1), S_f] = -ia^*(\hat{k} f, 2), \quad [a^*(f, 2), S_f] = ia^*(\hat{k} f, 1).$$

One sees that

$$e^{i\theta \hat{\sigma} \cdot (J_f + S_f)} H_f e^{-i\theta \hat{\sigma} \cdot (J_f + S_f)} = H_f,$$

$$e^{i\theta \hat{\sigma} \cdot (J_f + S_f)} P_f e^{-i\theta \hat{\sigma} \cdot (J_f + S_f)} = RP_f,$$

$$e^{i\theta \hat{\sigma} \cdot (J_f + S_f)} A_\varphi e^{-i\theta \hat{\sigma} \cdot (J_f + S_f)} = RA_\varphi.$$

Define the total angular momentum by

$$J_{\text{total}} = J_f + S_f + \frac{1}{2} \sigma.$$ 

It follows that

$$e^{i\theta \hat{p} \cdot J_{\text{total}}} H_p e^{-i\theta \hat{p} \cdot J_{\text{total}}} = \frac{1}{2} ((R \sigma) \cdot (R p - R P_f - e R A_\varphi))^2 + H_f = H_p.$$ 

In particular $E(p) = E(R p).$ Moreover taking $\vec{n} = \hat{p} = p/|p|$ we have

$$e^{i\theta \hat{p} \cdot J_{\text{total}}} H_p e^{-i\theta \hat{p} \cdot J_{\text{total}}} = H_p.$$ 

Formally we may say that $H_p$ has a "field angular momentum+ helicity+SU(2)" symmetry. It is easily seen that $\sigma(\hat{p} \cdot (J_f + S_f)) = \mathbb{Z}$ and $\sigma(\hat{p} \cdot \sigma) = \{-1, 1\}.$ Thus

$$\sigma(\hat{p} \cdot J_{\text{total}}) = \mathbb{Z} + \frac{1}{2}.$$
which is independent of $p$. Thus $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$ and $H_p$ are decomposable as

$$\mathbb{C}^2 \otimes \mathcal{F} = \bigoplus_{z \in \mathbb{Z} + \frac{1}{2}} \mathcal{H}(z),$$

and

$$H_p = \bigoplus_{z \in \mathbb{Z} + \frac{1}{2}} H_p(z).$$

As our main result we state

**Theorem 2.1** Suppose $|e| < e_0$ with some constant $e_0$ given in (3.3), and $\Delta(p) > 0$. Then $H_p$ has two orthogonal ground states, $\psi_\pm$, with $\psi_\pm \in \mathcal{H}(\pm 1/2)$.

We emphasize that all our estimates on the allowed ranges for $p$ and $e$ do not depend on $m_{\text{ph}}$ if we take $\omega(k) = \sqrt{|k|^2 + m_{\text{ph}}^2}$.

### 3 A proof of Theorem 2.1

In what follows $\psi_p = \begin{pmatrix} \psi_{p+} \\ \psi_{p-} \end{pmatrix}$ denotes an arbitrary ground state of $H_p$. The number operator is defined by

$$N_f = \sum_{j=1,2} \int a^*(k,j)a(k,j)dk.$$

The following lemma is shown in [15]

**Lemma 3.1** Suppose $\Delta(p) > 0$. Then

$$(\psi_p, N_f \psi_p) \leq 2e^2 \int \frac{|k|^2/4 + 6E(p)}{(E(p-k) + \omega(k) - E(p))^2} \frac{|\hat{\varphi}(k)|^2}{\omega(k)}dk ||\psi_p||^2.$$

We set

$$\theta(p) = 2 \int \frac{|k|^2/4 + 6E(p)}{(E(p-k) + \omega(k) - E(p))^2} \frac{|\hat{\varphi}(k)|^2}{\omega(k)}dk.$$

Let $P_\Omega$ be the projection onto $\{\mathbb{C}\Omega\}$.

**Lemma 3.2** Suppose that $\Delta(p) > 0$ and $e^2 < 1/\theta(p)$. Then $(\psi_p, P_\Omega \psi_p) > 0$.

**Proof:** Since $P_\Omega + N_f \geq 1$, we have

$$(\psi_p, P_\Omega \psi_p) \geq ||\psi_p||^2 - ||N_f^{1/2} \psi_p||^2 > (1 - e^2 \theta(p)) ||\psi_p||^2.$$

Thus the lemma follows.
Let $\varphi_{+} = \begin{pmatrix} \Omega \\ 0 \end{pmatrix}$ and $\varphi_{-} = \begin{pmatrix} 0 \\ \Omega \end{pmatrix}$, which are the ground states of $H_{p_0}$ with $p = (0, 0, 1)$ and $\varphi_{\pm} \in \mathcal{H}(\pm 1/2)$. Let us denote by $P$ the projection onto $\{c_{1}\varphi_{1} + c_{2}\varphi_{2}, c_{1}, c_{2} \in \mathbb{C}\}$.

Let $\{\phi_{i}\}$ be a base of the space spanned by ground states of $H_{p}$ and $\{\psi_{j}\}$ that of the complement.

**Lemma 3.3** Suppose $e^2 < 1/(3\theta(p))$. Then $\text{Tr}P_{p} \leq 2$.

**Proof:** For $\psi = \begin{pmatrix} \psi_{+} \\ \psi_{-} \end{pmatrix}$, since $(\psi, P\psi) = |(\Omega, \psi_{+})|^2 + |(\Omega, \psi_{-})|^2 = (\psi, (1 \otimes P_{1})\psi)$, we have $(\psi, (P + 1 \otimes N_{1})\psi) = (\psi, 1 \otimes (P_{1} + N_{1})\psi) \geq ||\psi||^2$. Hence $P + N_{1} \geq 1$. Then

$$\text{Tr}(P_{p}(1 - P)) = \sum_{\phi \in \{\phi_{i}\} \oplus \{\psi_{j}\}} (\phi, P_{p}(1 - P)\phi) = \sum_{\phi \in \{\phi_{i}\}} (\phi_{i}(1 - P)\phi) \leq \sum_{\phi \in \{\phi_{i}\}} (\phi_{i}, N_{1}\phi) = \sum_{\phi \in \{\phi_{i}\}} (\phi_{i}, P_{p}N_{1}\phi) = \text{Tr}(P_{p}N_{1}).$$

Thus $\text{Tr}(P_{p}(1 - P)) \leq \text{Tr}(P_{p}N_{1})$. It follows that

$$\text{Tr}(P_{p}P) = \sum_{\phi \in \{\phi_{i}\} \oplus \{\psi_{j}\}} (\phi, P_{p}P\phi) = \sum_{\phi \in \{\phi_{i}\}} (\phi, P_{p}\phi) \leq 2.$$ 

Thus $\text{Tr}(P_{p}P) \leq 2$. Moreover we have $\text{Tr}(P_{p}N_{1}) \leq e^2\theta(p)\text{Tr}P_{p}$, since

$$\text{Tr}(P_{p}N_{1}) = \sum_{\phi \in \{\phi_{i}\} \oplus \{\psi_{j}\}} (\phi, P_{p}N_{1}\phi) = \sum_{\phi \in \{\phi_{i}\}} (\phi, N_{1}\phi) \leq e^2\theta(p)\sum_{\phi \in \{\phi_{i}\}} (\phi, \phi) = e^2\theta(p)\text{Tr}P_{p}. $$

Then $\text{Tr}P_{p} - \text{Tr}(P_{p}P) = \text{Tr}P_{p}(1 - P) \leq \text{Tr}(P_{p}N_{1}) \leq e^2\theta(p)\text{Tr}P_{p}$. Hence it follows that $(1 - e^2\theta(p))\text{Tr}P_{p} \leq \text{Tr}(P_{p}P) \leq 2$. We have

$$\text{Tr}P_{p} \leq \frac{2}{1 - e^2\theta(p)} < 3.$$ 

Thus the lemma follows. \(\square\)

We say that $\psi \in \mathcal{F}$ is real, if $\psi^{(n)}(k_{1}, j_{1}, \cdots, k_{n}, j_{n})$ is a real-valued function on $L^2(\mathbb{R}^{3n} \times \{1, 2\}^n)$ for all $n \geq 0$. The set of real $\psi$ is denoted by $\mathcal{F}_{\text{real}}$. We define the set of reality-preserving operators $\mathcal{O}_{\text{real}}(\mathcal{F})$ as follows:

$$\mathcal{O}_{\text{real}}(\mathcal{F}) = \{A|A : \mathcal{F}_{\text{real}} \cap D(A) \longrightarrow \mathcal{F}_{\text{real}}\}.$$
It is seen that $H_I$ and $P_I$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Since, for all $k \in \mathbb{R}$ and $z \in \mathbb{R}^3$,
\[
((H_{p0} + z)^{k}\psi)^{(n)}(k_1, j_1, \cdots, k_n, j_n) = \left(\frac{1}{2} \left( p - \sum_{i=1}^{n} k_i \right)^2 + \sum_{i=1}^{n} \omega(k_i) + z \right)^{k} \psi^{(n)}(k_1, j_1, \cdots, k_n, j_n),
\]
$(H_{p0} + z)^{k}$ is also in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Moreover $A_\varphi$ and $iB_\varphi$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$.

**Lemma 3.4** Suppose $|e| < e_*$. Let $x \in \mathbb{C}^2$. Then there exists $a(t) \in \mathbb{R}$ independent of $x$ such that for $t \geq 0$
\[
(x \otimes \Omega, e^{-t(H_{\rho} - E(p))}x \otimes \Omega)_\mathcal{H} = a(t)(x, x)_{\mathbb{C}^2}.
\]

**Proof:** Note that $\| H_{\rho}(1 + H_{p0})^{-1} \| < 1$ for $|e| < e_*$. Then, by spectral theory, one has
\[
e^{-t(H_{\rho} - E(p))} = \lim_{n \to \infty} \left( 1 + \frac{t}{n}(H_{\rho} - E(p)) \right)^{-n}
\]
= \lim_{n \to \infty} \lim_{k \to \infty} \left\{ \left( 1 + \frac{t}{n} H_{\rho} \right)^{-1/2} \left( \sum_{k=0}^{m} \left( -\frac{t}{n} \bar{H}_{\rho} \right)^{k} \right) \left( 1 + \frac{t}{n} H_{\rho} \right)^{-1/2} \right\}^{n}.
\]
Here
\[
\bar{H}_{\rho} = \bar{H}_{\rho} I + i \sigma \cdot \bar{B},
\]
\[
\bar{B} = \left( 1 + \frac{t}{n} H_{\rho} \right)^{-1/2} (iB_\varphi) \left( 1 + \frac{t}{n} H_{\rho} \right)^{-1/2},
\]
\[
\bar{H}_{\rho} I = \left( 1 + \frac{t}{n} H_{\rho} \right)^{-1/2} (H_{\rho} - E(p)) \left( 1 + \frac{t}{n} H_{\rho} \right)^{-1/2},
\]
\[
H_{\rho} = -e(p - P_I) \cdot A_\varphi + \frac{e^2}{2} A_\varphi^2.
\]
It is seen that
\[
\bar{H}_{\rho}^2 = \bar{H}_{\rho} I + \bar{B} \cdot \bar{B} + i \sigma \cdot (\bar{H}_{\rho} I \bar{B} + \bar{B} \bar{H}_{\rho} I - \bar{B} \wedge \bar{B}) = M + i \sigma \cdot L.
\]

Here both of $M = \bar{H}_{\rho} I \bar{B} + \bar{B} \bar{H}_{\rho} I - \bar{B} \wedge \bar{B}$ and $L = \bar{H}_{\rho} I \bar{B} + \bar{B} \bar{H}_{\rho} I - \bar{B} \wedge \bar{B}$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Moreover
\[
\bar{H}_{\rho}^3 = \bar{H}_{\rho} I M - \bar{B} L + i \sigma \cdot (\bar{B} M + \bar{H}_{\rho} I L - \bar{B} \wedge L),
\]
where both of $\bar{H}_{\rho} I M - \bar{B} L$ and $\bar{B} M + \bar{H}_{\rho} I L - \bar{B} \wedge L$ are also in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Thus, repeating above procedure, one obtains
\[
\sum_{k=0}^{m} \left( -\frac{t}{n} \bar{H}_{\rho} \right)^{k} = a_m + i \sigma \cdot b_m,
\]
where $a_m$ and $b_m$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Hence there exist $a_{nm} \in \mathcal{O}_{\text{real}}(\mathcal{F})$ and $b_{nm} \in \mathcal{O}_{\text{real}}(\mathcal{F})$ such that

$$\left\{ \left(1 + \frac{t}{n} H_{\Phi}\right)^{-1/2} \left( \sum_{k=0}^{m} \left( -\frac{t}{n} \overline{H_{\Phi}} \right)^{k} \right) \left(1 + \frac{t}{n} H_{\Phi}\right)^{-1/2} \right\}^{n} = a_{nm} + i \sigma \cdot b_{nm}.$$  

Finally

$$(x \otimes \Omega, e^{-t(H_{\Phi}-E(p))}x \otimes \Omega) = \lim_{n \to \infty} \lim_{k \to \infty} (x, x)(\Omega, a_{nm}\Omega) + i \lim_{n \to \infty} \lim_{k \to \infty} (x, \sigma x)(\Omega, b_{nm}\Omega).$$

Since the left-hand side is real, the second term of the right-hand side vanishes and $a(t) = \lim_{n \to \infty} \lim_{k \to \infty} (\Omega, a_{nm}\Omega)$ exists, which establishes the desired result.  

**Lemma 3.5** Suppose $|e| < e_*$ and $|e| < 1/\sqrt{\theta(p)}$. Then there exists $a > 0$ such that

$$PP_{p}P = aP.$$  

**Proof:** Note that $P_{p} = s - \lim_{t \to \infty} e^{-t(H_{\Phi}-E(p))}$. Thus by Lemma 3.4,

$$(x \otimes \Omega, P_{p}x \otimes \Omega) = \lim_{t \to \infty} (x \otimes \Omega, e^{-t(H_{\Phi}-E(p))}x \otimes \Omega) = \lim_{t \to \infty} a(t)(x, x)$$

for all $x \in \mathbb{C}^2$. Since by Lemma 3.2, $(x \otimes \Omega, P_{p}x \otimes \Omega) \neq 0$ for some $x \in \mathbb{C}^2$, $\lim_{t \to \infty} a(t)$ exists and it does not vanish. For arbitrary $\phi_1, \phi_2 \in \mathcal{H}$, the polarization identity leads to $(\phi_1, PP_{p}P\phi_2) = a(\phi_1, P\phi_2)$. The lemma follows.  

**Lemma 3.6** Suppose $|e| < e_*$ and $|e| < 1/\sqrt{\theta(p)}$. Then $\text{Tr} P_{p} \geq 2$.

**Proof:** Suppose $\text{Tr} P_{p} = 1$. Let $P = |\varphi_{+}\rangle\langle\varphi_{+}| + |\varphi_{-}\rangle\langle\varphi_{-}|$ and $P_{p} = |\psi_{p}\rangle\langle\psi_{p}|$. Lemma 3.5 yields that

$$PP_{p}P = (|\varphi_{+}\rangle\langle\varphi_{+}| + |\varphi_{-}\rangle\langle\varphi_{-}|)\psi_{p}\langle\psi_{p}|(|\varphi_{+}\rangle\langle\varphi_{+}| + |\varphi_{-}\rangle\langle\varphi_{-}|)$$

$$= |(\varphi_{+}, \psi_{p})|^2|\varphi_{+}\rangle\langle\varphi_{+}| + |(\varphi_{-}, \psi_{p})|^2|\varphi_{-}\rangle\langle\varphi_{-}|$$

$$+ (\varphi_{+}, \psi_{p})(\psi_{p}, \varphi_{-})|\varphi_{-}\rangle\langle\varphi_{+}| + (\varphi_{-}, \psi_{p})(\psi_{p}, \varphi_{+})|\varphi_{-}\rangle\langle\varphi_{-}|$$

$$= a(|\varphi_{+}\rangle\langle\varphi_{+}| + |\varphi_{-}\rangle\langle\varphi_{-}|).$$

It follows that $(\varphi_{+}, \psi_{p})(\psi_{p}, \varphi_{-}) = 0$. Let us assume $(\psi_{p}, \varphi_{-}) = 0$. It implies in terms of (3.2) that $|(\varphi_{+}, \psi_{p})|^2|\varphi_{+}\rangle\langle\varphi_{+}| = a(|\varphi_{+}\rangle\langle\varphi_{+}| + |\varphi_{-}\rangle\langle\varphi_{-}|)$. This contradicts $(\varphi_{+}, \psi_{p}) \neq 0$ and $a \neq 0$. Thus the lemma follows.  

$\square$
We define
\[ e_0 = \inf \left\{ |e| \left| |e| < 1 / \sqrt{3 \theta(p)}, |e| < e_* \right\} \]. \hspace{1cm} (3.3)

A proof of Theorem 2.1

By Lemma 3.6, Tr\(P_p \geq 2\), and by Lemma 3.3, Tr\(P_p \leq 2\). Hence Tr\(P_p = 2\) follows. Without loss of generalization we may assume that \(p = (0, 0, 1)\). Then \(\phi_{\pm} \in \mathcal{H}(\pm 1/2)\).

Let \(\psi_{\pm}\) be ground states of \(H_p\) such that \(\psi_+ \in \mathcal{H}(z)\) and \(\psi_- \in \mathcal{H}(z')\) with some \(z, z' \in \mathbb{Z} + 1/2\). Since \(PP_p P = a P\) we have \((\phi_{\pm}, P_p \phi_{\pm}) = a > 0\). Let \(Q_{\pm}\) be the projections to \(\mathcal{H}(\pm 1/2)\). Then \(Q_+ P_p \phi_+ \neq 0\) and \(Q_- P_p \phi_- \neq 0\). The alternative \(Q_+ \psi_+ \neq 0\) or \(Q_- \psi_- \neq 0\) holds, or the alternative \(Q_- \psi_+ \neq 0\) or \(Q_+ \psi_- \neq 0\) holds. We may set \(Q_+ \psi_+ \neq 0\). Then \(\psi_+ \in \mathcal{H}(+1/2)\) and \(\psi_- \in \mathcal{H}(-1/2)\). The theorem follows. \(\square\)

4 Confining potentials

In this section we set \(\omega(k) = |k|\) and
\[ H = \frac{1}{2}(-i\nabla_x - eA(x))^2 + H_f - \frac{e}{2} \sigma B(x) + V. \]

Let \(V\) be relatively bounded with respect to \(-\Delta/2\) with a relative bound strictly smaller than one. It has been established in [10, 11] that \(H\) is self-adjoint on \(D(-\Delta) \cap D(H_f)\) and bounded from below, for arbitrary \(e\). A confining potential \(V\) breaks the total momentum invariance,
\[ [P_{\text{total}}, H] \neq 0. \hspace{1cm} (4.1) \]

Existence of ground states of \(H\) is expected by (4.1). Actually by many authors it has been established that \(H\) has ground states, e.g., [1, 6, 7, 8, 14, 13], and in a spinless case, the ground state is unique [9].

Let \(H_0 = H_{el} + H_f\) and \(H_{el} = \frac{1}{2}p^2 + V\). We set \(E = \inf \sigma(H), E_{el} = \inf \sigma(H_{el})\) and \(\Sigma_{el} = \inf \sigma_{\text{ess}}(H_{el})\).

We define a class of external potentials.

**Definition 4.1** (1) We say \(V = Z + W \in V_{\exp}\) if the following (i)-(iv) hold, (i) \(Z \in L_{1\text{oc}}^{1}(\mathbb{R}^3)\), (ii) \(Z > -\infty\), (iii) \(W < 0\), (iv) \(W \in L^p(\mathbb{R}^3)\) for some \(p > 3/2\).

(2) We say \(V \in V(m), m \geq 1\), if (i) \(V \in V_{\exp}\), (ii) \(Z(x) \geq \gamma |x|^{2m}\), outside a compact set for some positive constant \(\gamma\).

(3) We say \(V \in V(0), m \geq 1\), if (i) \(V \in V_{\exp}\), (ii) \(\lim \inf_{|x| \to \infty} Z(x) > \inf \sigma(H)\).
We assume that $V$ satisfies that (1) $\|Vf\| \leq a\|f^2/2\| + b\|f\|$ with some $a < 1$ and some $b \geq 0$, (2) $V \in V(m)$ with some $m \geq 0$, (3) $V(x) = V(-x)$, (4) $\Sigma_{e1} - E_{e1} > 0$ and the ground state $\phi_0$ of $H_{e1}$ is unique and real.

(1) guarantees self-adjointness of $H$, (2) derives a boundedness of $\|x|\psi_0\|$ for ground states $\psi_0$ of $H$, and (3) will be needed to estimate a lower bound of the multiplicity of ground states of $H$. (4) ensures that $H$ has ground states and $H_0$ has twofold ground states. Actually $H_0$ has the two ground states, $\phi_+ = \left( \phi_0 \otimes \Omega, 0 \right)$ and $\phi_- = \left( 0, \phi_0 \otimes \Omega \right)$.

Let $P_{\phi_0}$ denote the projection onto $\{C\phi_0\}$. Define

$$P = P_{\phi_0} \otimes P_{\Omega}, \quad Q = P_{\phi_0}^\perp \otimes P_{\Omega}.$$ 

Furthermore $P_e$ denotes the projection onto the space spanned by ground states of $H$. Let $\psi$ be arbitrary ground state of $H$. It is proven in [1] that

$$\|N_{it}^{1/2}\psi\|^2 \leq \theta_1(e)\|x|\psi\|^2,$$

and in [2, 12] that

$$\|x|^k\psi\|^2 \leq \theta_2(e)\|\psi\|^2.$$ 

Then together with (4.2) and (4.3), we have

$$\|N_{it}^{1/2}\psi\|^2 \leq \theta_1(e)\theta_2(e)\|\psi\|^2.$$ 

Suppose $\Sigma_{e1} - E > 0$. Then there exists $\theta_3(e)$ such that

$$\|Q\psi\|^2 \leq \theta_3(e)\|\psi\|^2.$$ 

Note that $\lim_{|e| \to 0} \theta_j(e) = 0$.

**Lemma 4.2** Suppose $\theta_1(e)\theta_2(e) + \theta_3(e) < 1$. Then $(\psi_0, P\psi_0) > 0$.

**Proof:** It follows from (4.4), (4.5) and $P \geq 1 - N_t - Q$. \(\square\)

**Lemma 4.3** Suppose $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then $Tr P_e \leq 2$.

**Proof:** It can be proven in the similar way as Lemma 3.3. \(\square\)
Next we estimate Tr$P_e$ from below using the realness argument used in the previous section. Let $F$ denote the Fourier transformation on $L^2(\mathbb{R}^3)$. We define the unitary operator $O$ on $\mathcal{H}$ by $O = (F \otimes 1)e^{ix \otimes P_f}$. Then $O$ maps $D(-\Delta) \cap D(H_f)$ onto $D(|x|^2) \cap D(H_f)$ with

$$H = OHO^{-1} = \frac{1}{2}(x - P_f - eA(0))^2 + \tilde{V} + H_t - \frac{e}{2}\sigma \cdot B(0).$$

Here $\tilde{V}$ is defined by

$$\tilde{V} f = FVF^{-1}f = \hat{V} \star f$$

where $\star$ denotes the convolution. By the assumption $V(x) = V(-x)$ we see that $\tilde{V}$ is a reality preserving operator. Let

$$\tilde{H}_0 = \frac{1}{2}(x - P_f)^2 + H_t + \tilde{V}.$$

**Lemma 4.4** We have $(\tilde{H}_0 - z)^{-n} \in \mathcal{O}_{\text{rea}}(L^2(\mathbb{R}^3; \mathcal{F}))$ for all $z \in \mathbb{R}$ with $z \not\in \sigma(\tilde{H}_0)$ and $n \in \mathbb{R}$.

**Proof:** We have

$$(\tilde{H}_0 - z)^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{-n} e^{-t\tilde{H}_0} e^{tz} dt,$$

where $\Gamma(\cdot)$ denotes the Gamma function. It is enough to prove $e^{-t\tilde{H}_0} \in \mathcal{O}_{\text{rea}}(L^2(\mathbb{R}^3; \mathcal{F}))$. Since by the Trotter product formula,

$$e^{-t\tilde{H}_0} = s \lim_{n \to \infty} (e^{-t/n}F^{-1}e^{-(t/n)V}F)^n,$$

and

$$e^{-s(P_f - z)^2} \in \mathcal{O}_{\text{rea}}(L^2(\mathbb{R}^3; \mathcal{F})),$$

it follows that $e^{-t\tilde{H}_0} \in \mathcal{O}_{\text{rea}}(L^2(\mathbb{R}^3; \mathcal{F}))$. The lemma follows. \hfill \Box

From this lemma it follows that $(\tilde{H}_0 - z)^{-1}, (\tilde{H}_0 - z)^{-1/2} \in \mathcal{O}_{\text{rea}}(L^2(\mathbb{R}^3; \mathcal{F}))$. We decompose $\tilde{H} = \tilde{H} - E$ as $\tilde{H} = \tilde{H}_0 + \tilde{H}_1$, where

$$\tilde{H}_1 = -\frac{e}{2}(x - P_f)A_\varphi(0) - \frac{e}{2}A_\varphi(0)(x - P_f) + \frac{e^2}{2}A_\varphi^2(0) - \frac{e}{2}\sigma B_\varphi(0) - E.$$

**Lemma 4.5** There exists $e_c > 0$ such that for all $|e| < e_c$, Tr$P_e \geq 2$. 

Proof: First we prove $PP e P = aP$ with some $a > 0$ in the similar way as Lemma 3.4 with $H_p$ and $H_{lp}$ replaced by $\tilde{H}$ and $\tilde{H}_1$, respectively. Then the lemma follows from the proof of Lemma 3.6.

**Theorem 4.6** Suppose $\Sigma_{\text{el}} - E > 0$, $|e| < e_c$ and $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then $\text{Tr} P_e = 2$.

**Proof:** It follows from Lemmas 4.3 and 4.5.

Suppose that $V$ is rotation invariant. Let

$$J_{\text{total}} = x \times (-i\nabla_x) + J_I + S_I + \frac{1}{2} \sigma.$$ 

Then we have for $\theta \in \mathbb{R}$, $\vec{n} \in \mathbb{R}^3$ with $|\vec{n}| = 1$,

$$e^{i\theta \vec{n} \cdot J_{\text{total}}} H e^{-i\theta \vec{n} \cdot J_{\text{total}}} = H.$$ 

Since $\sigma(\vec{n} \cdot J_{\text{total}}) = Z + 1/2$ for each $\vec{n}$, $\mathcal{H}$ and $H$ are decomposable as $\mathcal{H} = \bigoplus_{z \in \mathbb{Z} + \frac{1}{2}} \mathcal{H}(z)$, and $H = \bigoplus_{z \in \mathbb{Z} + \frac{1}{2}} H(z)$. In the same way as the proof of Theorem 2.1 one can prove the following corollary.

**Corollary 4.7** Suppose that $V$ is translation invariant, and $\Sigma_{\text{el}} - E > 0$, $|e| < e_c$ and $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. Then $H$ has two orthogonal ground states, $\psi_\pm$, with $\psi_\pm \in \mathcal{H}(\pm 1/2)$.

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**References**


