A SUBCLASS OF ANALYTIC FUNCTIONS WITH TWO FIXED POINTS

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ABSTRACT. Making use of operator of fractional calculus a subclass $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$ of univalent functions with fixed point in the unit disk $E$ is introduced and obtained coefficient-estimates distortion theorem. Lastly we investigated Hadamard product property and linear combination function of $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$.

1. INTRODUCTION

Let $A$ denote the class of functions of the form

\begin{equation}
(1.1) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \geq 0),
\end{equation}

which are analytic in unit disc $E = \{ z : |z| < 1 \}$. Silvermann ([4]) studied the class of functions of the form

\[ f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geq 0), \]

where either

\begin{equation}
(1.2) \quad f(z_0) = z_0 (-1 < z_0 < 1; z_0 \neq 0) \quad \text{or} \quad f'(z_0) = 1 (-1 < z_0 < 1).
\end{equation}

Recently, Uralegadi and Somanatha([6]) studied the class of functions of the form

\begin{equation}
(1.3) \quad f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geq 0) \quad \text{with} \quad \frac{(1-t)f(z_0)}{z_0} + tf'(z_0) = 1,
\end{equation}

where $-1 < z_0 < 1, \ 0 \leq t \leq 1$. A function $f(z)$ is said to be convex of order $\alpha$, if

\[ \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \ (z \in E : 0 \leq \alpha < 1). \]

We denote by $C^*(\alpha)$ the class of convex functions of order $\alpha (0 \leq \alpha < 1)$.

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We now recall the following definition of a generalized fractional operator introduced by Srivastava et al([5]).

**Definition 1** For real numbers $\eta(\eta > 0), \gamma$ and $\delta$, the generalized fractional integral operator $I_{0,z}^{\eta,\gamma,\delta}$ of order $\eta$ is defined for a function $f(z)$, by

$$I_{0,z}^{\eta,\gamma,\delta} f(z) = \frac{z^{-\eta-1}}{\Gamma(\eta)} \int_{0}^{z} (z-\xi)^{\eta-1} F(\eta + \gamma, -\delta; \eta; 1 - \xi/z) f(\xi) d\xi,$$

where $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin with the order

$$f(z) = O(|z|^\epsilon), (z \to 0), (\epsilon < \max\{0, \gamma, -\delta\} - 1),$$

(1.4) \hspace{1cm} F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} (z \in D),$$

and $(v)_n$ being the pochhammer symbol defined by

$$(v)_n = \frac{\Gamma(v+n)}{\Gamma(v)},$$

if $n = 0$ then $(v)_n = 1$ and if $n \in N = \{1, 2, \cdots\}$ then $v_n = v(v+1) \cdots (v+n-1)$, provided further that the multiplicity of $(z-\xi)^{-\eta}$ is removed as Definition 1 above.

**Definition 2** For real numbers $\eta(0 \leq \eta < 1), \gamma$, and $\delta$, the generalized fractional derivative operator $J_{0,z}^{\eta,\gamma,\delta}$ of order $\eta$ is defined for a function $f(z)$, by

$$J_{0,z}^{\eta,\gamma,\delta} f(z) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dz} \left\{ z^{\eta-\gamma} \int_{0}^{z} (z-\xi)^{-\eta} F\left(\gamma - \eta, -\delta; 1 - \eta; 1 - \xi/z\right) f(\xi) d\xi \right\},$$

(1.5) \hspace{1cm} \text{where $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin, and the multiplicity of $(z-\xi)^{-\eta}$ is removed as Definition 1 above.}

It follows readily from Definition 2, $J_{0,z}^{\eta,\eta,\delta} f(z) = D_{z}^{\eta} f(z)$ $(0 \leq \eta < 1)$, where operator $D_{z}^{\eta}$ is fractional derivatives operator which is defined by Owa([2]). Furthermore, in terms of Gamma functions, we have the following Lemma.

**Lemma 3.** ([5]) If $0 \leq \eta < 1$ and $n > \gamma - \delta - 2$, then

$$J_{0,z}^{\eta,\gamma,\delta} z^n = \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+2)}{\Gamma(n-\gamma+1)\Gamma(n-\eta+\delta+2)} z^{n-\gamma}.$$
Lemma 4. If the form of a function $f(z)$ defined by (1.2) and satisfying (1.3), then

\[\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}z^{\gamma-1}J_{0,z}^{\eta,\gamma,\delta}f(z) = a_1 - \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1},\]

where we denote $\Psi_n(\eta, \gamma, \delta) = \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+2)}{\Gamma(n-\eta+\delta+2)}$.

proof. By Lemma 3, we get (1.7). ⊓⊔

We will define the following definition.

Definition 5 A function $f(z)$ defined by (1.2) and satisfying (1.3) is said to be in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$ if

\[\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}z^{\gamma-1}J_{0,z}^{\eta,\gamma,\delta}f(z) - a_1 - (1+\mu)\alpha < \beta, z \in E,\]

where

\[0 \leq \eta < 1, \eta - \delta < 3, \gamma - \delta < 3, 0 \leq \mu \leq 1, 0 \leq \alpha < 1, 0 < \beta \leq 1 \text{ and } \alpha < a_1.\]

Furthermore, by specializing the parameters $\alpha, \beta, \mu, \eta, \gamma, \delta, t$, we obtain the following subclasses studied by various authors,

1. $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, 1; 0) = P^*(\alpha, \beta, \mu, \eta)$ (Jochi [1]);
2. $\varphi(\alpha, \beta, \mu, 1, 1, \delta, 1; 0) = P^*(\alpha, \beta, \mu)$ (Owa and Aouf [3]);

The main purpose of this paper is to investigate coefficient-inequalities, distortion theorem and radius problem of functions in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$. And, we obtain Hadamard product property and linear combination function.

2. A Coefficient Theorem

We begin by starting our first result as,

Theorem 2.1. A function $f(z)$ is in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, t, \delta; z_0)$ if and only if

\[\sum_{n=2}^{\infty} \left\{(1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n - [(1-t)+tn]z_0^{n-1}\right\} a_n \leq (1+\mu)\beta(1-\alpha),\]

where $\Psi_n(\mu, \gamma, \delta)$ is in (1.7).

proof. Suppose that $f(z)$ is in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, t, \delta; z_0)$, so that condition (1.8) readily yields.

\[\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}z^{\gamma-1}J_{0,z}^{\eta,\gamma,\delta}f(z) - a_1 - (1+\mu)\alpha < \beta.\]
Using Lemma 4, we obtain that

\[
- \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1} \mu \left\{ a_1 - \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1} \right\} + a_1 - (1 + \mu) \alpha < \beta \ (z \in E).
\]

Since \(|\Re(z)| \leq |z|\), for any \(z\), we have

\[
(2.2) \quad \mathcal{R} \left\{ \frac{\sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1}}{(1 + \mu) (a_1 - \alpha) - \mu \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1}} \right\} < \beta.
\]

Choose values of \(z\) on the real axis so that \(\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z)\) is real, upon clearing the denominator in (2.2) and letting \(z \to 1\) through the real values, we get

\[
(2.3) \quad \sum_{n=2}^{\infty} (1 + \mu \beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1} \\
\quad \leq (1 + \mu) \beta (a_1 - \alpha).
\]

Finally, substituting \(a_1 = 1 + \sum_{n=2}^{\infty} [(1-t)+tn] a_n |z_0|^{n-1}\) in (2.3), we get (2.1).

Conversely, assume that the inequality (2.1) holds true. Consider

\[
\left| \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) - a_1 \right| - \beta \mu \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) + a_1 - (1 + \mu) \alpha \\
\leq \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n |z|^{n-1} \\
- (1 + \mu) \beta (a_1 - \alpha) + \beta \mu \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n |z|^{n-1} \\
\leq \sum_{n=2}^{\infty} (1 + \mu \beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n - (1 + \mu) \beta (a_1 - \alpha) \leq 0,
\]

by the hypothesis. Hence, a function \(f(z)\) is in the class \(\varphi(\alpha, \beta, \mu, \eta, \gamma, t; z_0)\). \(\square\)

**Corollary 2.2.** Let the function \(f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n\) defined by (1.3) be in the class \(\varphi(\alpha, \beta, \mu, \eta, \gamma, t; z_0)\). Then

\[
a_n \leq \frac{(1 + \mu \beta) (1 - \alpha)}{(1 + \mu \beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n + (1 - a_1)}.
\]
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The assertion (2.1) of Theorem 2.1 is sharp extremal function being

\( f(z) = a_1 z - \frac{(1 + \mu) \beta (1 - \alpha)}{(1 + \mu \beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n + 1 - a_1} z^n. \)

Where \( a_1 = 1 + \sum_{n=2}^{\infty} [(1-t) + tn] a_n z_0^{n-1} \).

3. Distortion Theorem

**Theorem 3.1.** If a function \( f(z) \) is in the class \( \varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0) \) with \( \eta > \gamma \), then

\[ |f(z)| \geq a_1 |z| - \frac{(2 - \gamma) (3 - \eta + \delta) (1 + \mu) \beta (a_1 - \alpha)}{2 (3 - \gamma + \delta) (1 + \mu \beta)} |z|^2, \]

and

\[ |f(z)| \leq a_1 |z| + \frac{(2 - \gamma) (3 - \eta + \delta) (1 + \mu) \beta (a_1 - \alpha)}{2 (3 - \gamma + \delta) (1 + \mu \beta)} |z|^2. \]

**Proof.** In view of inequality (2.1) and the fact that \( \Psi_n(\eta, \gamma, \delta) \) is non-decreasing for \( \eta \geq \gamma \), we have

\[ (1 + \mu) \beta (a_1 - \alpha) \]
\[ \geq \sum_{n=2}^{\infty} (1 + \mu \beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+2)}{\Gamma(n-\gamma+1)\Gamma(n-\eta+\delta+2)} a_n \]
\[ \geq \frac{2 (3 - \gamma + \delta) (1 + \mu \beta)}{(2 - \gamma) (3 - \eta + \delta)} \sum_{n=2}^{\infty} a_n. \]

Therefore, we obtain

\[ |f(z)| \geq a_1 |z| - \sum_{n=p+1}^{\infty} a_n |z|^n \geq a_1 |z| - |z|^2 \sum_{n=p+1}^{\infty} a_n \]
\[ \geq a_1 |z| - \frac{(2 - \gamma) (3 - \eta + \delta) (1 + \mu) \beta (a_1 - \alpha)}{2 (3 - \gamma + \delta) (1 + \mu \beta)} |z|^2, \]

\[ \sum_{n=2}^{\infty} a_n \leq \frac{(2 - \gamma) (3 - \eta + \delta) (1 + \mu) \beta (a_1 - \alpha)}{2 (3 - \gamma + \delta) (1 + \mu \beta)}, \]

and we have

\[ |f(z)| \geq a_1 |z| - \frac{(2 - \gamma) (3 - \eta + \delta) (1 + \mu) \beta (a_1 - \alpha)}{2 (3 - \gamma + \delta) (1 + \mu \beta)} |z|^2. \]

Similiarly,

\[ |f(z)| \leq a_1 |z| + \frac{(2 - \gamma) (3 - \eta + \delta) (1 + \mu) \beta (a_1 - \alpha)}{2 (3 - \gamma + \delta) (1 + \mu \beta)} |z|^2. \]

The proof is complete. \( \square \)
Theorem 3.2. If a function $f(z)$ is in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$, then

\[
\left| J_{0, z}^{\eta, \gamma, \delta} f(z) \right| \geq \frac{\Gamma(2 - \gamma) \Gamma(3 - \eta + \delta)}{\Gamma(3 - \gamma + \delta)} \left\{ a_1 |z|^{1-\gamma} - \frac{(1 + \mu) \beta(a_1 - \alpha)}{(1 + \mu \beta)} |z|^{2-\gamma} \right\},
\]

and

\[
\left| J_{0, \acute{z}}^{\eta, \gamma, \delta} f(z) \right| \leq \frac{\Gamma(2 - \gamma) \Gamma(3 - \eta + \delta)}{\Gamma(3 - \gamma + \delta)} \left\{ a_1 |z|^{1-\gamma} + \frac{(1 + \mu) \beta(a_1 - \alpha)}{(1 + \mu \beta)} |z|^{2-\gamma} \right\},
\]

for $z \in D_0$ where $D_0$ equals to $E$ if $\gamma \leq 1$, and $D_0$ equals to $E^*$ if $1 < \gamma < n$.

proof. By using in equality (1.8) and Theorem 2.1, we obtain that

\[
\left| \frac{\Gamma(3 - \gamma + \delta)}{\Gamma(2 - \gamma) \Gamma(3 - \eta + \delta)} z^\gamma J_{0, z}^{\eta, \gamma, \delta} f(z) \right| \geq a_1 |z| - \sum_{n=2}^{\infty} \frac{\Gamma(2 - \gamma) \Gamma(3 - \eta + \delta)}{\Gamma(3 - \gamma + \delta)} \Psi_n a_n |z|^n \geq a_1 |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(2 - \gamma) \Gamma(3 - \eta + \delta)}{\Gamma(3 - \gamma + \delta)} \Psi_n a_n \geq a_1 |z| - |z|^2 \frac{(1 + \mu) \beta(a_1 - \alpha)}{(1 + \mu \beta)},
\]

which is equivalent to (3.4). Similary, We obtain that

\[
\left| J_{0, \acute{z}}^{\eta, \gamma, \delta} f(z) \right| \leq \frac{\Gamma(2 - \gamma) \Gamma(3 - \eta + \delta)}{\Gamma(3 - \gamma + \delta)} \left\{ a_1 |z|^{1-\gamma} + \frac{(1 + \mu) \beta(a_1 - \alpha)}{(1 + \mu \beta)} |z|^{2-\gamma} \right\}.
\]

The proof is complete. ☐

Corollary 3.3. Let a function $f(z)$ be in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$ with $\eta > \gamma$. Then, in view of Theorem 3.1, $f(z)$ is included in a disk with its center at origin and radius $r$ given by

\[
r = a_1 + \frac{(2 - \gamma)(3 - \eta + \delta)}{2(3 - \gamma + \delta)} \frac{(1 + \mu) \beta(a_1 - \alpha)}{(1 + \mu \beta)},
\]

and $J_{0, z}^{\eta, \gamma, \delta} f(z)$ is included in a disk with its center at origin and radius $R$ given by

\[
R = \frac{\Gamma(3 - \gamma + \delta)}{\Gamma(2 - \gamma) \Gamma(3 - \eta + \delta)} \left\{ a_1 + \frac{(1 + \mu) \beta(a_1 - \alpha)}{(1 + \mu \beta)} \right\}.
\]

4. Radius of convexity
Theorem 4.1. Let \( f(z) \) be in the class \( \varphi(\alpha, \beta, \mu, \gamma, \delta, t; z_0) \). Then \( f(z) \) is convex in the disk
\[
|z| < r(\alpha, \beta, \mu, \gamma, \delta, t; z_0)
\]
\[
= \inf_{n \geq 2} \left\{ \frac{(1 + \mu \beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n + [(1 + \mu) \beta (1 - \alpha) - 1] [(1 - t) + tn] z_0^{n-1}}{n^2 (1 + \mu) \beta (1 - \alpha)} \right\}^{\frac{1}{n-1}}
\]

The result is sharp for the function given by

\[
|zf''(z)| \leq 1
\]

for \( r(\alpha, \beta, \mu, \gamma, \delta, t; z_0) \).

\[\text{proof.}\] It is sufficient to prove that \( |zf''(z)| \leq 1 \) for \( r(\alpha, \beta, \mu, \gamma, \delta, t; z_0) \).

A simple calculation gives us
\[
|zf''(z)| = \left| - \sum_{n=2}^{\infty} n(n-1)a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}.
\]

Clearly \( |zf''(z)| \leq 1 \), if
\[
\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1} \leq a_1 - \sum_{n=2}^{\infty} n|a_n| |z|^{n-1}.
\]

Using \( a_1 = 1 + \sum_{n=2}^{\infty} [(1 - t) tn] a_1 z_0^{n-1} \) in (4.3), we are led to
\[
\sum_{n=2}^{\infty} \{n^2|z|^{n-1} - [(1 - t) + tn] z_0^{n-1}\} a_n \leq 1.
\]

By Theorem 2.1, we have
\[
\sum_{n=2}^{\infty} \left\{ \frac{(1 + \mu \beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n - [(1 - t) + tn] z_0^{n-1}}{(1 + \mu) \beta (1 - \alpha)} \right\} a_n \leq 1.
\]

Hence (4.4) will hold, if
\[
n^2|z|^{n-1} - [(1 - t) + tn] z_0^{n-1} \leq \frac{(1 + \mu \beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n - [(1 - t) + tn] z_0^{n-1}}{(1 + \mu) \beta (1 - \alpha)},
\]
or equivalently
\[
|z|^{n-1} \leq \frac{(1 + \mu \beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n + [(1 + \mu) \beta (1 - \alpha) - 1] [(1 - t) + tn] z_0^{n-1}}{n^2 (1 + \mu) \beta (1 - \alpha)}.
\]
which in turn implies the assertion of the theorem. □

5. Property of Hadamard product

Let the function $f_j(z)$ ($j = 1, 2$) defined by $f_j(z) = a_{1,j}z - \sum_{n=2}^{\infty} a_{n,j}z^n$, we define the hadamard product $f(z) \ast g(z)$ by

$$ (f_1 \ast f_2)(z) = a_{1,1}a_{1,2}z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}z^n. $$

Theorem 5.1. Let the functions $f_j(z)$ ($j = 1, 2$) defined by (5.1) be in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta; z_0)$ with $\mu > \gamma$. Then $f_1(z) \ast f_2(z)$ is in the class $\varphi(v, \beta, \mu, \eta, \gamma, \delta; z_0)$ where

$$ v = v(\alpha, \beta, \mu, \eta, \gamma, \delta) = a_{1,1}a_{1,2} - \frac{(2 - \gamma)(3 - \eta + \delta)(1 + \mu)\beta(a_{1,1} - \alpha)(a_{1,2} - \alpha)}{2(3 - \gamma + \delta)(1 + \mu\beta)}. $$

Proof. Suppose that $f_1(z)$ and $f_2(z)$ are in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta; z_0)$, by using Theorem 2.1, we have

$$ \sum_{n=2}^{\infty} \frac{(1 + \mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_{n,1}}{(1 + \mu) \beta (a_{1,1} - \alpha)} \leq 1, $$

and

$$ \sum_{n=2}^{\infty} \frac{(1 + \mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_{n,2}}{(1 + \mu) \beta (a_{1,2} - \alpha)} \leq 1. $$

From (5.3) and (5.4), using Cauchy-Schwarze inequality, we have

$$ \sum_{n=2}^{\infty} \frac{(1 + \mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n}{(1 + \mu) \beta} \sqrt{\frac{a_{n,1}a_{n,2}}{(a_{1,1} - \alpha)(a_{1,2} - \alpha)}} \leq 1. $$

Hence, we find the largest $v$ such that

$$ \sum_{n=2}^{\infty} \frac{(1 + \mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n}{(1 + \mu) \beta (a_{1,1}a_{1,2} - v)} \leq 1, $$
or equivalently

\[ \sqrt{a_{n,1}a_{n,2}} \leq \frac{a_{1,1}a_{1,2} - v}{\sqrt{(a_{1,1} - \alpha)(a_{1,2} - \alpha)}}. \]

So, it is sufficient to find the largest \( v \) such that

\[ \frac{(1 + \mu) \beta \sqrt{(a_{1,1} - \alpha)(a_{1,2} - \alpha)}}{(1 + \mu \beta) \frac{\Gamma(2 - \gamma) \Gamma(3 - \eta + \delta)}{\Gamma(3 - \gamma + \delta)} \Psi_n} \leq \frac{a_{1,1}a_{1,2} - v}{\sqrt{(a_{1,1} - \alpha)(a_{1,2} - \alpha)}}. \]

Hence (5.8) yields

\[ v \leq a_{1,1}a_{1,2} - \frac{(1 + \mu) \beta (a_{1,1} - \alpha)(a_{1,2} - \alpha)}{(1 + \mu \beta) \frac{\Gamma(2 - \gamma) \Gamma(3 - \eta + \delta)}{\Gamma(3 - \gamma + \delta)} \Psi_n}, \]

for \( \mu \geq \gamma \), we have

\[ v \leq a_{1,1}a_{1,2} - \frac{(2 - \gamma)(3 - \eta + \delta)(1 + \mu \beta)(a_{1,1} - \alpha)(a_{1,2} - \alpha)}{2(3 - \gamma + \delta)(1 + \mu \beta)}, \]

which proves the assertion of this theorem.

6. Linear combination of the function in the class \( \varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0) \)

**Theorem 6.1.** Let \( T(\alpha, \beta, \mu, \eta, \gamma, \delta) = \frac{(1 + \mu) \beta(1 - \alpha)}{(1 + \mu \beta) \frac{\Gamma(2 - \gamma) \Gamma(3 - \eta + \delta)}{\Gamma(3 - \gamma + \delta)} \Psi_n} \), and let us put

\[ f_{\alpha}(z) = a_{1,1}z - T(\alpha, \beta, \mu, \eta, \gamma, \delta)z^{n}, n = 2, 3, \ldots, \]

and \( f_1(z) = a_{1,1}z \). Then \( f(z) \in \varphi(\alpha_j, \beta_j, \mu_j, \eta, \gamma, \delta, t_j; z_0) \) if and only if

\[ f(z) = \sum_{n=1}^{\infty} t_n f_{\alpha}(z), z \in E, \]

where \( \sum_{n=1}^{\infty} t_n = 1, t_n \geq 0 \) for \( n = 1, 2, 3, \ldots \).

**proof.** Let \( f(z) \in \varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0) \). Then, by Corollary 2.2, \( |a_n| \leq T(\alpha, \beta, \mu, \eta, \gamma, \delta) \).

Let us put

\[ t_n = T(\alpha, \beta, \mu, \eta, \gamma, \delta)^{-1} a_n, n = 2, 3, \cdots, \]

and \( t_1 = 1 - \sum_{n=2}^{\infty} t_n \). By assumption, we have \( t_n \geq 0, n = 2, 3, \cdots \), and \( t_1 \geq 0 \). Thus

\[ \sum_{n=1}^{\infty} t_n f_{\alpha}(z) = t_1 f_1(z) + \sum_{n=2}^{\infty} t_n f_{\alpha}(z) \]

\[ = \left(1 - \sum_{n=2}^{\infty} t_n\right) a_{1,1}z + \sum_{n=2}^{\infty} t_n \{a_{1,1}z - T(\alpha, \beta, \mu, \eta, \gamma, \delta)z^n\} \]

\[ = a_{1,1}z - \sum_{n=2}^{\infty} a_n z^n = f(z). \]
Conversely, let us function $f(z)$ satisfy (6.2). Since

$$f(z) = \sum_{n=1}^{\infty} t_{n}f_{n}(z) = t_{1}f_{1}(z) + \sum_{n=2}^{\infty} t_{n}f_{n}(z)$$

(6.5)

$$= \left(1 - \sum_{n=2}^{\infty} t_{n}\right) a_{1}z + \sum_{n=2}^{\infty} t_{n} \left\{ a_{1}z - T(\alpha, \beta, \mu, \eta, \gamma, \delta) z^{n}\right\},$$

$$= a_{1}z - \sum_{n=2}^{\infty} t_{n} T(\alpha, \beta, \mu, \eta, \gamma, \delta) z^{n} = a_{1}z - \sum_{n=2}^{\infty} a_{n}z^{n}$$

which proves the assertion of this theorem. □

REFERENCES


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