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<td>Pescar, Virgil; Owa, Shigeyoshi</td>
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Univalence of certain integral operators

Virgil Pescar and Shigeyoshi Owa

Abstract

Let \( A_n \) be the class of functions \( f(z) \) which are analytic and \( n \)-fold symmetric in the open unit disk \( U \). The integral operator \( G_\alpha(z) \) for \( f(z) \in A_n \) is considered. The object of the present paper is to derive univalence conditions of the integral operator \( G_\alpha(z) \) for \( f(z) \in A_n \).

1 Introduction

Let \( A_n \) denote the class of functions \( f(z) \) of the form

\[
 f(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1} \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots \})
\]

which are analytic and \( n \)-fold symmetric in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). We denote by \( S_n \) the subclass of \( A_n \) consisting of functions \( f(z) \) which are univalent in \( U \). Many authors studied the problem of integral operators for functions \( f(z) \) in the class \( S_1 \). In this sense, the following useful result is due to Pfaltzgraff [3].

**Theorem 1.1.** If \( f(z) \) is univalent in \( U \) and \( \alpha \) is complex number with \( |\alpha| \leq \frac{1}{4} \), then the integral operator \( G_\alpha(z) \) given by

\[
 G_\alpha(z) = \int_0^z (f'(t))^\alpha dt
\]

is also univalent in \( U \).

Further, Pascu and Pescar [2] gave

**Theorem 1.2.** If \( f(z) \in S_1 \) and \( \alpha \) is a complex number with \( |\alpha| \leq \frac{1}{4n} \), then the integral operator \( G_{\alpha,n}(z) \) given by

\[
 G_{\alpha,n}(z) = \int_0^z (f'(t))^\alpha dt
\]

is also in the class \( S_1 \) for all positive integer \( n \).

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2 Properties of integral operators

To discuss our problems for integral operators, we need to recall here the following lemma due to Becker [1].

Lemma 2.1. If \( f(z) \in A_1 \) satisfies

\[
(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (z \in \mathbb{U}),
\]
then \( f(z) \in S_1 \).

Applying the above lemma, we derive

Theorem 2.1. If \( f(z) \in A_1 \) satisfies the inequality (2) for all \( z \in \mathbb{U} \), then the integral operator \( G_\alpha(z) \) defined by (1) belongs to the class \( S_1 \) for all \( \alpha (|\alpha| \leq 1) \).

Proof. Note that \( G_\alpha(z) \in A_1 \) for \( f(z) \in A_1 \) and that

\[
\frac{zf''(z)}{f'(z)} = \frac{1}{\alpha} \frac{zG''_\alpha(z)}{G'_\alpha(z)}.
\]

It follows that

\[
(1 - |z|^2) \left| \frac{zG''_\alpha(z)}{G'_\alpha(z)} \right| = |\alpha|(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq |\alpha| \leq 1
\]
for \( z \in \mathbb{U} \). Thus, using Lemma 2.1, we have \( G_\alpha(z) \in S_1 \).

Next, we prove

Corollary 2.1. If \( f(z) \in A_1 \) satisfies

\[
\left| \frac{f''(z)}{f'(z)} \right| \leq 1 \quad (z \in \mathbb{U}),
\]
then the integral operator \( G_\alpha(z) \) defined by (1) is in the class \( S_1 \) with \( |\alpha| \leq \frac{3\sqrt{3}}{2} \).

Proof. In view of the proof of Theorem 2.1, we see that

\[
(1 - |z|^2) \left| \frac{zG''_\alpha(z)}{G'_\alpha(z)} \right| \leq |\alpha|(1 - |z|^2)|z| \leq 1,
\]
because \( |\alpha| \leq \frac{3\sqrt{3}}{2} \) and

\[
\max_{|\iota| \leq 1} (1 - |z|^2)|z| = \frac{2}{3\sqrt{3}}.
\]
Thus, by Lemma 2.1, we prove that \( G_\alpha(z) \in S_1 \).

Finally, we show
Theorem 2.2. If \( f(z) \in A_n \) satisfies
\[
\left| \frac{f''(z)}{f'(z)} \right| \leq |z|^{n-1} \quad (z \in \mathbb{U}),
\]
then the integral operator \( G_\alpha(z) \) defined by (1) belongs to the class \( S_n \) with
\[
|\alpha| \leq \frac{(n+2)^{n+1}}{2n^2}.
\]

Proof. Since
\[
\frac{zf''(z)}{f'(z)} = \frac{1}{\alpha} \frac{zG''_\alpha(z)}{G'_\alpha(z)} = n(n+1)a_{n+1}z^n + \cdots,
\]
we have that
\[
(1 - |z|^2) \left| \frac{zG''_\alpha(z)}{G'_\alpha(z)} \right| = |\alpha|(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right|
\]
\[
\leq |\alpha|(1 - |z|^2)|z|^n \quad (z \in \mathbb{U}).
\]
Note that
\[
|\alpha| \leq \frac{(n+2)^{n+1}}{2n^2}
\]
and
\[
(1 - |z|^2)|z|^n \leq \frac{2n^2}{(n+2)^{n+1}} \quad (z \in \mathbb{U}).
\]
This gives us that
\[
(1 - |z|^2) \left| \frac{zG''_\alpha(z)}{G'_\alpha(z)} \right| \leq 1 \quad (z \in \mathbb{U}).
\]
Further, it is easy to see that \( G_\alpha(z) \in A_n \). This completes the proof of the theorem.

Remark. For \( n = 1 \), Theorem 2.2 becomes Theorem 2.1.

References


V. Pescar
Department of Mathematics
Transilvania University of Brasov
2200 Brasov
Romania

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka, 577-8502
Japan