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# A SUBCLASS OF ANALYTIC FUNCTIONS WITH TWO FIXED POINTS

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ABSTRACT. Making use of operator of fractional calculus a subclass $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$ of univalent functions with fixed point in the unit disk E is introduced and obtained coefficient-estimates distortion theorem. Lastly we investigated Hadamard product property and linear combination function of $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$.

## 1. INTRODUCTION

Let $A$ denote the class of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n(a_n\geq 0), \tag{1.1}$$

which are analytic in unit disc $E=\{z:|z|<1\}$. Silvermann ([4]) studied the class of functions of the form

$$f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n\ (a_n\geq 0),$$

where either

$$f(z_0)=z_0(-1<z_0<1;z_0\neq 0)\quad\text{or}\quad f'(z_0)=1(-1<z_0<1). \tag{1.2}$$

Recently, Uralegadi and Somanatha([6]) studied the class of functions of the form

$$f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n\ (a_n\geq 0)\quad\text{with}\quad\frac{(1-t)f(z_0)}{z_0}+tf'(z_0)=1, \tag{1.3}$$

where $-1<z_0<1$, $0\leq t\leq 1$. A function $f(z)$ is said to be convex of order $\alpha$, if

$$\Re\left\{1+\frac{zf''(z)}{f(z)}\right\}>\alpha\,(z\in E:0\leq\alpha<1).$$

We denote by $C^*(\alpha)$ the class of convex functions of order $\alpha$ $(0\leq\alpha<1)$.

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We now recall the fallowing definition of a generalized fractional operator introduced by Srivastava et al([5]).

**Definition 1** For real numbers $\eta(\eta > 0), \gamma$ and $\delta$, the generalized fractional integral operator $I_{0,z}^{\eta,\gamma,\delta}$ of order $\eta$ is defined for a function $f(z)$, by

$$I_{0,z}^{\eta,\gamma,\delta} f(z) = \frac{z^{-\eta-1}}{\Gamma(\eta)} \int_0^z (z-\xi)^{\eta-1} F(\eta+\gamma, -\delta; \eta; 1-\frac{\xi}{z}) f(\xi) d\xi,$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O\left(|z|^{\varepsilon}\right), (z \to 0), (\varepsilon < \max\{0, \gamma, -\delta\} - 1),$$

(1.4) $$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (z \in D),$$

and $(v)_n$ being the pochhammer symbol defined by

$$(v)_n = \frac{\Gamma(v+n)}{\Gamma(v)},$$

if $n = 0$ then $(v)_n = 1$ and if $n \in N = \{1, 2, \cdots\}$ then $v_n = v(v+1)\cdots(v+n-1)$, provided further that the multiplicity of $(z-\xi)^{\eta-1}$ is removed requiring $\log(z-\xi)$ to be real $(z-\xi) > 0$.

**Definition 2** For real numbers $\eta(0 \leq \eta < 1)$, $\gamma$, and $\delta$, the generalized fractional derivative operator $J_{0,z}^{\eta,\gamma,\delta}$ of order $\eta$ is defined for a function $f(z)$, by

(1.5)
$$J_{0,z}^{\eta,\gamma,\delta} f(z) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dz} \left\{ z^{\eta-\gamma} \int_0^z (z-\eta)^{-\eta} F\left(\gamma-\eta, -\delta; 1-\eta; 1-\frac{\xi}{z}\right) f(\xi)\, d\xi \right\},$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\eta}$ is removed as Definition 1 obove.

It follows readily form Definition 2, $J_{0,z}^{\eta,\eta,\delta} f(z) = D_z^{\eta} f(z)\, (0 \leq \eta < 1)$, where oprator $D_z^{\eta}$ is fractional derivatives operator which is defined by Owa([2]). Furthermore, in terms of Gamma functions, we have the following Lemma.

**Lemma 3.** *([5]) If* $0 \leq \eta < 1$ *and* $n > \gamma - \delta - 2$, *then*

(1.6) $$J_{0,z}^{\eta,\gamma,\delta} z^n = \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+2)}{\Gamma(n-\gamma+1)\Gamma(n-\eta+\delta+2)} z^{n-\gamma}.$$

**Lemma 4.** *If the form of a function $f(z)$ defined by (1.2) and satisfying (1.3), then* (1.7)

$$\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) = a_1 - \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1},$$

*where we denote* $\Psi_n(\eta,\gamma,\delta) = \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+2)}{\Gamma(n-\gamma+1)\Gamma(n-\eta+\delta+2)}$.

*proof.* By Lemma3, we get a (1.7). □

We will define the following definition.

**Definition 5** A function $f(z)$ defined by (1.2) and satisfying (1.3) is said to be in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$ if

$$\left| \frac{\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) - a_1}{\mu \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) + a_1 - (1+\mu)\alpha} \right| < \beta, z \in E, \tag{1.8}$$

where

$$0 \le \eta < 1, \eta-\delta < 3, \gamma-\delta < 3, 0 \le \mu \le 1, 0 \le \alpha < 1, 0 < \beta \le 1 \text{ and } \alpha < a_1.$$

Furthermore, by specializing the parameters $\alpha,\beta,\mu,\eta,\gamma,\delta,t$, we obtain the following subclasses studied by various authors,

(1) $\varphi(\alpha,\beta,\mu,\eta,\eta,\delta,1;0) = P^*(\alpha,\beta,\mu,\eta)$( Jochi [1]);
(2) $\varphi(\alpha,\beta,\mu,1,1,\delta,1;0) = P^*(\alpha,\beta,\mu)$ (Owa and Aouf [3]);

The main purpose of this paper is to investigate coefficient-inequalites, distortion theorem and radius problem of functions in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$. And, we obtain Hadmard product property and linear combination function.

## 2. A Coefficient Theorem

We begin by starting our first result as,

**Theorem 2.1.** *A function $f(z)$ is in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,t,\delta;z_0)$ if and only if* (2.1)

$$\sum_{n=2}^{\infty} \left\{ (1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n - [(1-t)+tn] z_0^{n-1} \right\} a_n \le (1+\mu)\beta(1-\alpha),$$

*where $\Psi_n(\mu,\gamma,\delta)$ is in (1.7).*

*proof.* Suppose that $f(z)$ is in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,t,\delta;z_0)$, so that condition (1.8) readily yields.

$$\left| \frac{\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) - a_1}{\mu \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) + a_1 - (1+\mu)\alpha} \right| < \beta.$$

Using Lemma 4, we obtain that

$$\left| \frac{-\sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1}}{\mu\left\{a_1 - \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1}\right\} + a_1 - (1+\mu)\alpha} \right| < \beta \ (z \in E).$$

Since $|\Re(z)| \leq |z|$, for any $z$, we have

$$\Re\left\{ \frac{\sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1}}{(1+\mu)(a_1-\alpha) - \mu \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1}} \right\} < \beta. \tag{2.2}$$

Choose values of $z$ on the real axis so that $\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z)$ is real, upon clearing the denominator in (2.2) and letting $z \to 1$ through the real values, we get

$$\sum_{n=2}^{\infty} (1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n \leq (1+\mu)\beta(a_1-\alpha). \tag{2.3}$$

Finally, substituting $a_1 = 1 + \sum_{n=2}^{\infty} [(1-t)+tn]\, a_n |z_0|^{n-1}$ in (2.3), we get (2.1).

Conversely, assume that the inequality (2.1) holds true. Consider

$$\begin{aligned}
&\left| \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) - a_1 \right| - \\
&\beta \left| \mu \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) + a_1 - (1+\mu)\alpha \right| \\
&\leq \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n |z|^{n-1} \\
&- (1+\mu)\beta(a_1-\alpha) + \beta\mu \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n |z|^{n-1} \\
&\leq \sum_{n=2}^{\infty} (1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n - (1+\mu)\beta(a_1-\alpha) \leq 0,
\end{aligned}$$

by the hypothesis. Hence, a function $f(z)$ is in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, t, \delta; z_0)$. $\square$

**Corollary 2.2.** *Let the function $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ defined by (1.3) be in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$. Then*

$$a_n \leq \frac{(1+\mu)\beta(1-\alpha)}{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n + (1-a_1)}. \tag{2.4}$$

*The assertion (2.1) of Theorem2.1 is sharp extremal function being*

$$f(z)=a_1z-\frac{(1+\mu)\,\beta\,(1-\alpha)}{(1+\mu\beta)\,\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\,\Psi_n+1-a_1}z^n. \tag{2.5}$$

Where $a_1=1+\sum\limits_{n=2}^{\infty}\left[(1-t)+tn\right]a_nz_0^{n-1}$.

## 3. Distortion Theorem

**Theorem 3.1.** *If a function $f(z)$ is in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$ with $\eta>\gamma$, then*

$$|f(z)|\geq a_1|z|-\frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)}|z|^2, \tag{3.1}$$

*and*

$$|f(z)|\leq a_1|z|+\frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)}|z|^2. \tag{3.2}$$

*proof.* In view of inequality (2.1) and the fact that $\Psi_n(\eta,\gamma,\delta)$ is non-decreasing for $\eta\geq\gamma$, we have

$$\begin{aligned}&(1+\mu)\,\beta\,(a_1-\alpha)\\&\geq\sum_{n=2}^{\infty}(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+2)}{\Gamma(n-\gamma+1)\Gamma(n-\eta+\delta+2)}a_n\\&\geq\frac{2(3-\gamma+\delta)(1+\mu\beta)}{(2-\gamma)(3-\eta+\delta)}\sum_{n=2}^{\infty}a_n.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}|f(z)|&\geq a_1|z|-\sum_{n=p+1}^{\infty}a_n|z|^n\geq a_1|z|-|z|^2\sum_{n=p+1}^{\infty}a_n\\&\geq a_1|z|-\frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)}|z|^2,\end{aligned}$$

$$\sum_{n=2}^{\infty}a_n\leq\frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)}, \tag{3.3}$$

and we have

$$|f(z)|\geq a_1|z|-\frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)}|z|^2.$$

Simillary,

$$|f(z)|\leq a_1|z|+\frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\,\beta\,(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)}|z|^2.$$

The proof is complete. □

**Theorem 3.2.** *If a function $f(z)$ is in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$, then*

$$(3.4)\quad \left|J_{0,z}^{\eta,\gamma,\delta} f(z)\right| \geq \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \left\{a_1 |z|^{1-\gamma} - \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)} |z|^{2-\gamma}\right\},$$

*and*

$$(3.5)\quad \left|J_{0,z}^{\eta,\gamma,\delta} f(z)\right| \leq \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \left\{a_1 |z|^{1-\gamma} + \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)} |z|^{2-\gamma}\right\},$$

*for $z \in D_0$ where $D_0$ equals to $E$ if $\gamma \leq 1$, and $D_0$ equals to $E^*$ if $1 < \gamma < n$.*

*proof.* By using in equality (1.8) and Theorem 2.1, we obtain that

$$\left|\frac{\Gamma(3-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)} z^{\gamma} J_{0,z}^{\eta,\gamma,\delta} f(z)\right| \geq a_1|z| - \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n |z|^n$$
$$\geq a_1|z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n \geq a_1|z| - |z|^2 \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)},$$

which is equivalent to (3.4). Simillary, We obtain that

$$\left|J_{0,z}^{\eta,\gamma,\delta} f(z)\right| \leq \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \left\{a_1 |z|^{1-\gamma} + \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)} |z|^{2-\gamma}\right\}.$$

The proof is complete. □

**Corollary 3.3.** *Let a function $f(z)$ be in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$ with $\eta > \gamma$. Then, in view of Theorem 3.1, $f(z)$ is included in a disk with its center at origin and radius $r$ given by*

$$(3.6)\qquad r = a_1 + \frac{(2-\gamma)(3-\eta+\delta)}{2(3-\gamma+\delta)} \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)},$$

*and $J_{0,z}^{\eta,\gamma,\delta} f(z)$ is included in a disk with its center at origin and radius $R$ given by*

$$(3.7)\qquad R = \frac{\Gamma(3-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)} \left\{a_1 + \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)}\right\}.$$

## 4. Radius of convexity

**Theorem 4.1.** *Let $f(z)$ be in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$. Then $f(z)$ is convex in the disk*

(4.1)

$$|z| < r(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0) = \inf_{\substack{n\geq 2\\ n\in N}} \left\{ \frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n + [(1+\mu)\beta(1-\alpha)-1][(1-t)+tn]z_0^{n-1}}{n^2(1+\mu)\beta(1-\alpha)} \right\}^{\frac{1}{n-1}}$$

*The result is sharp for the function given by*

*proof.* It is sufficient to prove that $\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$ for $r(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$.

A simple calculation gives us

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{-\sum\limits_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{a_1 - \sum\limits_{n=2}^{\infty} na_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{a_1 - \sum\limits_{n=2}^{\infty} na_n|z|^{n-1}}.$$

Clearly $\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$, if

$$\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1} \leq a_1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}. \tag{4.3}$$

Using $a_1 = 1 + \sum\limits_{n=2}^{\infty}[(1-t)tn]a_1 z_0^{n-1}$ in (4.3), we are led to

$$\sum_{n=2}^{\infty}\left\{n^2|z|^{n-1} - [(1-t)+tn]z_0^{n-1}\right\}a_n \leq 1. \tag{4.4}$$

By Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\left\{\frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n - [(1-t)+tn]z_0^{n-1}}{(1+\mu)\beta(1-\alpha)}\right\}a_n \leq 1. \tag{4.5}$$

Hence (4.4) will hold, if

(4.6)

$$n^2|z|^{n-1} - [(1-t)+tn]z_0^{n-1} \leq \frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n - [(1-t)+tn]z_0^{n-1}}{(1+\mu)\beta(1-\alpha)},$$

or equivalently

$$|z|^{n-1} \leq \frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n + [(1+\mu)\beta(1-\alpha)-1][(1-t)+tn]z_0^{n-1}}{n^2(1+\mu)\beta(1-\alpha)} \tag{4.7}$$

which in turm implies the assertion of the theorem. □

## 5. Property of Hadamard product

Let the function $f_j(z)\,(j=1,2)$ defined by $f_j(z)=a_{1,j}z-\sum_{n=2}^{\infty}a_{n,j}z^n$, we define the hadamard product $f(z)*g(z)$ by

$$(f_1*f_2)(z)=a_{1,1}a_{1,2}z-\sum_{n=2}^{\infty}a_{n,1}a_{n,2}z^n. \tag{5.1}$$

**Theorem 5.1.** *Let the functions $f_j(z)(j=1,2)$ defined by (5.1) be in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$ with $\mu>\gamma$. Then $f_1(z)*f_2(z)$ is in the class $\varphi(v,\beta,\mu,\eta,\gamma,\delta,t;z_0)$ where*

(5.2)

$$v=v(\alpha,\beta,\mu,\eta,\gamma,\delta)=a_{1,1}a_{1,2}-\frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_{1,1}-\alpha)(a_{1,2}-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)}.$$

*proof.* Suppose that $f_1(z)$ and $f_2(z)$ are in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$, by using Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_{n,1}}{(1+\mu)\beta(a_{1,1}-\alpha)}\le 1, \tag{5.3}$$

and

$$\sum_{n=2}^{\infty}\frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n a_{n,2}}{(1+\mu)\beta(a_{1,2}-\alpha)}\le 1. \tag{5.4}$$

From (5.3) and (5.4), using Cauchy-Schwarze inequality, we have

$$\sum_{n=2}^{\infty}\frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n}{(1+\mu)\beta}\sqrt{\frac{a_{n,1}a_{n,2}}{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}}\le 1. \tag{5.5}$$

Hence, we find the largest $v$ such that

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n}{(1+\mu)\beta(a_{1,1}a_{1,2}-v)}a_{n,1}a_{n,2}\\ &\le\sum_{n=2}^{\infty}\frac{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n}{(1+\mu)\beta}\sqrt{\frac{a_{n,1}a_{n,2}}{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}}\le 1,\end{aligned} \tag{5.6}$$

or equivalently

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{a_{1,1}a_{1,2}-v}{\sqrt{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}}. \tag{5.7}$$

So, it is sufficient to find the largest $v$ such that

$$\frac{(1+\mu)\beta\sqrt{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}}{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n} \leq \frac{a_{1,1}a_{1,2}-v}{\sqrt{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}}. \tag{5.8}$$

Hence (5.8) yields $v \leq a_{1,1}a_{1,2} - \frac{(1+\mu)\beta(a_{1,1}-\alpha)(a_{1,2}-\alpha)}{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n}$. Since $\Psi_n(\mu,\gamma,\delta)$ is non-decreasing for $\mu \geq \gamma$, we have

$$v \leq a_{1,1}a_{1,2} - \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_{1,1}-\alpha)(a_{1,2}-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)},$$

which proves the assertion of this theorem. □

## 6. Linear combination of the function in the class $\varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$

**Theorem 6.1.** *Let* $T(\alpha,\beta,\mu,\eta,\gamma,\delta) = \frac{(1+\mu)\beta(1-\alpha)}{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n+1-a_1}$, *and let us put*

$$f_n(z) = a_1 z - T(\alpha,\beta,\mu,\eta,\gamma,\delta)\, z^n, n = 2,3,\cdots, \tag{6.1}$$

*and* $f_1(z) = a_1 z$. *Then* $f(z) \in \varphi(\alpha_j,\beta_j,\mu_j,\eta,\gamma,\delta,t_j;z_0)$ *if and only if*

$$f(z) = \sum_{n=1}^{\infty} t_n f_n(z), z \in E, \tag{6.2}$$

*where* $\sum_{n=1}^{\infty} t_n = 1, t_n \geq 0$ *for* $n = 1,2,3,\cdots$.

*proof.* Let $f(z) \in \varphi(\alpha,\beta,\mu,\eta,\gamma,\delta,t;z_0)$. Then, by Corollary 2.2, $|a_n| \leq T(\alpha,\beta,\mu,\eta,\gamma,\delta)$.

Let us put

$$t_n = T(\alpha,\beta,\mu,\eta,\gamma,\delta)^{-1} a_n, n = 2,3,\cdots, \tag{6.3}$$

and $t_1 = 1 - \sum_{n=2}^{\infty} t_n$. By assumption, we have $t_n \geq 0, n = 2,3,\cdots$, and $t_1 \geq 0$. Thus

$$\begin{aligned}\sum_{n=1}^{\infty} t_n f_n(z) &= t_1 f_1(z) + \sum_{n=2}^{\infty} t_n f_n(z)\\ &= \left(1 - \sum_{n=2}^{\infty} t_n\right) a_1 z + \sum_{n=2}^{\infty} t_n \{a_1 z - T(\alpha,\beta,\mu,\eta,\gamma,\delta)\, z^n\}\\ &= a_1 z - \sum_{n=2}^{\infty} a_n z^n = f(z).\end{aligned} \tag{6.4}$$

Conversely, Let us function $f(z)$ satisfy (6.2). Since

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} t_n f_n(z) = t_1 f_1(z) + \sum_{n=2}^{\infty} t_n f_n(z) \\ &= \left(1 - \sum_{n=2}^{\infty} t_n\right) a_1 z + \sum_{n=2}^{\infty} t_n \left\{a_1 z - T(\alpha,\beta,\mu,\eta,\gamma,\delta) z^n\right\}, \qquad (6.5) \\ &= a_1 z - \sum_{n=2}^{\infty} t_n T(\alpha,\beta,\mu,\eta,\gamma,\delta) z^n = a_1 z - \sum_{n=2}^{\infty} a_n z^n \end{aligned}$$

which proves the assertion of this theorem. □

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