On a strongly starlikeness criteria

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Abstract

H. Silverman [Internat. J. Math. Math. Sci. 22(1999), 75-79] investigated and obtained some results for the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. In this paper, we obtain a sufficient condition of functions for strongly starlikeness of order $\beta$.

1 Introduction

Let $S$ denote the class of functions $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$ that are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in S$ is said to be starlike of order $\alpha$ if and only if

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha$$

for some $0 \leq \alpha < 1$, and for all $z \in U$. The class of starlike functions of order $\alpha$ is denoted by $S^*(\alpha)$. Further, a function $f(z) \in S$ is said to be convex of order $\alpha$ if and only if

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha$$

for some $0 \leq \alpha < 1$, and for all $z \in U$. Also we denote by $C(\alpha)$ the subclass of $S$ consisting of all convex functions of order $\alpha$ in $U$.

On the other hand, a function $f(z)$ in $S$ is said to be strongly starlike of order $\beta$ if it satisfies

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \beta$$

for some $0 < \beta \leq 1$, and for all $z \in U$. We say that $f(z) \in SS^*(\beta)$ if $f(z)$ is strongly starlike of order $\beta$ in $U$.

Silverman [2] investigated the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. Let $G_\beta$ be the subclass of $S$ consisting of functions $f(z) \in S$ which satisfy

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\[ \left| \left\{ \frac{1 + zf''(z)}{f'(z)} \right\} - 1 \right| < b \quad (z \in \mathbb{U}) \]

for some real \( b \).

For this class \( \mathcal{G}_b \), Silverman obtained the following result.

**Theorem A ([2]).** If \( 0 < b \leq 1 \), then

\[ \mathcal{G}_b \subset S^* \left( \frac{2}{1 + \sqrt{1 + 8b}} \right). \]

The result is sharp for all \( b \).

In this paper, we consider the strongly starlikeness for functions \( f(z) \) belonging to \( \mathcal{G}_b \).

## 2 Strongly starlikeness

To discuss the strongly starlikeness of functions \( f(z) \) in \( \mathcal{G}_b \), we have to recall here the following result by Nunokawa [1].

**Lemma.** Let \( p(z) \) be analytic in \( \mathbb{U} \) with \( p(0) = 1 \) and \( p(z) \neq 0 \) (\( z \in \mathbb{U} \)). Suppose that there exists a point \( z_0 \in \mathbb{U} \) such that

\[ |\arg(p(z))| < \frac{\pi}{2} \beta \quad (|z| < |z_0|) \]

and

\[ |\arg(p(z_0))| = \frac{\pi}{2} \beta, \]

where \( \beta > 0 \). Then we have

\[ \frac{z_0 p'(z_0)}{p(z_0)} = iK\beta, \]

where

\[ K \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = \frac{\pi}{2} \beta \]

and

\[ K \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = -\frac{\pi}{2} \beta, \]

where \( p(z_0)^{\frac{1}{2}} = \pm ia \) and \( a > 0 \).

Our main result is contained in

**Theorem 1.** If \( f(z) \) belongs to the class \( \mathcal{G}_{b(\beta)} \) with

\[ b(\beta) = \frac{\beta}{\sqrt{(1 - \beta)^{1-\beta}(1 + \beta)^{1+\beta}}} \quad (0 < \beta \leq 1), \]
then $f(z) \in SS^{*}(\beta)$.

**Proof.** Let us define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)}.$$ 

Then it follows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)^2}.$$

If there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\beta$$

for $|z| < |z_0|$ and

$$|\arg(p(z_0))| = \frac{\pi}{2}\beta,$$

then, applying Lemma, we have that

$$\left| \frac{z_0p'(z_0)}{p(z_0)^2} \right| = \left| iK\beta \frac{1}{(\pm ia)^\beta} \right| = \beta |K|a^{-\beta} \geq \frac{\beta}{2} \left( a^{1-\beta} + \frac{1}{a^{1+\beta}} \right).$$

Define the function $g(a)$ by

$$g(a) = a^{1-\beta} + \frac{1}{a^{1+\beta}} \quad (a > 0; 0 < \beta \leq 1).$$

Since

$$g'(a) = \frac{1}{a^{2+\beta}}((1-\beta)a^2 - (1+\beta)),$$

$g(a)$ takes its minimum value at $a = \sqrt{\frac{1+\beta}{1-\beta}}$. This implies that

$$\left| \frac{z_0p'(z_0)}{p(z_0)^2} \right| \geq \frac{\beta}{2} g \left( \sqrt{\frac{1+\beta}{1-\beta}} \right)$$

$$= \frac{\beta}{2} \left\{ \left( \frac{1+\beta}{1-\beta} \right)^{\frac{1-\beta}{2}} + \left( \frac{1-\beta}{1+\beta} \right)^{\frac{1+\beta}{2}} \right\}$$

$$= \frac{\beta}{\sqrt{(1-\beta)^{1-\beta}(1+\beta)^{1+\beta}}},$$

which contradicts our condition $f(z) \in G_{b(\beta)}$ of the theorem. Thus we complete the proof of the theorem.

Considering the case of $\beta = 1$ in the proof of Theorem 1, we have
Corollary 1. If \( f(z) \in \mathcal{G}_b \) with \( b = \frac{1}{2} \), then \( f(z) \in SS^*(1) \), or \( f(z) \) is strongly starlike in \( U \).

Taking \( \beta = \frac{1}{2} \) in Theorem 1, we have

Corollary 2. If \( f(z) \in \mathcal{G}_b \) with \( b = \frac{1}{\sqrt{3\sqrt{3}}} = 0.438691 \cdots \), then \( f(z) \in SS^*(\frac{1}{2}) \).

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