Properties of certain p-valently convex functions

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Abstract

A subclass $C_p(\lambda, \mu) (p \in \mathbb{N}, 0 < \lambda < 1, -\lambda \leq \mu < 1)$ of p-valently convex functions in the open unit disk $\mathbb{U}$ is introduced. The object of the present paper is to discuss some interesting properties of functions belonging to the class $C_p(\lambda, \mu)$.

1 Introduction

Let $A_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \cdots \})$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in $A_p$ is said to be p-valently convex of order $\alpha$ if it satisfies

$$\Re \left\{1 + \frac{zf''(z)}{f'(z)}\right\} > p\alpha \quad (z \in \mathbb{U})$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $K_p(\alpha)$ the subclass of $A_p$ consisting of functions which are p-valently convex of order $\alpha$ in $\mathbb{U}$. In particular, we denote by $K_1(0) = K$.

A function $f(z) \in A_1$ is said to be uniformly convex in $\mathbb{U}$ if $f(z)$ is in the class $K$ and has the property that the image arc $f(\gamma)$ is convex for every circular arc $\gamma$ contained in $\mathbb{U}$ with center at $t \in \mathbb{U}$. We also denote by $UK$ the subclass of $A_1$ consisting of all uniformly convex functions in $\mathbb{U}$. Goodman [2] has introduced the class $UK$ and given that $f(z) \in A_1$ belongs to the class $UK$ if and only if

$$\Re \left\{1 + (z - t)\frac{f''(z)}{f'(z)}\right\} \geq 0 \quad ((z, t) \in \mathbb{U} \times \mathbb{U}).$$

Ma and Minda [3] and Rønning [5] have showed a more applicable characterization for $UK$. We state this as

**Theorem A.** Let $f(z) \in A_1$. Then $f(z) \in UK$ if and only if

$$\Re \left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{U})$$

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In view of Theorem A, Owa [4] considered a subclass $\mathcal{UK}(\mu)(-1 < \mu < 1)$ of $A_1$. A function $f(z) \in A_1$ is said to be a member of the class $\mathcal{UK}(\mu)(-1 < \mu < 1)$ if and only if
\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} - \mu > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}).
\]

In the present paper we investigate the following subclass of $A_p$.

**Definition.** A function $f(z) \in A_p$ is said to be a member of the class $C_p(\lambda, \mu)$ if
\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} - p\mu > \lambda \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \quad (z \in \mathbb{U})
\]
for some $\lambda (0 < \lambda < 1)$ and $\mu (-\lambda \leq \mu < 1)$.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| \leq |z|$ and $f(z) = g(w(z))$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

In proving our results, we need the following lemmas.

**Lemma 1.1.** Let
\[
f(z) = \sum_{n=1}^{\infty} a_n z^n \prec g(z)
\]
and $g(z) \in \mathcal{K}$. Then $|a_n| \leq 1$ ($n = 1, 2, 3, \cdot \cdot \cdot$).

We note that Lemma 1.1 can be seen in [1].

**Lemma 1.2.** A function $f(z)$ in $A_p$ belongs to the class $\mathcal{K}_p(\alpha)$ ($0 \leq \alpha < 1$) if
\[
\sum_{n=1}^{\infty} (p+n)\{n+p(1-\alpha)\}|a_{p+n}| \leq p^2(1-\alpha).
\]

**Proof.** If the inequality (2) holds true, then we have that
\[
\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| = \left| \frac{\sum_{n=1}^{\infty} n(p+n)a_{p+n}z^n}{p + \sum_{n=1}^{\infty} (p+n)a_{p+n}z^n} \right|
\leq \frac{\sum_{n=1}^{\infty} n(p+n)|a_{p+n}|}{p - \sum_{n=1}^{\infty} (p+n)|a_{p+n}|} \leq p(1 - \alpha)
\]
for $z \in \mathbb{U}$. From (3), we easily seen that $f(z) \in \mathcal{K}_p(\alpha)$. 
2 Subordination properties

Our first result for properties of functions $f(z) \in A_p$ is contained in

Theorem 2.1. A function $f(z) \in C_p(\lambda, \mu)$ if and only if

$$1 + \frac{zf''(z)}{f'(z)} < h(z)$$

with

$$h(z) = p + \frac{p(1-\mu)}{2\sin^2 \beta} \left\{ \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^{\frac{2\beta}{\pi}} + \left( \frac{1-\sqrt{z}}{1+\sqrt{z}} \right)^{\frac{2\beta}{\pi}} - 2 \right\} \quad (\beta = \arccos \lambda). \quad (4)$$

Proof. Let $1 + \frac{zf''(z)}{f'(z)} = w$ and $w = u + iv$. Then the inequality (1) can be written as

$$u - p\mu > \lambda \sqrt{(u-p)^2 + v^2}. \quad (5)$$

By computing, we find that the inequality (5) is equivalent to

$$\left( u + \frac{p(\lambda^2 - \mu)}{1-\lambda^2} \right)^2 - \frac{\lambda^2}{1-\lambda^2}v^2 > \left( \frac{p\lambda(1-\mu)}{1-\lambda^2} \right)^2 \quad (6)$$

and

$$u > \frac{p(\lambda + \mu)}{1+\lambda}. \quad (7)$$

Thus the domain of the values of $1 + \frac{zf''(z)}{f'(z)}$ for $z \in U$ is

$$\mathbb{D} = \{w = u + iv : u and v satisfy (6) with (7)\}.$$ 

In order to prove our theorem, it suffices to show that the function $h(z)$ given by (4) maps $\mathbb{U}$ conformally onto the domain $\mathbb{D}$.

Consider the transformations

$$w_1 = \frac{1 - \lambda^2}{p(1-\mu)}w + \frac{\lambda^2 - \mu}{1-\mu}$$

and

$$t = \frac{1}{2} \left( w_2^{\frac{\beta}{\pi}} + w_2^{-\frac{\beta}{\pi}} \right),$$

where $\beta = \arccos \lambda$ and $w_2 = w_1 + \sqrt{w_1^2 - 1}$ is the inverse function of

$$w_1 = \frac{w_2 + \frac{1}{w_2}}{2}.$$

It is easy to verify that composite function $t = t(w)$ maps $\mathbb{D}^+$ defined by

$$\mathbb{D}^+ = \{w = u + iv : u and v satisfy (6) with (7) and v > 0\}$$
conformally onto the upper half plane \(\text{Im}(t) > 0\) so that \(w = p\) corresponds to \(t = 1\) and \(w = \frac{p(t + \mu)}{1 + \lambda}\) to \(t = -1\). With the help of the symmetry principle, this function \(t = t(w)\) maps \(\mathbb{D}\) conformally onto the domain

\[ G = \{ t : |\arg(t + 1)| < \pi \}. \]

Since

\[ t = 2 \left( \frac{1+z}{1-z} \right)^2 - 1 \]

maps \(U\) onto \(G\), we see that

\[ w = p + \frac{p(1 - \mu)}{2(1 - \lambda^2)} \left\{ (t + \sqrt{t^2 - 1})^2 + (t + \sqrt{t^2 - 1})^{-2} - 2 \right\} \]

\[ = p + \frac{p(1 - \mu)}{2\sin^2\beta} \left\{ \frac{(1 + \sqrt{z})^2}{1 - \sqrt{z}} + \frac{(1 - \sqrt{z})^2}{1 + \sqrt{z}} - 2 \right\} \]

\[ = h(z) \]

maps \(U\) onto \(D\) with \(h(0) = p\). Hence the proof of the theorem is completed.

\[ \square \]

Theorem 2.1 gives the following corollaries.

Corollary 2.1. If \(f(z) \in \mathcal{C}_p(\lambda, \mu)\), then \(f(z) \in \mathcal{K}_p \left( \frac{\lambda + \mu}{1 + \lambda} \right)\) and the order \(\frac{\lambda + \mu}{1 + \lambda}\) is sharp with the extremal function

\[ f_0(z) = p \int_0^z \left( \frac{1}{t_2} \exp \int_0^{t_2} \frac{h(t_1) - p}{t_1} \, dt_1 \right) \, dt_2, \quad (8) \]

where \(h(z)\) is given by (4).

Proof. Using (7) in the proof of Theorem 2.1 and noting that

\[ \text{Re} \left( 1 + \frac{zf'''(z)}{f'(z)} \right) = \text{Re}(h(z)) \to p\frac{\lambda + \mu}{1 + \lambda} \]

as \(z = \text{Re}(z) \to -1\), we have the corollary.

\[ \square \]

Corollary 2.2. If \(f(z) \in \mathcal{C}_p(\lambda, \mu)\) and \(-\lambda < \mu < \lambda < 1\), then

\[ \left| \arg \left( 1 + \frac{zf'''(z)}{f'(z)} \right) \right| < \arctan \left( \frac{1 - \mu}{\sqrt{\lambda^2 - \mu^2}} \right) \quad (z \in \mathcal{U}). \quad (9) \]

The bound in (9) is sharp with the extremal function \(f_0(z)\) given by (8).
Proof. Let the function \( h(z) \) be defined by (4). Then \( h(\mathbb{U}) = \mathbb{D} \) and an easy calculation yields that
\[
\min\{\theta : |\arg(h(z))| < \theta \ (z \in \mathbb{U})\} = \arctan\left(\frac{1 - \mu}{\sqrt{\lambda^2 - \mu^2}}\right)
\]
for \(-\lambda < \mu < \lambda < 1\). Therefore the corollary follows immediately from Theorem 2.1.

Next we derive

**Theorem 2.2.** Let \( f(z) \in C_p(\lambda, \mu) \) and \( h(z) \) be defined by (4). Then
\[
\frac{f'(z)}{pz^{p-1}} < \exp \int_0^z \frac{h(t) - p}{t} \, dt
\]
and
\[
\left| \frac{f'(z)}{pz^{p-1}} \right| < \exp \int_0^1 \frac{h(\rho) - p}{\rho} \, d\rho \quad (z \in \mathbb{U}).
\]
The bound in (11) is sharp with the extremal function \( f_0(z) \) given by (8).

**Proof.** Since the function \( h(z) - p \) is univalent and starlike (with respect to the origin), by Theorem 2.1 and the result due to Suffridge [6, Theorem 3], we have
\[
\log \left(\frac{f'(z)}{pz^{p-1}}\right) = \int_0^z \left(\frac{f''(t)}{f'(t)} - \frac{p - 1}{t}\right) \, dt < \int_0^z \frac{h(t) - p}{t} \, dt,
\]
which implies the subordination (10).

Furthermore, noting that the univalent function \( h(z) \) maps the disk \(|z| < \rho \ (0 < \rho \leq 1)\) onto the domain which is convex and symmetric with respect to the real axis, we deduce that
\[
\text{Re} \int_0^z \frac{h(t) - p}{t} \, dt = \int_0^1 \frac{\text{Re}\{h(\rho z) - p\}}{\rho} \, d\rho < \int_0^1 \frac{h(\rho) - p}{\rho} \, d\rho
\]
for \( z \in \mathbb{U} \). Thus the inequality (11) follows from (12) and (13).

**Remark.** If we let \( \beta = \frac{\pi}{4} \) and \( x = \left(\frac{1 + \sqrt{\rho}}{1 - \sqrt{\rho}}\right)^{\frac{1}{2}} \ (0 \leq \rho < 1) \), then
\[
\int_0^1 \left\{ \left(\frac{1 + \sqrt{\rho}}{1 - \sqrt{\rho}}\right)^{2\beta} + \left(\frac{1 - \sqrt{\rho}}{1 + \sqrt{\rho}}\right)^{2\beta} - 2 \right\} \frac{d\rho}{\rho} = 8 \int_1^{+\infty} \left(\frac{x}{x^2 + 1} - \frac{1}{x + 1}\right) \, dx = 4\log 2.
\]
Thus, as the special case of Theorem 2.2, we have that if \( f(z) \in C_p\left(\frac{1}{\sqrt{2}}, \mu\right) \left(-\frac{1}{\sqrt{2}} \leq \mu < 1\right) \), then
\[
\left| \frac{f'(z)}{pz^{p-1}} \right| < 16^p(1-\mu) \quad (z \in \mathbb{U}),
\]
and the result is sharp.
3 Coefficient inequalities

Theorem 3.1. If

\[ f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \]

belongs to \( C_p(\lambda, \mu) \), then

\[ |a_{p+1}| \leq \frac{8p^2 (1 - \mu)}{p + 1} \left( \frac{\beta}{\pi \sin \beta} \right)^2 \quad (\beta = \arccos \lambda). \quad (14) \]

The result is sharp.

Proof. It can be easily verified that

\[ 1 + \frac{zf''(z)}{f'(z)} = p + \left( 1 + \frac{1}{p} \right) a_{p+1}z + \cdots \quad (15) \]

and

\[ h(z) = p + \frac{p(1 - \mu)}{2 \sin^2 \beta} \left( \frac{8 \beta}{\pi} + \frac{8 \beta}{\pi} \left( \frac{2 \beta}{\pi} - 1 \right) \right) z + \cdots \]

\[ = p + 8p(1 - \mu) \left( \frac{\beta}{\pi \sin \beta} \right)^2 z + \cdots, \quad (16) \]

where \( h(z) \) is given by (4). Since

\[ f(z) = z^p + a_{p+1} z^{p+1} + \cdots \in C_p(\lambda, \mu), \]

it follows from (15), (16) and Theorem 2.1 that

\[ \frac{\pi^2}{8p(1 - \mu)} \left( \frac{\sin \beta}{\beta} \right)^2 \left( 1 + \frac{zf''(z)}{f'(z)} - p \right) = \frac{p + 1}{8p^2 (1 - \mu)} \left( \frac{\pi \sin \beta}{\beta} \right)^2 a_{p+1}z + \cdots \]

\[ < \frac{\pi^2}{8p(1 - \mu)} \left( \frac{\sin \beta}{\beta} \right)^2 (h(z) - p). \]

In view of

\[ \frac{\pi^2}{8p(1 - \mu)} \left( \frac{\sin \beta}{\beta} \right)^2 (h(z) - p) \in K, \]

we get (14) by using Lemma 1.1. Also the bound in (14) ia sharp for the function \( f_0(z) \) given by (8).

\[ \square \]

Next we see

Theorem 3.2. If the function

\[ f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \]

...
belonging to the class $A_p$ satisfies

$$\sum_{n=1}^{\infty} (p+n)(n(1+\lambda)+ p(1-\mu))|a_{p+n}| \leqq p^2(1-\mu),$$

(17)

then $f(z)$ belongs to the class $C_p(\lambda, \mu)$.

Proof. Applying the inequality (17), we deduce that

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) - p\mu - \lambda \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \geqq p(1-\mu) - (1+\lambda) \left| 1 + \frac{zf''(z)}{f'(z)} - p \right|$$

$$= p(1-\mu) - (1+\lambda) \left| \frac{\sum_{n=1}^{\infty} n(p+n)a_{p+n}z^n}{p + \sum_{n=1}^{\infty} (p+n)a_{p+n}z^n} \right|$$

$$\geqq p(1-\mu) - (1+\lambda) \left( \frac{\sum_{n=1}^{\infty} n(p+n)|a_{p+n}|}{p - \sum_{n=1}^{\infty} (p+n)|a_{p+n}|} \right)$$

$$\geqq 0,$$

which shows that $f(z) \in C_p(\lambda, \mu)$.

By using Theorem 3.2 and Corollary 2.1, we easily have

Corollary 3.1. Let

$$f(z) = z^p + \sum_{n=1}^{\infty} (-1)^{n+1} |a_{p+n}|z^{p+n}$$

be in the class $A_p$. Then $f(z)$ belongs to the class $C_p(\lambda, \mu)$ if and only if $f(z) \in K_p \left( \frac{\lambda + \mu}{1+\lambda} \right)$.

Finally, we derive

Theorem 3.3. A function $f(z) = z^p + a_{p+n}z^{p+n}$ ($n \in \mathbb{N}$) is in the class $C_p(\lambda, \mu)$ if and only if

$$|a_{p+n}| \leqq \frac{p^2(1-\mu)}{(p+n)(n(1+\lambda) + p(1-\mu))}.$$  

(18)

Proof. In view of Theorem 3.2, it suffices to show the only if part. Let us suppose that $f(z) \in C_p(\lambda, \mu)$. Then

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) - p\mu - \lambda \left| 1 + \frac{zf''(z)}{f'(z)} - p \right|$$
\begin{equation}
= p(1 - \mu) + \text{Re} \left( \frac{n(p + n)a_{p+n}z^n}{p + (p + n)a_{p+n}z^n} \right) - \lambda \left| \frac{n(p + n)a_{p+n}z^n}{p + (p + n)a_{p+n}z^n} \right| > 0.
\end{equation}

Writing \( a_{p+n} = |a_{p+n}|e^{i\theta} (\neq 0) \) and letting \( z \to e^{i\frac{x-i\theta}{n}} (z \in \mathbb{U}) \), we have \( a_{p+n}z^n \to -|a_{p+n}| \) and it follows from (19) that

\[ p(1 - \mu) - (1 + \lambda) \frac{n(p + n)|a_{p+n}|}{p - (p + n)|a_{p+n}|} \geq 0, \]

which implies the inequality (18).

\[ \square \]

\section*{References}


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