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ON SOME PROPERTIES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let \( p(z) \) be analytic in \(|z| < 1\), \( p(0) = 1 \), \( p(z) \neq 0 \) in \(|z| < 1\) and suppose that \(|p(z)|\) takes its maximum and minimum value at points \( z = z_0 \) and \( z = z_1 \) respectively on the closed disc \(|z| = |z_0| = |z_1| \leq r (0 < r < 1)\). Then we have
\[
\frac{z_0 p'(z_0)}{p(z_0)} = m \geq 0
\]
and
\[
\frac{z_1 p'(z_1)}{p(z_1)} = n \leq 0.
\]

1. INTRODUCTION.

In [2], Jack proved the following theorem.

**Theorem A.** Let \( w(z) \) be analytic in \( \mathbb{E} = \{z : |z| < 1\} \) and suppose that \( w(0) = 0 \). If \(|w(z)|\) takes its maximum value on the circle \(|z| = r < 1\) at a point \( z = z_0 \), then we have
\[
\frac{z_0 w'(z_0)}{w(z_0)} = k
\]
where \( k \) is real number and \( k \geq 1 \).


Furthermore, Miller and Mocanu [3] generalized Theorem A and obtained the following theorem.

**Theorem B.** Let \( w(z) = \sum_{k=n}^{\infty} a_k z^k \) be analytic in \( \mathbb{E}, n \in \mathbb{N}, w(z) \neq 0 \). If \( z_0 = r_0 e^{i\theta_0} (0 < r_0 < 1) \) and
\[
|w(z_0)| = \max_{|z_0| \leq r_0} |w(z)|
\]
then
\[
\frac{z_0 w'(z_0)}{w(z_0)} = m
\]
and
\[
1 + \Re \frac{z_0 w''(z_0)}{w'(z_0)} \geq m
\]
where \( 1 \leq n \leq m \).

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It is a purpose of the present paper to obtain some similar but a little different results from Theorem A and B.

2. MAIN RESULT.

Theorem. Let \( p(z) \) be analytic in \( \mathbb{E} \), \( p(0) = 1 \), \( p(z) \neq 0 \) in \( \mathbb{E} \) and suppose that

\[
\max_{|z| \leq r} |p(z)| = |p(z_0)|
\]

and

\[
\min_{|z| \leq r} |p(z)| = |p(z_1)|
\]

where \( 0 < r < 1 \) and \( |z_0| = |z_1| = r \). Then we have

\[
\frac{z_0p'(z_0)}{p(z_0)} = m \geq 0,
\]

\[
\frac{z_1p'(z_1)}{p(z_1)} = n \leq 0,
\]

\[
1 + \Re \frac{z_0p''(z_0)}{p'(z_0)} \geq m
\]

and

\[
1 + \Re \frac{z_1p''(z_1)}{p'(z_1)} \geq n
\]

where \( m \) and \( n \) are real, \( 0 \leq m \) and \( n \leq 0 \).

Proof. Let us put

\[
g(z) = \left( \frac{R+1}{R-1} \right) \left( \frac{R-p(z)}{R+p(z)} \right) = A \left( \frac{R-p(z)}{R+p(z)} \right), \quad g(0) = 1
\]

where \( R = |p(z_0)| \) and \( A = (R+1)/(R-1) \). Then we have

\[
p(z) = R \left( \frac{A-g(z)}{A+g(z)} \right).
\]

When \( |p(z)| \) takes its maximum value at a point \( z = z_0 \), then we have \( \Re g(z) > 0 \) for \( |z| < |z_0| \) and \( \Re g(z_0) = 0 \). Putting \( |z| = |z_0|, z = |z_0|e^{i\theta} \) and \( 0 \leq \theta \leq 2\pi \), we have

\[
\frac{zp'(z)}{p(z)} = \frac{d\arg p(z)}{d\theta} - i \frac{d\log |p(z)|}{d\theta} = -\frac{zg'(z)}{A - g(z)} - \frac{zg'(z)}{A + g(z)} = -\frac{2zg'(z)}{A^2 - g(z)^2}.
\]
Therefore we have
\[
\frac{z_0 p'(z_0)}{p(z_0)} = -\frac{2z_0 g'(z_0)}{A^2 - g(z_0)^2} = \left(\frac{d \arg p(z)}{d\theta}\right)_{z=z_0} - i \left(\frac{1}{|p(z)|}\right) \left(\frac{d|p(z)|}{d\theta}\right)_{z=z_0}
\]
\[
= \left(\frac{d \arg p(z)}{d\theta}\right)_{z=z_0} = m \geq 0
\]
where \(m\) is a real and \(0 \leq m\). By logarithmic differentiation of (1), we have
\[
1 + \frac{zp''(z)}{p'(z)} = \frac{zp'(z)}{p(z)} + 1 + \frac{zg'(z)}{g(z)} - \frac{2zg'(z)g(z)}{A^2 - g(z)^2}.
\]
Then we have
\[
1 + \text{Re} \frac{z_0 p''(z_0)}{p'(z_0)} = \text{Re} \frac{z_0 p'(z_0)}{p(z_0)} + 1 + \text{Re} \frac{z_0 g''(z_0)}{g'(z_0)} = \text{Re} \frac{2z_0 g'(z_0)g(z_0)}{A^2 - g(z_0)^2}
\]
\[
= m + 1 + \text{Re} \frac{z_0 g''(z_0)}{g'(z_0)} = m + 1 + \text{Re} \frac{z_0 g'(z_0)}{g'(z_0)}
\]
From the hypothesis, we have
\[
\text{Re} g(z) > 0 \quad \text{for} \quad |z| < |z_0|
\]
and
\[
\text{Re} g(z_0) = 0,
\]
then from the geometrical property of \(g(z)\), we have
\[
1 + \text{Re} \frac{z_0 g''(z_0)}{g'(z_0)} \geq 0.
\]
This shows that
\[
1 + \text{Re} \frac{z_0 p''(z_0)}{p'(z_0)} \geq m.
\]
On the other hand, let us put
\[
h(z) = \left(\frac{p(z) - l}{p(z) + l}\right) \left(\frac{1 + l}{1 - l}\right), \quad h(0) = 1
\]
where \(0 < l = \min_{|z| \leq |z_1|} |p(z)| < 1\), then we have
\[
p(z) = l \left(\frac{B + h(z)}{B - h(z)}\right)
\]
where \(B = (1 + l)/(1 - l)\). From the hypothesis of the theorem, we have
\[
\text{Re} h(z) > 0 \quad \text{for} \quad |z| < |z_1|
\]
and
\[
\text{Re} h(z_1) = 0.
\]
From (2), we have

\[
\frac{zp'(z)}{p(z)} = \frac{2zh'(z)}{B^2 - h(z)^2}.
\]

(3)

By the same reason as the above, we have on the circle \(|z| = |z_1|e^{i\theta}\) and \(0 \leq \theta \leq 2\pi\)

\[
\left( \frac{d|p(z)|}{d\theta} \right)_{z=z_1} = 0
\]

and from the geometrical property, we have

\[
\left( \frac{d\arg p(z)}{d\theta} \right)_{z=z_1} \leq 0.
\]

This shows that

\[
\frac{z_1p'(z_1)}{p(z_1)} = \text{Re} \frac{z_1p'(z_1)}{p(z_1)} = n \leq 0.
\]

where \(n\) is a real number. From (3), we have

\[
1 + \frac{zp''(z)}{p'(z)} = \frac{zp'(z)}{p(z)} + 1 + \frac{zh''(z)}{h'(z)} + \frac{2zh'(z)h(z)}{B^2 - h(z)^2}
\]

and therefore we have

\[
1 + \text{Re} \frac{z_1p''(z_1)}{p'(z_1)} = n + 1 + \text{Re} \frac{z_1h''(z_1)}{h'(z_1)} - \text{Re} \frac{2z_1h'(z_1)h(z_1)}{B^2 - h(z_1)^2}
\]

\[
= n + 1 + \text{Re} \frac{z_1h''(z_1)}{h'(z_1)} - n\text{Re} h(z_1)
\]

\[
= n + 1 + \text{Re} \frac{z_1h''(z_1)}{h'(z_1)}.
\]

Applying the same reason as the above, we have

\[
1 + \text{Re} \frac{z_1h''(z_1)}{h'(z_1)} \geq 0
\]

and this shows that

\[
1 + \text{Re} \frac{z_1p''(z_1)}{p'(z_1)} \geq n
\]

where \(n\) is a real number and \(n \leq 0\). This completes the proof. \(\Box\)

REFERENCES


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