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<th>Integral means of holomorphic mappings in $C^n$ (Inequalities in Univalent Function Theory and Its Applications)</th>
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<tr>
<td>Author(s)</td>
<td>Tsurumi, Kazuyuki; Sekine, Tadayuki</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2002, 1276: 102-108</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42309">http://hdl.handle.net/2433/42309</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Integral means of holomorphic mappings in $C^n$

Kazuyuki Tsurumi  
(東京電機大工 鶴見和之)

Tadayuki Sekine  
(日本大学薬 関根忠行)

Let $f(z)$ and $g(z)$ be holomorphic functions in the unit disk $U$ with $f(0)=g(0)=0$.

The function $g(z)$ is said to be subordinate to the function $f(z)$ if there exists a function $\psi(z)$ holomorphic in $U$ such that $|\psi(z)| \leq |z|$ for $z \in U$ and $g(z) = f(\psi(z))$. Let $g(z)$ be a holomorphic function in $U$. For $0 < p < \infty$ and $0 \leq r < 1$, let us put

$$M_p(r, \varphi) := \left( \frac{1}{2\pi} \int_0^{2\pi} |\varphi(xe^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

Then we have the theorem

**Theorem A** (The subordination theorem of Littlewood [2], p.191)

Suppose that $f$ and $g$ are holomorphic in $U$ and $f(0)=g(0)=0$ and that $g$ is subordinate to $f$, then we have

$$M_p(r, g) \leq M_p(r, f) \quad (0 < p < \infty, \ 0 \leq r < 1)$$

The purpose of this note is to extend this theorem to the case of $C^n$.

§ 1. Preliminaries

Let us denote a point $z$ of the space $C^n$ by the column vector
$z:= \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

and $z^*$ denotes the conjugate transposed vector of $z$. The norm of $z$ is denoted by

$$|z| := \sqrt{z^*z}.$$ Denoted by $B_n(r, z_0)$ the ball in $C^n$ with radius $r$ and center $z_0$, i.e,

$$B_n(r, z_0) := \{ z \in C^n \mid |z - z_0| < r \},$$

and let $B := B_n(1, 0)$, $S(r) := \partial B_n(r, 0)$ (the boundary of $B_n(r, 0)$) and $S := S(r)$.

For $z \in C^n$, we will use the following differential forms and operators for the proof of our theorem:

$$dz = \begin{pmatrix} dz_1 \\ \vdots \\ dz_n \end{pmatrix}, \quad d\bar{z} = \begin{pmatrix} d\bar{z}_1 \\ \vdots \\ d\bar{z}_n \end{pmatrix}.$$  

$$\omega(z) := dz_1 \wedge \cdots \wedge dz_n \quad (n\text{-times})$$

$$\eta(z) := \sum_{j=1}^{n} (-1)^{j+1} z_j dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_n \quad (\text{Leray form})$$

$$d\sigma(z) := \frac{1}{(2i)^n} \{ \eta(z) \wedge \omega(z) + (-1)^{q(n)} \omega(z) \wedge \eta(z) \}. \quad (q(n) = \frac{n(n-1)}{2})$$

Let $z = r\zeta$, $r := |z|$ and $|\zeta| = 1$. Then we have

$$\sum_{j=1}^{n} c_j d\bar{z}_j + \cdots + c_n d\bar{z}_n + \zeta_1 d\bar{z}_1 + \cdots + \zeta_n d\bar{z}_n = 0.$$  

Thus $d\sigma(z) = r^{2n-1} d\sigma(\zeta)$, and the form $dS(\zeta) := \frac{1}{\nu(S)} d\sigma(\zeta)$ is the normalized rotation invariant surface measure on $S$, where

$$\nu(S) = \frac{2\pi^n}{(n-1)!} \quad (\text{the area of } S).$$

$$dv(z) := \frac{1}{(2i)^n} \omega(z) \wedge \omega(z) \quad (\text{the volume element of } C^n).$$

Then we get

$$dv(z) := \frac{1}{(2i)^n} \{ \eta(\zeta) \wedge \omega(\zeta) + (-1)^{q(n)} \omega(\zeta) \wedge \eta(\zeta) \} \wedge r^{2n-1} dr.$$
Let us set
\[
\frac{\partial}{\partial z} := (\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}), \quad \frac{\partial}{\partial \zeta} := \left( \frac{\partial}{\partial \zeta_1}, \ldots, \frac{\partial}{\partial \zeta_n} \right)
\]

\[
\Delta := 4 \frac{\partial^2}{\partial \overline{z} \partial z} = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial \overline{z_j} \partial z_j}
\]

\[
H := \frac{\partial^2}{\partial \overline{z} \partial z} = \begin{pmatrix}
\frac{\partial^2}{\partial \overline{z}_1 \partial z_1} & \cdots & \frac{\partial^2}{\partial \overline{z}_1 \partial z_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2}{\partial \overline{z}_n \partial z_1} & \cdots & \frac{\partial^2}{\partial \overline{z}_n \partial z_n}
\end{pmatrix}
\]

Let \( u(z) \) be a real function in a domain \( D \) in \( \mathbb{C}^n \). The function \( u(z) \) is said to be subharmonic in \( D \) if the following three conditions hold:

1. \(-\infty \leq u(z) < \infty\)
2. \( u(z) \) is upper semicontinuous in \( D \)
3. For any point \( z_0 \in D \), we can take an \( r > 0 \) such that

\[
B(r,z_0) \subset D \quad \text{and} \quad u(z_0) \leq \frac{1}{V(S)} \int_{S(r,z_0)} u(\zeta) d\sigma(\zeta)
\]

Suppose that \( u(z) \) is a function of the class \( C^2 \) in \( D \). Then

\[ u(z) \text{ is } \begin{cases} \text{subharmonic} \\ \text{plurisubharmonic} \end{cases} \]

\[ \Downarrow \uparrow \]

\[ \Delta u = 4 \frac{\partial^2 u}{\partial z \partial \overline{z}} \geq 0 \]

\[ H(u) = \frac{\partial^2 u}{\partial \overline{z} \partial z} \text{ (complex hessian) is positive definite.} \]

Thus the plurisubharmonic function is subharmonic.
The mapping function \( f(z) \) from a domain in \( C^n \) to \( C^n \) is denoted by the column vector
\[
\begin{pmatrix}
  f_1(z) \\
  \vdots \\
  f_n(z)
\end{pmatrix}.
\]

The mapping \( f(z) \) is said to be holomorphic if each component functions \( f_j(z) \) \((j = 1 \cdots n)\) are holomorphic. Let \( H_{n,m} \) be the family of holomorphic mappings from \( B_n \) to \( C_m \) and suppose that \( f(z) \) and \( g(z) \) are belonging to \( H_{n,m} \) and that \( f(0) = g(0) = 0 \).

The mapping \( g(z) \) is said to be subordinate to \( f(z) \) if there exists a holomorphic mapping \( \Psi(z) \) from \( B_n \) to \( B_n \) such that \( |\Psi(z)| \leq |z| \) \((z \in B_n)\) and \( g(z) = f(\Psi(z)) \).

For a mapping \( f(z) \in H_{n,m} \), we set
\[
M_p(r, f) := \left( \frac{1}{v(S)} \int_{S} |f(z)|^p d\sigma(z) \right)^{\frac{1}{p}} \quad (0 < r < 1, \ 0 < p < \infty).
\]

§ 2. Theorems for subharmonic functions

For the ball \( B_n(z_0, r) \subset C^n \), let us put
\[
K(\xi, z) := \frac{1}{v(S)} \frac{r^2 - |z - z_0|^2}{r^2 - |\xi - z_0|^2}
\]

Then the following theorems hold:

Theorem B ([3], p.32, Theorem 1.16)

Let \( \varphi(z) \) be a continuous function on \( S(r, z_0) \). Let us put
\[
u(z) := \int_{S(r, z_0)} K(\xi, z) \varphi(\xi) d\sigma(\xi) \quad (z \in B(r, z_0))
\]

Then the function \( \nu(z) \) is the solution of the problem of Dirichlet for \( B(r, z_0) \) with the boundary value \( \varphi(z) \).
Theorem C ([3], p.52, Theorem 2.7)

Let \( \varphi(z) \) be a subharmonic function on a domain \( D \) in \( C_n \) and \( u(z) \) is not equal to \(-\infty\). Suppose that \( B(z_0, r) \subset D \). Let us put

\[
V(z) = \chi_{B(r,t_0)}(z) \int_{S(r,t_0)} K(\xi, z) \varphi(\xi) d\sigma(\xi) + \chi_{D-B(r,t_0)}(z) u(z)
\]

where \( \chi_A \) denotes the characteristic function for \( A \). Then \( V(x) \) is subharmonic in \( D \) and harmonic in \( B(\gamma, \tilde{r}) \), and we have \( u(x) \leq V(x) \) in \( B(r,z_0) \).

Main Theorem. Let \( \varphi(z) \) be a subharmonic function in \( B \) and let \( \psi(z) \) be a holomorphic mapping from \( B \) to \( B \) such that \( |\psi(z)| \leq |z| \). Then we have

\[
\int_{S, d(r,0)} \varphi(\psi(z)) d\sigma(z) \leq \int_{S, d(r,0)} \varphi(z) d\sigma(z)
\]

As the corollary of our theorem, we obtain the Subordination Theorem of Littlewood for \( C^n \).

Corollary 1. Let \( f \) and \( g \) be holomorphic mappings of \( H_{n,m} \) such that \( f(0) = g(0) = 0 \). Suppose that \( g \) is subordinate to \( f \), then we have

\[
M_p(r, g) \leq M_p(r, f) \quad (0 < p < \infty, \ 0 \leq r < 1)
\]

From Corollary 1, we get the following:

Corollary 2. Let the functions \( f(z) \) and \( g(z) \) be the same as in Corollary 1, then we have

\[
\int_{B_n} \|g(z)\|^p dv(z) \leq K \int_{B_n} \|f(z)\|^p dv(z)
\]

where \( K \) is a constant.
References


