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Univalence of certain integral operators

Virgil Pescar and Shigeyoshi Owa

Abstract

Let $A_n$ be the class of functions $f(z)$ which are analytic and $n$-fold symmetric in the open unit disk $U$. The integral operator $G_\alpha(z)$ for $f(z) \in A_n$ is considered. The object of the present paper is to derive univalence conditions of the integral operator $G_\alpha(z)$ for $f(z) \in A_n$.

1 Introduction

Let $A_n$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{nk} z^{nk} \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic and $n$-fold symmetric in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $S_n$ the subclass of $A_n$ consisting of functions $f(z)$ which are univalent in $U$. Many authors studied the problem of integral operators for functions $f(z)$ in the class $S_1$. In this sense, the following useful result is due to Pfaltzgraff [3].

Theorem 1.1. If $f(z)$ is univalent in $U$ and $\alpha$ is complex number with $|\alpha| \leq \frac{1}{4}$, then the integral operator $G_\alpha(z)$ given by

$$G_\alpha(z) = \int_0^z (f'(t))^\alpha \, dt \quad (1)$$

is also univalent in $U$.

Further, Pascu and Pescar [2] gave

Theorem 1.2. If $f(z) \in S_1$ and $\alpha$ is a complex number with $|\alpha| \leq \frac{1}{4n}$, then the integral operator $G_{\alpha,n}(z)$ given by

$$G_{\alpha,n}(z) = \int_0^z (f'(t))^\alpha \, dt$$

is also in the class $S_1$ for all positive integer $n$.

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2 Properties of integral operators

To discuss our problems for integral operators, we need to recall here the following lemma due to Becker [1].

**Lemma 2.1.** If \( f(z) \in A_1 \) satisfies

\[
(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (z \in U),
\]

then \( f(z) \in S_1 \).

Applying the above lemma, we derive

**Theorem 2.1.** If \( f(z) \in A_1 \) satisfies the inequality (2) for all \( z \in U \), then the integral operator \( G_\alpha(z) \) defined by (1) belongs to the class \( S_1 \) for all \( \alpha (|\alpha| \leq 1) \).

**Proof.** Note that \( G_\alpha(z) \in A_1 \) for \( f(z) \in A_1 \) and that

\[
\frac{zf''(z)}{f'(z)} = \frac{1}{\alpha} \frac{zG_\alpha''(z)}{G_\alpha'(z)}. \]

It follows that

\[
(1 - |z|^2) \left| \frac{zG_\alpha''(z)}{G_\alpha'(z)} \right| = |\alpha|(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq |\alpha| \leq 1
\]

for \( z \in U \). Thus, using Lemma 2.1, we have \( G_\alpha(z) \in S_1 \).

Next, we prove

**Corollary 2.1.** If \( f(z) \in A_1 \) satisfies

\[
\left| \frac{f''(z)}{f'(z)} \right| \leq 1 \quad (z \in U),
\]

then the integral operator \( G_\alpha(z) \) defined by (1) is in the class \( S_1 \) with \( |\alpha| \leq \frac{3\sqrt{3}}{2} \).

**Proof.** In view of the proof of Theorem 2.1, we see that

\[
(1 - |z|^2) \left| \frac{zG_\alpha''(z)}{G_\alpha'(z)} \right| \leq |\alpha|(1 - |z|^2)|z| \leq 1,
\]

because \( |\alpha| \leq \frac{3\sqrt{3}}{2} \) and

\[
\max_{|\alpha| \leq 1} (1 - |z|^2)|z| = \frac{2}{3\sqrt{3}}.
\]

Thus, by Lemma 2.1, we prove that \( G_\alpha(z) \in S_1 \).

Finally, we show
Theorem 2.2. If \( f(z) \in \mathcal{A}_n \) satisfies
\[
\left| \frac{f''(z)}{f'(z)} \right| \leq |z|^n - 1 \quad (z \in \mathbb{U}),
\]
then the integral operator \( G_\alpha(z) \) defined by (1) belongs to the class \( S_n \) with
\[
|\alpha| \leq \frac{(n + 2)^{n+1}}{2n^3}.
\]

Proof. Since
\[
\frac{zf''(z)}{f'(z)} = \frac{1}{\alpha} \frac{zG''_\alpha(z)}{G'_\alpha(z)} = n(n+1)a_{n+1}z^n + \cdots,
\]
we have that
\[
(1 - |z|^2) \left| \frac{zG''_\alpha(z)}{G'_\alpha(z)} \right| = |\alpha|(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq |\alpha|(1 - |z|^2)|z|^n \quad (z \in \mathbb{U}).
\]

Note that
\[
|\alpha| \leq \frac{(n + 2)^{n+1}}{2n^3}
\]
and
\[
(1 - |z|^2)|z|^n \leq \frac{2n^3}{(n + 2)^{n+1}} \quad (z \in \mathbb{U}).
\]

This gives us that
\[
(1 - |z|^2) \left| \frac{zG''_\alpha(z)}{G'_\alpha(z)} \right| \leq 1 \quad (z \in \mathbb{U}).
\]

Further, it is easy to see that \( G_\alpha(z) \in \mathcal{A}_n \). This completes the proof of the theorem.

Remark. For \( n = 1 \), Theorem 2.2 becomes Theorem 2.1.

References


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