数理解析研究所講究録 1275 巻 2002 年 155-170

Holographic Renormalization Group*

MASAFUMI FUKUMA†, SO MATSUURA‡ AND TADAKATSU SAKAI§¶

Yukawa Institute for Theoretical Physics,
Kyoto University, Kyoto 606-8502, Japan

ABSTRACT

The holographic renormalization group (RG) is a manifestation of the idea that the radial direction of a $(d+1)$-dimensional space M_{d+1} with asymptotic anti-de Sitter (AdS) geometry should behave as a scaling parameter of a d-dimensional field theory whose conformal fixed point exists at the boundary of M_{d+1}. We give a review of recent developments in this field, and show that the Hamilton-Jacobi equation for such gravity system describes RG flows of the field theory in a simple and correct manner. We further investigate the situation where stringy corrections are taken into account, which turn Einstein gravity into higher-derivative gravity. We clarify the meaning of these corrections in terms of the holographic renormalization group, and derive a Hamilton-Jacobi-like equation that determines the generating functional of the boundary field theory. Using the expected duality between a higher-derivative gravity system and $\mathcal{N}=2$ superconformal field theory in four dimensions, we demonstrate that the resulting Weyl anomaly is consistent with the field theoretic result.

---

*Talk given by M. Fukuma at the RIMS Symposium "Applications of RG Methods in Mathematical Sciences," July 25–27, 2001, RIMS, Kyoto Univ.

†E-mail: fukuma@yukawa.kyoto-u.ac.jp

‡E-mail: matsu@yukawa.kyoto-u.ac.jp

§E-mail: tsakai@post.tau.ac.il

¶Address after Sep 1: School of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel

1 Introduction

The AdS/CFT correspondence [1] states that a gravitational theory on the $(d+1)$-dimensional anti-de Sitter space (AdS_{d+1}) has a dual description in terms of a conformal field theory (CFT) on the d-dimensional boundary. One of the most significant aspects of the AdS/CFT correspondence is that it further gives us a framework to study the renormalization group (RG) structure of the boundary field theories. In this scheme of the "holographic RG," the extra radial coordinate in the bulk is regarded as parametrizing the RG flow of the dual boundary field theory, and the evolution of bulk fields along the radial direction is considered as describing the RG flow of the coupling constants in the boundary field theory.

On the other hand, there have been several attempts to confirm the validity of the duality beyond the classical Einstein gravity approximation. The AdS/CFT correspondence is believed to be a duality between string theories and a certain class of quantum field theories. In this sense, the AdS/CFT correspondence, and so the structure of the holographic RG, must exist even when a gravity theory is subject to stringy corrections, which turn the theory into higher-derivative gravity. We discuss that such corrections correspond to the introduction of coupling constants which are coupled to highly irrelevant operators, and show that one can explicitly calculate the fixed-point action in the presence of these irrelevant operators.

The organization of this proceeding is as follows. In §2, we give a review of the flow equation that is obtained from the Hamilton-Jacobi equation [2]. In §3, we describe a prescription for solving the flow equation and make some sample calculations to confirm the RG interpretation of the flow equation. In §4, we review the general theory for a higher-derivative system, and apply it to higher-derivative gravity. We derive a Hamilton-Jacobi-like equation which is interpreted as a flow equation. §5 is devoted to a conclusion.

2 Hamilton-Jacobi equation and the flow equation

In this section, we briefly review the formulation of the holographic RG based on the Hamilton-Jacobi equation [2].

We consider Einstein gravity with bulk scalars $\phi^i(x,r)$ on a $(d+1)$-dimensional manifold M_{d+1} with boundary $\Sigma_d = \partial M_{d+1}$. The action is given by

$$\begin{aligned}
&\boldsymbol{S}_{d+1}[\boldsymbol{G}_{MN}(x,r),\phi^i(x,r)] \\
&= \int_{M_{d+1}} d^{d+1}X\sqrt{\boldsymbol{G}}\left(V(\phi) - \boldsymbol{R} + \frac{1}{2}\,L_{ij}(\phi)\,\boldsymbol{G}^{MN}\,\partial_M\phi^i\,\partial_N\phi^j\right) - 2\int_{\Sigma_d} d^dx\,\sqrt{G}\,K\,.
\end{aligned} \tag{2.1}$$

Here $X^M = (x^\mu, r)$ $(\mu,\nu = 1,2,\cdots,d;\ r_0 \le r < \infty)$ are local coordinates on M_{d+1}, and we assume that M_{d+1} has only one boundary Σ_d at $r = r_0$. To develop a Hamiltonian formalism for this system, it is convenient to introduce the ADM parametrization of the metric:

$$\begin{aligned}
ds^2 &= \boldsymbol{G}_{MN}\,dX^M dX^N \\
&= N(x,r)^2dr^2 + G_{\mu\nu}(x,r)\Big(dx^\mu + \lambda^\mu(x,r)dr\Big)\Big(dx^\nu + \lambda^\nu(x,r)dr\Big),
\end{aligned} \tag{2.2}$$

where N and λ^μ are the lapse and the shift function, respectively. The action is then expressed as

$$\begin{aligned}
&\boldsymbol{S}_{d+1}[G_{\mu\nu}(x,r),\phi^i(x,r),N(x,r),\lambda^\mu(x,r)] \\
&= \int_{r_0}^\infty dr\int d^dx\,\sqrt{G}\,\Big[N\left(V(\phi) - R + K_{\mu\nu}K^{\mu\nu} - K^2\right) \\
&\quad + \frac{1}{2N}\,L_{ij}(\phi)\left(\left(\dot\phi^i - \lambda^\mu\partial_\mu\phi^i\right)\left(\dot\phi^j - \lambda^\mu\partial_\mu\phi^j\right) + N^2G^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\right)\Big] \\
&\equiv \int_{r_0}^\infty dr\int d^dx\,\sqrt{G}\,\mathcal{L}_{d+1}[G,\phi,N,\lambda],
\end{aligned} \tag{2.3}$$

where $\cdot\ = \partial/\partial r$. Here R and ∇_μ are the scalar curvature and the covariant derivative with respect to $G_{\mu\nu}$, respectively, and $K_{\mu\nu}$ is the extrinsic curvature defined by

$$K_{\mu\nu} = \frac{1}{2N}\left(\dot G_{\mu\nu} - \nabla_\mu\lambda_\nu - \nabla_\nu\lambda_\mu\right), \qquad K = G^{\mu\nu}\,K_{\mu\nu}. \tag{2.4}$$

Since the conjugate momenta are given by

$$\Pi^{\mu\nu} = K^{\mu\nu} - G^{\mu\nu}K, \qquad \Pi_i = \frac{1}{N}L_{ij}(\phi)\left(\dot\phi^j - \lambda^\mu\,\partial_\mu\phi^j\right), \tag{2.5}$$

the action (2.3) can be rewritten into the first-order form by the Legendre transformation,

$$\boldsymbol{S}_{d+1}[G_{\mu\nu},\phi^i,\Pi^{\mu\nu},\Pi_i,N,\lambda^\mu] \equiv \int_{r_0}^\infty dr\int d^dx\,\sqrt{G}\left[\Pi^{\mu\nu}\dot G_{\mu\nu} + \Pi_i\dot\phi^i - N\mathcal{H} - \lambda_\mu\mathcal{P}^\mu\right],$$

$$
\begin{aligned}
\mathcal{H} &\equiv \Pi_{\mu\nu}^2 - \frac{1}{d-1}\left(\Pi_\mu^\mu\right)^2 + \frac{1}{2}L^{ij}(\phi)\,\Pi_i\,\Pi_j - V(\phi) + R - \frac{1}{2}L_{ij}(\phi)\,G^{\mu\nu}\,\partial_\mu\phi^i\,\partial_\nu\phi^j, \\
\mathcal{P}^\mu &\equiv -2\,\nabla_\nu\Pi^{\mu\nu} + \Pi_i\,\nabla^\mu\phi^i. \qquad (2.7)
\end{aligned}
$$

Here N and λ^μ simply behave as Lagrange multipliers, giving the Hamiltonian and momentum constraints:

$$
\frac{1}{\sqrt{G}}\frac{\delta \mathbf{S}_{d+1}}{\delta N} = \mathcal{H} = 0, \qquad (2.8)
$$

$$
\frac{1}{\sqrt{G}}\frac{\delta \mathbf{S}_{d+1}}{\delta \lambda_\mu} = \mathcal{P}^\mu = 0. \qquad (2.9)
$$

Let $\overline{G}_{\mu\nu}(x,r)$ and $\bar{\phi}^i(x,r)$ be the classical solutions of the bulk action with the boundary conditions,[1]

$$
\overline{G}_{\mu\nu}(x,r\!=\!r_0) = G_{\mu\nu}(x), \qquad \bar{\phi}^i(x,r\!=\!r_0) = \phi^i(x). \qquad (2.10)
$$

We also define $\overline{\Pi}^{\mu\nu}(x,r)$ and $\overline{\Pi}_i(x,r)$ to be the classical solutions of $\Pi^{\mu\nu}(x,r)$ and $\Pi_i(x,r)$, respectively. Then, substituting these classical solutions into the bulk action, we obtain the *classical action* which is a functional of the boundary values, $G_{\mu\nu}(x)$ and $\phi^i(x)$:

$$
S[G_{\mu\nu}(x),\phi(x)] \equiv \mathbf{S}_{d+1}\left[\overline{G}_{\mu\nu}(x,r),\, \bar{\phi}^i(x,r),\, \overline{\Pi}^{\mu\nu}(x,r),\, \overline{\Pi}_i(x,r),\, N(x,r),\, \lambda^\mu(x,r)\right]. \qquad (2.11)
$$

The Hamilton-Jacobi equation shows that the classical conjugate momenta evaluated at $r=r_0$ are given by

$$
\Pi^{\mu\nu}(x) \equiv \overline{\Pi}^{\mu\nu}(x,r_0) = \frac{-1}{\sqrt{G}}\frac{\delta S}{\delta G_{\mu\nu}(x)}, \qquad \Pi_i(x) \equiv \overline{\Pi}_i(x,r_0) = \frac{-1}{\sqrt{G}}\frac{\delta S}{\delta \phi^i(x)}. \qquad (2.12)
$$

Substituting (2.12) into the Hamiltonian constraint (2.8), we thus obtain the *flow equation* [2]:[2]

$$
\{S,S\}(x) = \mathcal{L}_d(x), \qquad (2.13)
$$

[1]One generally needs two boundary conditions for each field, since the equation of motion is a second-order differential equation in r. Here, one of the two is assumed to be already fixed by demanding the regular behavior of the classical solutions inside M_{d+1} ($r \to +\infty$) [1].

[2]The classical action does not depend on the coordinate r_0 explicitly. This can be proved also by the Hamilton-Jacobi equation, since the Hamiltonian is a linear combination of constraints and thus vanishes for the classical solutions. This reflects the invariance of the gravity system under diffeomorphisms in the r direction. The momentum constraint (2.9) ensures the invariance of S under a d-dimensional diffeomorphism along the fixed time slice $r=r_0$.

$$\{S,S\}(x) \equiv \left(\frac{1}{\sqrt{G}}\right)^2 \left[-\frac{1}{d-1}\left(G_{\mu\nu}\frac{\delta S}{\delta G_{\mu\nu}}\right)^2 + \left(\frac{\delta S}{\delta G_{\mu\nu}}\right)^2 + \frac{1}{2}L^{ij}(\phi)\frac{\delta S}{\delta \phi^i}\frac{\delta S}{\delta \phi^j}\right], \tag{2.14}$$

$$\mathcal{L}_d(x) \equiv V(\phi) - R + \frac{1}{2}L_{ij}(\phi)\, G^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j. \tag{2.15}$$

3 Solution to the flow equation and its RG interpretation

In this section we give a prescription for solving the flow equation [2][3], and reveal the RG structure in the flow equation.

3.1 Solution to the flow equation

In a most naive form of the AdS/CFT correspondence, we take $r_0 = -\infty$ and assume that the classical metric $G_{MN}(x,r)$ is AdS: $ds^2 = G_{MN}\, dX^M\, dX^N = dr^2 + \exp(-2r/l)\,(dx^\mu)^2$ (l is called the "radius" of the AdS although the AdS space is noncompact). Then the scalar fields $\phi^i(x)$ are interpreted as the sources coupled to scaling operators $\mathcal{O}_i(x)$ of the boundary CFT, and the classical action $S[G_{\mu\nu}(x) = \exp(-2r_0/l)\delta_{\mu\nu}, \phi^i(x)]$ is regarded as the generating functional of the CFT: $S = \left\langle \int d^dx\, \phi^i(x)\mathcal{O}_i(x)\right\rangle_{\rm CFT}$. However, since there appears divergence in the integration around $r \sim -\infty$, we need to set r_0 to be finite, which turns out to be introducing a UV cutoff into the boundary field theory. Furthermore, if we take into account back-reactions from the scalar fields to the metric, we still should leave arbitrariness to the boundary values of the metric, $G_{\mu\nu}(x)$.

We thus are led to decompose the classical action into two parts:[3]

$$\frac{1}{2\kappa_{d+1}^2}S[G(x),\phi(x)] = \frac{1}{2\kappa_{d+1}^2}S_{\rm loc}[G(x),\phi(x)] + \Gamma[G(x),\phi(x)]. \tag{3.1}$$

Now $\Gamma[G,\phi]$ is the non-local part of $S[G,\phi]$, which is interpreted as the generating functional of the d-dimensional field theory in the presence of the background metric $G_{\mu\nu}(x)$,

[3]We have recovered the $(d+1)$-dimensional Newton constant $2\kappa_{d+1}^2$.

while $S_{\rm loc}[G,\phi]$ is the local counter term, which can be expressed as an integral of differential polynomials of $G_{\mu\nu}(x)$ and $\phi^i(x)$:

$$\begin{aligned} S_{\rm loc}[G(x),\phi(x)] &= \int d^dx\,\sqrt{G(x)}\,\mathcal{L}_{\rm loc}(x) \\ &= \int d^dx\,\sqrt{G(x)}\sum_{w=0,2,4,\cdots}\left[\mathcal{L}_{\rm loc}(x)\right]_w. \end{aligned} \tag{3.2}$$

Here we have arranged the sum over local terms according to the weight w that is defined additively from the following rule [3]:

	weight
$G_{\mu\nu}(x)$, $\phi^i(x)$, $\Gamma[G,\phi]$	0
∂_μ	1
R, $R_{\mu\nu}$, $\partial_\mu\phi^i\partial_\nu\phi^j$, $\cdots$	2
$\delta\Gamma/\delta G_{\mu\nu}(x)$, $\delta\Gamma/\delta\phi^i(x)$	d

The last line is a natural consequence of the relation $w(\Gamma[G,\phi])=0$, since $\delta\Gamma=\int d^dx$ $(\delta\phi(x)\times\delta\Gamma/\delta\phi(x)+\cdots)$. Then, substituting the above equation into the flow equation (2.13) and comparing terms of the same weight, we obtain a sequence of equations that relate the off-shell bulk action (2.6) to the classical action (3.1). They take the following form:

$$\sqrt{G}\,\mathcal{L}_d = \Big[\{S_{\rm loc},S_{\rm loc}\}\Big]_0+\Big[\{S_{\rm loc},S_{\rm loc}\}\Big]_2, \tag{3.3}$$

$$0 = \Big[\{S_{\rm loc},S_{\rm loc}\}\Big]_w \quad (w=4,6,\cdots,d-2), \tag{3.4}$$

$$0 = 2\Big[\{S_{\rm loc},\Gamma\}\Big]_d+\Big[\{S_{\rm loc},S_{\rm loc}\}\Big]_d, \tag{3.5}$$

$$\vdots$$

Eqs. (3.3) and (3.4) determine $[\mathcal{L}_{\rm loc}]_w$ $(w=0,2,\cdots,d-2)$, which together with eq. (3.5) in turn determine the non-local functional Γ.

By parametrizing $[\mathcal{L}_{\rm loc}]_0$ and $[\mathcal{L}_{\rm loc}]_2$ as

$$[\mathcal{L}_{\rm loc}]_0 = W(\phi), \tag{3.6}$$

$$[\mathcal{L}_{\rm loc}]_2 = -\Phi(\phi)\,R+\frac{1}{2}\,M_{ij}(\phi)\,G^{\mu\nu}\,\partial_\mu\phi^i\,\partial_\nu\phi^j, \tag{3.7}$$

one can easily solve (3.3) to obtain[4]

$$\begin{aligned}
V(\phi) &= -\frac{d}{4(d-1)}W(\phi)^2 + \frac{1}{2}L^{ij}(\phi)\,\partial_i W(\phi)\,\partial_j W(\phi)\,, &(3.8)\\
-1 &= \frac{d-2}{2(d-1)}W(\phi)\,\Phi(\phi) - L^{ij}(\phi)\,\partial_i W(\phi)\,\partial_j \Phi(\phi)\,, &(3.9)\\
\frac{1}{2}L_{ij}(\phi) &= -\frac{d-2}{4(d-1)}W(\phi)\,M_{ij}(\phi) - L^{kl}(\phi)\,\partial_k W(\phi)\,\Gamma^{(M)}_{l;ij}(\phi)\,, &(3.10)\\
0 &= W(\phi)\,\nabla^2\,\Phi(\phi) + L^{ij}(\phi)\,\partial_i W(\phi)\,M_{jk}(\phi)\,\nabla^2\phi^k\,. &(3.11)
\end{aligned}$$

Here $\partial_i = \partial/\partial\phi^i$, and $\Gamma^{(M)k}_{ij}(\phi) \equiv M^{kl}(\phi)\,\Gamma^{(M)}_{l;ij}(\phi)$ is the Christoffel symbol constructed from $M_{ij}(\phi)$.

The equation (3.5) becomes

$$\frac{1}{\sqrt{G}}\left[-2G_{\mu\nu}\frac{\delta\Gamma}{\delta G_{\mu\nu}} + \beta^i(\phi)\frac{\delta\Gamma}{\delta\phi^i}\right] = \frac{-2(d-1)}{2\kappa_{d+1}^2 W(\phi)}\Big[\{S_{\rm loc},\, S_{\rm loc}\}\Big]_d, \tag{3.12}$$

where

$$\beta^i(\phi) = \frac{2(d-1)}{W(\phi)}L^{ij}(\phi)\frac{\partial W(\phi)}{\partial\phi^j}. \tag{3.13}$$

In the following subsections, eq. (3.12) will be shown to describe the RG flow of the generating functional of the boundary field theory.

We conclude this subsection with a comment on the term $[\mathcal{L}_{\rm loc}]_d$ in the expansion (2.1). From the equation (3.5), this term would add some local terms to the right hand side of (3.12). However, the contribution from $[\mathcal{L}_{\rm loc}]_d$ always takes a form of a total derivative. This can be understood by observing that possible contributions from $[\mathcal{L}_{\rm loc}]_d$ vanish for constant dilatations [4]. We have neglected such total derivative in the expression (3.12).

3.2 RG flow and classical trajectory

We consider the classical solution

$$\overline{G}_{\mu\nu}(r,x) = \frac{1}{a(r)^2}\delta_{\mu\nu}, \qquad \overline{\phi}^i(r,x) = \phi^i(a(r)), \tag{3.14}$$

with the boundary condition

$$\overline{G}_{\mu\nu}(r=r_0,x) = \frac{1}{a^2}\,\delta_{\mu\nu}, \qquad \overline{\phi}^i(r=r_0,x) = \phi^i\ (\text{const.}). \tag{3.15}$$

[4]The expression for $d=4$ can be found in Ref. [2].

From (2.12), the boundary values of the conjugate momenta are evaluated as

$$\Pi_{\mu\nu}(x) = -\frac{1}{2}a^{-2}W(\phi)\delta_{\mu\nu}, \qquad \Pi_i(x) = -\frac{\partial W(\phi)}{\partial \phi^i}. \tag{3.16}$$

On the other hand, from (2.5), $\Pi_{\mu\nu}$ and Π_i are expressed as

$$\Pi_{\mu\nu}(x) = (d-1)\frac{\dot{a}}{a^3}\delta_{\mu\nu}, \qquad \Pi_i(x) = L_{ij}\dot{\phi}^j \tag{3.17}$$

Combining these equations, we can verify

$$a\frac{d}{da}\phi^i(a) = \frac{2(d-1)}{W(\phi)}L^{ij}(\phi)\frac{\partial W(\phi)}{\partial \phi^j}, \tag{3.18}$$

which agrees with the function (3.13). Since a gives a unit length of the d-dimensional space, eq. (3.18) shows that the classical trajectory $\overline{\phi}^i(r,x)$ can be interpreted as the RG flow of the boundary field theory with the functions $\beta^i(\phi)$ being the RG beta functions. One can further show [2] that the Callan-Symanzik equation holds for the correlation functions defined by $\langle \mathcal{O}_{i_1}(x_1)\cdots\mathcal{O}_{i_n}(x_n)\rangle\,(a,\phi) \equiv \delta^n S/\delta\phi^{i_1}(x_1)\cdots\delta\phi^{i_n}(x_n)\big|_{(3.15)}$.

3.3 Weyl anomaly

Since $\Gamma[G,\phi]$ is regarded as the generating functional of the boundary field theory, the first term of the equation (3.12) should give the vacuum expectation value of the trace of the energy-momentum tensor of the boundary field theory. Thus, for the configuration $\beta^i = 0$, the right hand side of the equation (3.12) expresses the Weyl anomaly of the boundary field theory:

$$-\frac{2}{\sqrt{G}}G_{\mu\nu}\frac{\delta\Gamma}{\delta G_{\mu\nu}} \equiv \langle T^\mu_\mu\rangle = \frac{-2(d-1)}{2\kappa^2_{d+1}W(\phi)}\Big[\{S_{\rm loc}, S_{\rm loc}\}\Big]_d. \tag{3.19}$$

As an example, we consider five-dimensional dilatonic gravity ($d=4$) with a single scalar field, setting $V = -d(d-1)/l^2 = -12/l^2$ and $L = 1$:

$$\mathcal{L}_4 = -\frac{12}{l^2} - R + \frac{1}{2}G^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi. \tag{3.20}$$

In this case, all the functions W, M and Φ do not depend on ϕ, and eqs. (3.8)–(3.10) are solved as

$$S_{\rm loc}[G,\phi] = \int d^4x\sqrt{G}\left(-\frac{6}{l} - \frac{l}{2}R + \frac{l}{2}G^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right). \tag{3.21}$$

We can further calculate $\left[\{S_{\rm loc}, S_{\rm loc}\}\right]_4$ easily and find

$$\begin{aligned}\langle T^\mu_\mu\rangle &= -\frac{2l^3}{2\kappa_5^2}\left(\frac{1}{24}R^2 - \frac{1}{8}R_{\mu\nu}R^{\mu\nu} - \frac{1}{24}R\,G^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\right.\\ &\left.+\frac{1}{8}R^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{48}\left(G^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right)^2 - \frac{1}{16}\left(\nabla^2\phi\right)^2\right). \end{aligned} \tag{3.22}$$

This is in exact agreement with the result of Ref. [6]. (See also [5].) If we assume $\phi(x) = \phi$ (const.) and take the background of $\mathrm{AdS}_5 \times S^5$, this also reproduces the correct large N limit of the four-dimensional $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory.

3.4 Scaling dimension

We assume that the scalars are normalized as $L_{ij}(\phi) = \delta_{ij}$ and that the bulk scalar potential $V(\phi)$ has the expansion

$$V(\phi) = 2\Lambda + \frac{1}{2}\sum_i m_i^2\,\phi_i^2 + \sum_{ijk} g_{ijk}\,\phi_i\phi_j\phi_k + \cdots, \tag{3.23}$$

with $\Lambda = -d(d-1)/2l^2$. Then it follows from (3.8) that W takes the form

$$W = -\frac{2(d-1)}{l} + \frac{1}{2}\sum_i \lambda_i\,\phi_i^2 + \sum_{ijk}\lambda_{ijk}\,\phi_i\phi_j\phi_k + \cdots, \tag{3.24}$$

with

$$l\lambda_i = \frac{1}{2}\left(-d + \sqrt{d^2 + 4\,m_i^2\,l^2}\right), \tag{3.25}$$

$$g_{ijk} = \left(\frac{d}{l} + \lambda_i + \lambda_j + \lambda_k\right)\lambda_{ijk}. \tag{3.26}$$

The beta functions can be evaluated easily and are found to be

$$\beta^i = -\sum_i l\lambda_i\,\phi_i - 3\sum_{jk}\lambda_{ijk}\,\phi_j\phi_k + \cdots. \tag{3.27}$$

Thus, equating the coefficient of the first term with $d - \Delta_i$, where Δ_i is the scaling dimension of the operator coupled to ϕ_i, we obtain

$$\Delta_i = d + l\lambda_i = \frac{1}{2}\left(d + \sqrt{d^2 + 4\,m_i^2\,l^2}\right). \tag{3.28}$$

This exactly reproduces the result given in Ref. [1].

4 Higher-derivative gravity and the holographic RG

In this section we consider $(d+1)$-dimensional classical higher-derivative gravity and discuss its RG interpretation [7]. We first review the general theory for classical mechanics of higher-derivative system and then apply it to the gravity case.

4.1 General theory of higher-derivative system

We consider a system of point particle with the action

$$S[q(r)] = \int_{t'}^{t} dr\, L\left(q, \dot{q}, \cdots, q^{(N+1)}\right) \quad \left(q^{(n)}(r) \equiv d^n q(r)/dr^n\right). \tag{4.1}$$

The action (4.1) can be rewritten into the first-order form by introducing the Lagrange multipliers $p, P_1, \cdots, P_{N-1}$, so that $q, Q^1 = \dot{q}, \cdots, Q^N = q^{(N)}$ can be regarded as independent canonical variables:

$$S[q, Q^1, \cdots, Q^N; p, P_1, \cdots, P_N] = \int_{t'}^{t} dr \left[p\,\dot{q} + \sum_{a=1}^{N} P_a \dot{Q}^a - H(q, Q^a; p, P_a)\right]. \tag{4.2}$$

Here we have carried out a Legendre transformation from $(Q^N, \dot{Q}^N)$ to (Q^N, P_N) through

$$P_N = \frac{\partial L}{\partial \dot{Q}^N}\left(q, Q^1, \cdots, Q^N, \dot{Q}^N\right). \tag{4.3}$$

The Hamiltonian is given by

$$\begin{aligned} H(q, Q^a; p, P_a) &= p\,Q^1 + P_1 Q^2 + \cdots + P_{N-1} Q^N + P_N\, \dot{Q}^N(q, Q^a; P_N) \\ &\quad - L\left(q, Q^1, \cdots, Q^N, \dot{Q}^N(q, Q^a; P_N)\right). \end{aligned} \tag{4.4}$$

The equation of motion consists of the usual Hamilton equations,

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{Q}^a = \frac{\partial H}{\partial P_a}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{P}_a = -\frac{\partial H}{\partial Q^a}, \tag{4.5}$$

and of the following constraint which must hold at the boundary, $r = t$ and $r = t'$:

$$p\,\delta q + \sum_a P_a\, \delta Q^a = 0 \quad (r = t, t'). \tag{4.6}$$

The latter requirement, (4.6), can be satisfied when we take either Dirichlet boundary conditions or Neumann boundary conditions,

$$\underline{\text{Dirichlet}}: \quad \delta q = 0, \quad \delta Q^a = 0 \quad (r = t, t'), \tag{4.7}$$

$$\underline{\text{Neumann}}: \quad p = 0, \quad P_a = 0 \quad (r = t, t'), \tag{4.8}$$

for each variable q and Q^a $(a=1,\cdots,N)$.

Although there are various choices of boundary conditions when solving (4.5), we adopt the following mixed boundary conditions:

$$\delta q = P_a = 0 \quad (r = t,\, t')\,. \tag{4.9}$$

The reason why we choose this condition is explained in the next subsection.

Under the condition (4.9), the classical solution is a function of the boundary value of q:

$$\bar{q} = \bar{q}(r,x;q,t;q',t') \qquad \Big(q = \bar{q}(r=t,x),\; q' = \bar{q}(r=t',x)\Big), \tag{4.10}$$

and thus the classical action becomes a function only of the boundary value of q;

$$S(t,q;t',q') \equiv \boldsymbol{S}[\bar{q}(r,x;q,t;q',t')]. \tag{4.11}$$

We will call $S(t,q;t',q')$ the "*reduced classical action.*"

Since we took the mixed boundary conditions, the reduced classical action does not obey the Hamilton-Jacobi equation in the usual form. However, one can prove the following theorem for any Lagrangian of the form

$$L(q^i,\dot{q}^i,\ddot{q}^i) = L_0(q^i,\dot{q}^i) + c\,L_1(q^i,\dot{q}^i,\ddot{q}^i). \tag{4.12}$$

<u>Theorem</u> [7]
Let $H_0(q,p)$ be the Hamiltonian corresponding to $L_0(q,\dot{q})$. Then the reduced classical action $S(t,q;\,t',q')=S_0(t,q;\,t',q')+c\,S_1(t,q;\,t',q')+\mathcal{O}(c^2)$ satisfies the following equation up to $\mathcal{O}(c^2)$:

$$-\frac{\partial S}{\partial t} = \widetilde{H}(q,p), \quad p_i = \frac{\partial S}{\partial q^i}, \quad \text{and} \quad +\frac{\partial S}{\partial t'} = \widetilde{H}(q',p'), \quad p'_i = -\frac{\partial S}{\partial q'^i}, \tag{4.13}$$

where

$$\begin{aligned}
\widetilde{H}(q,p) &\equiv H_0(q,p) - c\,L_1(q,f_1(q,p),f_2(q,p)),\\
f_1^i(q,p) &\equiv \{H_0,q^i\} = \frac{\partial H_0}{\partial p_i},\\
f_2^i(q,p) &\equiv \{H_0,\{H_0,q^i\}\} = \frac{\partial^2 H_0}{\partial p_i \partial q^j}\frac{\partial H_0}{\partial p_j} - \frac{\partial^2 H_0}{\partial p_i \partial p_j}\frac{\partial H_0}{\partial q^j}.\\
&\left(\{F(q,p),G(q,p)\} \equiv \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q^i}\right)
\end{aligned} \tag{4.14}$$

4.2 RG interpretation of the mixed boundary conditions

The mixed boundary conditions we took in the preceding subsection, can be understood in terms of the holographic renormalization group. To explain this, we consider a toy model that has the Lagrangian of the form (4.12):

$$L = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\mu^2 q^2 + \frac{c}{2}\ddot{q}^2 . \tag{4.15}$$

Its first-order form reads

$$L = p\,\dot{q} + P\dot{Q} - H(q, Q;\, p, P), \tag{4.16}$$

with

$$H(q, Q;\, p, P) = -\frac{1}{2}\mu^2 q^2 - \frac{1}{2}Q^2 + Qp + \frac{1}{2c}P^2. \tag{4.17}$$

By performing an almost diagonal canonical transformation, the Lagrangian can be rewritten into the following form with a normalized kinetic term:

$$L = p'\dot{q}' + P'\dot{Q}' - H'(q', p';\, Q', P'), \tag{4.18}$$

where

$$H'(q', Q';\, p', P') = \frac{1}{2}p'^2 + \frac{1}{2}P'^2 - \frac{1}{2}m^2 q'^2 - \frac{1}{2}M^2 Q'^2, \tag{4.19}$$

with

$$\begin{aligned} m^2 &= \frac{1 - \sqrt{1 - 4c\,\mu^2}}{2c} = \mu^2(1 + \mathcal{O}(c)), \\ M^2 &= \frac{1 + \sqrt{1 - 4c\,\mu^2}}{2c} = \frac{1}{c}(1 + \mathcal{O}(c)). \end{aligned} \tag{4.20}$$

Since a bulk scalar mode with mass M is coupled to a scaling operator with scaling dimension $\Delta = \frac{1}{2}\left(d + \sqrt{d + 4M^2}\right)$, the relation (4.20) shows that the mode $Q' \sim Q$ is coupled to a highly irrelevant operator with large scaling dimension when $c \ll 1$. Thus, even if we take the boundary value of Q arbitrarily, the flow of (q, Q) converges rapidly to the renormalized trajectory. This implies that in order to take a continuum limit, we only need to consider the flow on the renormalized trajectory. This can be achieved by taking the boundary value which realizes the condition that the β function for the very massive mode vanishes, but this is nothing but our mixed boundary condition since $P \sim \dot{Q}$.

4.3 Application to higher-derivative gravity

We apply the formalism developed in the preceding subsections, to higher-derivative gravity that has the Lagrangian of the form (4.12). Since higher-derivative terms stem from integrating over string excitation mode with mass of order α', eq. (4.12) implies that we are taking account of stringy corrections up to $c \sim \alpha'$.

We consider classical pure gravity on M_{d+1} whose action takes generically the form

$$\boldsymbol{S} = \boldsymbol{S}_B + \boldsymbol{S}_b\,. \tag{4.21}$$

Here $\boldsymbol{S}_B$ is the bulk action and $\boldsymbol{S}_b$ is the boundary action:[5]

$$\boldsymbol{S}_B = \int_{M_{d+1}} d^{d+1}X\sqrt{\widehat{G}}\left[2\Lambda - \widehat{R} - a\widehat{R}^2 - b\widehat{R}_{MN}^2 - c\widehat{R}_{MNPQ}^2\right], \tag{4.22}$$

$$\boldsymbol{S}_b = \int_{\Sigma_d} d^dx\sqrt{G}\Big[2K + x_1\,RK + x_2\,R_{\mu\nu}K^{\mu\nu} + x_3\,K^3 + x_4\,KK_{\mu\nu}^2 + x_5\,K_{\mu\nu}^3\Big]. \tag{4.23}$$

Using the ADM parametrization, we can express the action in the form:

$$\boldsymbol{S} = \int_{M_{d+1}} d^{d+1}X\sqrt{G}\left[\mathcal{L}_{d+1}^{(0)}(g,\dot{g};N,\lambda^\mu) + \mathcal{L}_{d+1}^{(1)}(g,\dot{g},\ddot{g};N,\lambda^\mu)\right]. \tag{4.24}$$

Applying *Theorem* to this system, we obtain the flow equation of the form

$$\{S,S\} + \{S,S,S,S\} = \mathcal{L}_d, \tag{4.25}$$

where $\{S,S\} \sim (\delta S/\delta g)^2$ and $\{S,S,S,S\} \sim (\delta S/\delta g)^4$, and their explicit form can be found in [7].

This equation can be solved in a way similar to that in section 3. The local part of the reduced classical action is

$$S_{\rm loc} = \int d^dx\sqrt{G}\Big[W - \Phi R + \cdots\Big], \tag{4.26}$$

with

$$\begin{aligned} W &= -\frac{2(d-1)}{l} - \frac{4(d+3)}{3l^3}\Big[d(d+1)a + db + 2c\Big], \\ \Phi &= \frac{l}{d-2} + \frac{2}{(d-2)\,l}\Big[d(d-5)a - 2b - 2c\Big], \end{aligned} \tag{4.27}$$

[5] We require the geometry to be asymptotically AdS near the boundary. To satisfy this condition, $x_1, \cdots .x_5$ must satisfy the condition $x_1 = 4a$, $x_2 = 2b$, $d^2\,x_3 + d\,x_4 + x_5 = -(4/3)\big(d(d+1)a + d\,b + 2c\big)$ and also $\Lambda = -d(d-1)/2l^2 + d(d-3)\big(d(d+1)a + db + 2c\big)/2l^4$ [7].

and the Weyl anomaly is

$$\langle T^i_i \rangle_G = \frac{2l^3}{2\kappa_5^2}\left[\left(\frac{-1}{24}+\frac{5a}{3l^2}+\frac{b}{3l^2}+\frac{c}{3l^2}\right)R^2+\left(\frac{1}{8}-\frac{5a}{l^2}-\frac{b}{l^2}-\frac{3c}{2l^2}\right)R_{ij}^2+\frac{c}{2l^2}R_{ijkl}^2\right]. \tag{4.28}$$

As a check, we consider $\mathcal{N}=2$ superconformal $USp(N)$ gauge theory in four dimensions which is thought of as the AdS/CFT dual of type IIB string theory on $AdS_5 \times S^5/\boldsymbol{Z}_2$ [8]. In this case, we set the values $a = b = 0$ and $c/2l^2 = 1/32N + \mathcal{O}(1/N^2)$, as determined in [9]. l and $1/2\kappa_5^2$ are

$$l = (8\pi g_s N)^{1/4}\left(1+\frac{\xi}{N}\right), \quad \frac{1}{2\kappa_5^2} = \frac{\mathrm{Vol}(S^5/\boldsymbol{Z}_2)\,(8\pi g_s N)^{5/4}}{2\kappa^2}\left(1+\frac{\eta}{N}\right), \tag{4.29}$$

where ξ and η represent possible but unknown corrections due to D7-O7 background [9]. Thus the Weyl anomaly (4.28) becomes

$$\begin{aligned}\langle T^i_i \rangle_g = \frac{N^2}{2\pi^2}\left(1+\frac{3\xi+\eta}{N}\right)&\left[\left(\frac{-1}{24}+\frac{1}{48N}\right)R^2+\left(\frac{1}{8}-\frac{3}{32N}\right)R_{ij}^2+\frac{1}{32N}R_{ijkl}^2\right] \\ &+\mathcal{O}(N^0).\end{aligned} \tag{4.30}$$

If $3\xi + \eta = 5/4$, our calculation reproduces the field theoretical result [10],

$$\langle T^i_i \rangle_g = \frac{N^2}{2\pi^2}\left[\left(\frac{-1}{24}-\frac{1}{32N}\right)R^2+\left(\frac{1}{8}+\frac{1}{16N}\right)R_{ij}^2+\frac{1}{32N}R_{ijkl}^2\right]+\mathcal{O}(N^0). \tag{4.31}$$

5 Conclusion

In this article, we discussed several aspects of the holographic RG. We found that the Hamilton-Jacobi equation for a gravity system is quite useful for exploring the structure of the holographic RG. From the flow equation, we derived the Weyl anomaly of the boundary field theory and also the scaling dimension of a scaling operator which is dual to a bulk scalar field. We also showed that the classical trajectory of a bulk field can actually be interpreted as the RG flow of the corresponding scaling operator.

We further discussed how higher-derivative gravity systems can be interpreted in the context of the AdS/CFT correspondence. Although higher-derivative gravity requires more boundary conditions for each bulk field than those in Einstein gravity, we pointed out that by choosing the Neumann boundary conditions for higher-derivative modes, the classical trajectory is interpreted as the renormalized trajectory in the presence of highly irrelevant operators. We further derived a Hamilton-Jacobi-like equation that determines the fixed-point action. Using this equation, we computed the $1/N$ correction to the Weyl anomaly of $\mathcal{N}=2$ $USp(N)$ superconformal field theory in four dimensions, on the basis of the holographic description in terms of type IIB string theory on $AdS_5 \times S^5/\boldsymbol{Z}_2$ [8].

In spite of the developments described here, deep understanding is still lacking about what kind of continuum field theories can be described in the scheme of the holographic RG, although it is widely believed that such field theories should have some kind of supersymmetry and also should include variables that have redundancy in their degrees of freedom (like gauge variables). Some developments in this direction are expected to be made in the near future.

References

[1] J. Maldacena, "*The large N limit of superconformal field theories and supergravity*," Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200;
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "*Gauge Theory Correlators from Non-Critical String Theory*," Phys. Lett. **B428** (1998) 105, hep-th/9802109;
E. Witten, "*Anti De Sitter Space And Holography*," Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.

[2] J. de Boer, E. Verlinde and H. Verlinde, "*On the Holographic Renormalization Group*," hep-th/9912012.

[3] M. Fukuma, S. Matsuura and T. Sakai, "*A Note on the Weyl Anomaly in the Holographic Renormalization Group*," Prog. Theor. Phys. **104** (2000) 1089, hep-th/0007062.

[4] M. Fukuma and T. Sakai, "*Comment on Ambiguities in the Holographic Weyl Anomaly*," Mod. Phys. Lett. **A15** (2000) 1073, hep-th/0007200.

[5] M. Henningson and K. Skenderis, "*The Holographic Weyl anomaly*," J.High Energy Phys. **07** (1998) 023, hep-th/9806087.

[6] S. Nojiri and S. Odintsov, "*Conformal Anomaly for Dilaton Coupled Theories from AdS/CFT Correspondence*," Phys. Lett. **B444** (1998) 92, hep-th/9810008;
S. Nojiri, S. Odintsov and S. Ogushi, "*Conformal Anomaly from d5 Gauged Supergravity and c-function Away from Conformity*," hep-th/9912191; "*Finite Action in d5 Gauged Supergravity and Dilatonic Conformal Anomaly for Dual Quantum Field Theory*," hep-th/0001122.

[7] M. Fukuma, S. Matsuura and T. Sakai, "*Higher-Derivative Gravity and the AdS/CFT Correspondence*," Prog. Theor. Phys. **105** (2001) 1017, hep-th/0103187

[8] A. Fayyazuddin and M. Spalinski "*Large N Superconformal Gauge Theories and Supergravity Orientifolds*," Nucl.Phys. **B535** (1998) 219, hep-th/9805096;
O. Aharony, A. Fayyazuddin and J. Maldacena, "*The Large N Limit of $\mathcal{N} = 1,2$ Field Theories from Three Branes in F-theory*," J.High Energy Phys. **9807** (1998) 013, hep-th/9806159.

[9] M. Blau, K. S. Narain and E. Gava "*On Subleading Contributions to the AdS/CFT Trace Anomaly*," J.High Energy Phys. **9909** (1999) 018, hep-th/9904179.

[10] M. J. Duff, "*Twenty Years of the Weyl Anomaly*," Class. Quant. Grav. **11** (1994) 1387, hep-th/9308075.

[11] S. Nojiri and S. D. Odintsov, "*On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence*," Int. J. Mod. Phys. **A15** (2000) 413 hep-th/9903033;
S. Nojiri and S. D. Odintsov, "*Finite gravitational action for higher derivative and stringy gravity*," Phys. Rev. **D62** (2000) 064018 hep-th/9911152.