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On Construction of Continuous Functions with Cusp Singularities

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1 Introduction

In this paper, we study various constructions of continuous functions on $\mathbf{R}$ which have the prescribed cusp singularities at each point. As applications, we get a generalization of the result given in our previous paper [7], which discuss the cusp singularities of the classical Weierstrass functions.

Let s be a positive number, which is not an integer and let x_0 be a point in $\mathbf{R}^n$. Then a function f on $\mathbf{R}^n$ belongs to the pointwise Hölder space $C^s(x_0)$, if there exists a polynomial P of degree less than s such that

$$|f(x)-P(x-x_0)|\leq C|x-x_0|^s$$

in a neighborhood of x_0. The pointwise Hölder exponent of a function f at a point x_0 in $\mathbf{R}^n$ is defined as

$$H(f,x_0)=\sup\{s>0; f\in C^s(x_0)\}.$$

If a continuous function f does not belong to $C^s(x_0)$ for every $s>0$, then $H(f,x_0)=0$.

However the pointwise Hölder exponent of a function f at a point x_0 in $\mathbf{R}^n$ is not stable under the pseudo-differential operators. Similarly it does not fully characterize the oscillatory behavior on a neighborhood of x_0. This implies that $f\in C^s(x_0)$ cannot be characterized by size estimates on the wavelet coefficients of f.

Here let us recall the definition of the weak scaling exponent characterizing the local oscillatory behavior.

$\mathcal{S}_0(\mathbf{R}^n)$ denotes the closed subspace of the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n} x^\alpha\psi(x)\,dx=0$$

for every multi-index α in $\mathbf{Z}_+^n$. Then a tempered distribution f belongs to $\Gamma^s(x_0)$, if for every ψ in $\mathcal{S}_0(\mathbf{R}^n)$, there exists a constant $C(\psi)$ such that

$$\left|\int_{\mathbf{R}^n} f(x)\frac{1}{a^n}\psi\left(\frac{x-x_0}{a}\right)\,dx\right| \le C(\psi)a^s, \quad 0<a\le 1.$$

The weak scaling exponent of a function f at a point x_0 in $\mathbf{R}^n$ is defined as

$$\beta(f,x_0)=\sup\{s\in\mathbf{R}; f \text{ locally belongs to } \Gamma^s(x_0)\}.$$

Since it is known that the pointwise Hölder space $C^s(x_0)$ is contained in local $\Gamma^s(x_0)$, it is obvious that

$$H(f,x_0)\le\beta(f,x_0).$$

Now we recall the definition of the two-microlocal spaces $C_{x_0}^{s,s'}$, which characterize this weak scaling exponent.

Let φ be a function in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ such that

$$\hat{\varphi}(\xi)=\begin{cases}1 & \text{on } |\xi|\le\dfrac{1}{2}\\ 0 & \text{on } |\xi|\ge 1\end{cases},$$

where $\hat{\varphi}$ is the Fourier transform of φ. For every non-negative integer j, we define the convolution operator $S_j(f)=f*\varphi_{\frac{1}{2^j}}$ where $\varphi_a(x)=\frac{1}{a^n}\varphi(\frac{x}{a})$, and the difference operator $\Delta_j=S_{j+1}-S_j$. Then

$$I=S_0+\sum_{j=0}^{\infty}\Delta_j.$$

Let $\psi=\varphi_{\frac{1}{2}}-\varphi$. Then $\psi\in\mathcal{S}_0(\mathbf{R}^n)$ and

$$\Delta_j(f)=f*\psi_{\frac{1}{2^j}}.$$

Let s and s' be two real numbers and x_0 a point in $\mathbf{R}^n$. Then a tempered distribution f belongs to the two-microlocal spaces $C_{x_0}^{s,s'}$, if there exists a constant C such that

$$|S_0(f)(x)|\le C(1+|x-x_0|)^{-s'}$$

and

$$|\Delta_j(f)(x)|\le C2^{-js}(1+2^j|x-x_0|)^{-s'}$$

for every $j\in\mathbf{Z}_+$ and $x\in\mathbf{R}^n$.

The following remarkable theorems with respect to the two-microlocal spaces $C^{s,s'}_{x_0}$ and $\Gamma^s(x_0)$ were given in [5].

Theorem A [5, Theorem 1.8.]. *Let s and s' be two real numbers and x_0 a point in $\mathbf{R}^n$ and let us assume two positive integers r and N satisfying*

$$r + s + \inf(s', n) > 0$$

and

$$N > \sup(s, s + s').$$

Let ψ be a function such that

$$|\partial^\alpha \psi(x)| \le \frac{C(q)}{(1+|x|)^q}, \quad |\alpha| \le r, \quad q \ge 1$$

and

$$\int_{\mathbf{R}^n} x^\beta \psi(x)\, dx = 0, \quad |\beta| \le N-1.$$

If a function or a distribution f belongs to the two-microlocal spaces $C^{s,s'}_{x_0}$, then we have

$$\left| \int_{\mathbf{R}^n} f(x) \frac{1}{a^n} \overline{\psi\left(\frac{x-b}{a}\right)}\, dx \right| \le C a^s \left(1 + \frac{|b-x_0|}{a}\right)^{-s'}, \quad 0 < a \le 1, \quad |b - x_0| \le 1.$$

Theorem B [5, Theorem 1.2.]. *Let s be a real number and let f be a function or a distribution defined on a neighborhood V of x_0.*

Then f locally belongs to $\Gamma^s(x_0)$ if and only if f locally belongs to the two-microlocal spaces $C^{s,s'}_{x_0}$ for some s'.

Several scientists have been interested in constructing irregular functions. The well-known example is the Weierstrass function [8]. It is an example of a nowhere differentiable continuous function. Hardy gave better estimates of the regularities for the Weierstrass function

$$\mathcal{W}_c(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \tag{1}$$

and its sine series

$$\mathcal{W}_s(x) = \sum_{n=0}^{\infty} a^n \sin(b^n \pi x), \tag{2}$$

where $0 < a < 1$, $b > 1$ and $ab \geq 1$ [3]. He proved that these functions do not possess finite derivatives at each point x and showed more precisely that if $ab > 1$ and $\xi = \frac{\log(\frac{1}{a})}{\log b}$, then these functions satisfy

$$\mathcal{W}_c(x+h) - \mathcal{W}_c(x) = O(|h|^\xi) \quad \text{and} \quad \mathcal{W}_s(x+h) - \mathcal{W}_s(x) = O(|h|^\xi)$$

for each x, but satisfy neither

$$\mathcal{W}_c(x+h) - \mathcal{W}_c(x) = o(|h|^\xi) \quad \text{nor} \quad \mathcal{W}_s(x+h) - \mathcal{W}_s(x) = o(|h|^\xi)$$

for any x.

Next let us recall the definition of the Takagi function [6]. Let θ^* be the 1-periodic function such that

$$\theta^*(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ 1-x & \text{if } \frac{1}{2} \leq x < 1 \end{cases}.$$

Then the Takagi function is defined by

$$\mathcal{T}(x) = \sum_{n=0}^{\infty} \frac{\theta^*(2^n x)}{2^n}.$$

It is another example of a nowhere differentiable continuous function.

Using the scaling exponents, Meyer defined two types of singularities of functions as follows [5]: a point x_0 in $\mathbf{R}^n$ is called a cusp singularity of a function f, when

$$H(f, x_0) = \beta(f, x_0) < \infty,$$

while a point x_0 in $\mathbf{R}^n$ is called an oscillating singularity of a function f, when

$$H(f, x_0) < \beta(f, x_0).$$

When a point x_0 is a cusp singularity of a function f, the pointwise Hölder exponent can be found by computing the size estimates on the wavelet coefficients of f inside the influence cone. Using this fact, we construct continuous functions which have a prescribed cusp singularity at each point x_0 in $\mathbf{R}$.

Daoudi and his team [2] studied the following problem which was raised by Lévy Véhel: *Let s be a function from $[0,1]$ to $[0,1]$. Under what conditions on s does there exist a continuous function f from $[0,1]$ to $\mathbf{R}$ such that $H(f,x) = s(x)$ for all x in $[0,1]$?*

They solved the problem as follows: "For a function s from $[0,1]$ to $[0,1]$, there exist a continuous function f on $[0,1]$ such that $H(f,x) = s(x)$ for all x in $[0,1]$ if and only if s is a function which can be represented as a limit inferior of a sequence of continuous functions on $[0,1]$." Further, they constructed such f by various methods, - as the Weierstrass type function, using Schauder bases and using Iterated Function System.

On the other hand, Andersson [1] proved a similar characterization for a function s from $\mathbf{R}$ to $[0,\infty]$ and constructed f satisfying $H(f,x)=s(x)$ for all x in $\mathbf{R}$ by a method using orthogonal wavelets.

In the rest of the paper we study, for a given function on $\mathbf{R}$, various constructions of a function f satisfying

$$H(f,x)=\beta(f,x)=s(x), \quad x\in\mathbf{R},$$

using orthonormal wavelets in Section 2 and as the Weierstrass type function in Section 3.

2 Construction Using Orthonormal Wavelets

In this section, using orthonormal wavelets, we construct a continuous function which has a prescribed cusp singularity at each point in $\mathbf{R}$.

The following Lemma 1 is used in the proof of Theorems 1 and 2.

Lemma 1. *Let s be a function from $\mathbf{R}$ to $[0,\infty]$, which is the lower limit of a sequence of real continuous functions $\{t_l\}_{l\in\mathbf{N}}$. Then there exists a sequence $\{s_l\}_{l\in\mathbf{Z}_+}$ of infinitely differentiable non-negative functions with compact supports such that*

(i) $s(x)=\liminf_{l\to\infty} s_l(x), \quad x\in\mathbf{R},$

(ii) *For each x_0 in $\mathbf{R}$, there exists a positive integer l_0 such that*

$$s_l(x)\geq\frac{1}{\sqrt{l+1}}, \quad l\geq l_0, \quad |x-x_0|\leq 1.$$

(iii) *There exists a sequence $\{C_k\}_{k\in\mathbf{Z}_+}\subset(0,\infty)$ such that*

$$\sup_{x\in\mathbf{R}}|s_l^{(k)}(x)|\leq C_k l^{k+1}, \quad l\in\mathbf{Z}_+,$$

where $s_l^{(k)}$ is the k-th derivative of s_l.

Proof. Let η be a non-negative infinitely differentiable function supported on $[-1,1]$ satisfying $\eta(x)=1$ if $|x|\leq\frac{1}{4}$, $\sup_{x\in\mathbf{R}}\eta(x)=1$ and $\int_{\mathbf{R}}\eta(x)\,dx=1$. If we put

$$\tilde{t}_l(x)=\eta\left(\frac{x}{l}\right)\min\left(\max\left(t_l(x),\frac{1}{\sqrt{l+1}}\right),l\right), \quad l\in\mathbf{N},$$

it is easy to see that $\{\tilde{t}_l\}_{l\in\mathbf{N}}$ satisfies

$$\liminf_{l\to\infty} \tilde{t}_l(x) = s(x), \quad x \in \mathbf{R},$$

$$\tilde{t}_l(x) \geq \frac{1}{\sqrt{l+1}}, \quad |x| \leq \frac{l}{4},$$

$$\tilde{t}_l(x) = 0, \quad |x| \geq l$$

and

$$\sup_{x\in\mathbf{R}} \tilde{t}_l(x) \leq l.$$

Since each $\tilde{t}_l$ is uniformly continuous, we can choose a strictly increasing sequence of positive integers $\{p_l\}_{l\in\mathbf{N}}$ such that

$$\sup_{|x-y|\leq\frac{1}{p_l}} |\tilde{t}_l(x) - \tilde{t}_l(y)| \leq \frac{1}{l}, \quad l \in \mathbf{N}.$$

Under these circumstances, we define $s_l(x)$ for $l \in \mathbf{Z}_+$ and $x \in \mathbf{R}$ by

$$s_l(x) = \begin{cases} 0 & \text{if } 0 \leq l < p_1 \\ \displaystyle\int_{\mathbf{R}} p_m \eta(p_m(x-y))\tilde{t}_m(y)\,dy & \text{if } p_m \leq l < p_{m+1}, \quad m \in \mathbf{N}. \end{cases}$$

If we put $C_k = \int_{\mathbf{R}} |\eta^{(k)}(x)|\,dx$ for $k \in \mathbf{Z}_+$, then $\{s_l\}_{l\in\mathbf{Z}_+}$ satisfies the required properties (i), (ii) and (iii). To prove (i) we have

$$\begin{aligned} |s_l(x) - \tilde{t}_m(x)| &= \left| \int_{\mathbf{R}} p_m \eta(p_m(x-y)) \left(\tilde{t}_m(y) - \tilde{t}_m(x)\right) dy \right| \\ &\leq \sup_{|x-y|\leq\frac{1}{p_m}} |\tilde{t}_m(y) - \tilde{t}_m(x)| \int_{\mathbf{R}} \eta(y)\,dy \\ &\leq \frac{1}{m}, \quad p_m \leq l < p_{m+1}. \end{aligned}$$

This proves the desired result. To prove (ii) we choose $m_0 \in \mathbf{N}$ such that $\frac{m_0}{4} - \frac{1}{m_0} \geq |x_0|+1$ and put $l_0 = p_{m_0}$. For a positive integer $l \geq l_0$, choose $m \in \mathbf{N}$ such that $p_m \leq l < p_{m+1}$. Then if $|x - x_0| \leq 1$, we have

$$\begin{aligned} s_l(x) &= \int_{\mathbf{R}} p_m \eta(p_m(x-y))\tilde{t}_m(y)\,dy \\ &\geq \inf_{|x-y|\leq\frac{1}{p_m}} \tilde{t}_m(y) \int_{\mathbf{R}} \eta(y)\,dy \end{aligned}$$

$$\geq \inf_{|y|\leq|x_0|+1+\frac{1}{m}} \tilde{t}_m(y)$$
$$\geq \inf_{|y|\leq\frac{m}{4}} \tilde{t}_m(y)$$
$$\geq \frac{1}{\sqrt{m+1}} \geq \frac{1}{\sqrt{l+1}}.$$

To prove (iii) we choose $m \in \mathbf{N}$, for a given $l \in \mathbf{N}$, such that $p_m \leq l < p_{m+1}$. Then we have

$$|s_l^{(k)}(x)| = \left| \int_{\mathbf{R}} p_m^{k+1} \eta^{(k)}(p_m(x-y)) \tilde{t}_m(y)\, dy \right|$$
$$\leq p_m^k \sup_{|x-y|\leq\frac{1}{p_m}} \tilde{t}_m(y) \int_{\mathbf{R}} |\eta^{(k)}(y)|\, dy$$
$$\leq C_k m p_m^k \leq C_k l^{k+1}.$$

■

Theorem 1. *Let s be a function from $\mathbf{R}$ to $[0,\infty]$, which is the lower limit of a sequence of continuous functions. Then there exists a sequence $\{s_l\}_{l\in\mathbf{Z}_+}$ of differentiable functions such that*

$$s(x) = \liminf_{l\to\infty} s_l(x), \quad x \in \mathbf{R} \tag{3}$$

and

$$\sup_{x\in\mathbf{R}} |s_l'(x)| \leq C_1 l^2, \quad l \in \mathbf{Z}_+. \tag{4}$$

Let ψ be an orthonormal wavelet in the Schwartz class $\mathcal{S}(\mathbf{R})$. If we define a continuous function f by

$$f(x) = \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} c(l,m) \psi(2^l x - m),$$

where

$$c(l,m) = \min(2^{-l s_l(\frac{m}{2^l})}, 2^{-\frac{l}{\log l}}),$$

then we have

$$H(f,x_0) = \beta(f,x_0) = s(x_0)$$

at each point x_0 in $\mathbf{R}$.

Proof. The existence of $\{s_l\}_{l\in\mathbf{Z}_+}$ satisfying (3) and (4) follows from Lemma 1. Since

$$\begin{aligned}\lim_{j\to\infty} \sup_{|x-y|\le 2^{-\frac{j}{(\log j)^2}}} |s_j(x)-s_j(y)| &\le \lim_{j\to\infty}\sup_{x\in\mathbf{R}}|s_j'(x)| \sup_{|x-y|\le 2^{-\frac{j}{(\log j)^2}}} |x-y| \\ &\le C_1 \lim_{j\to\infty} j^2 2^{-\frac{j}{(\log j)^2}} \\ &= 0,\end{aligned}$$

$H(f,x_0)=s(x_0)$ at each point $x_0\in\mathbf{R}$ (cf. [1] p.441, proof of Theorem 1.). We only need to compute the value of $\beta(f,x_0)$.

Let us assume f locally belongs to $\Gamma^s(x_0)$. Then by Theorem B, f locally belongs to $C^{s,s'}_{x_0}$ for some $s'<0$. On the other hand, $\psi\in\mathcal{S}_0(\mathbf{R})$ (cf. [4, 2. Corollary 3.7.]). By Theorem A, there exist two constants $C\in(0,\infty)$ and $\delta\in(0,\frac{1}{2})$ such that

$$\left|\int f(x)\frac{1}{a}\overline{\psi\left(\frac{x-b}{a}\right)}\,dx\right| \le Ca^s\left(1+\frac{|b-x_0|}{a}\right)^{-s'}, \quad 0<a\le\delta,\quad |b-x_0|\le\delta. \tag{5}$$

Let j_0 be a positive integer such that $\frac{1}{2^{j_0}}\le\delta$. For every $j\ge j_0$, there exists $k_j\in\mathbf{Z}$ such that $\frac{k_j}{2^j}\le x_0<\frac{k_j+1}{2^j}$ and we define a_j and b_j by $a_j=\frac{1}{2^j}$ and $b_j=\frac{k_j}{2^j}$. Then $|b_j-x_0|\le a_j$ and by (5), we have

$$\left|\int f(x)2^j\overline{\psi(2^jx-k_j)}\,dx\right| \le \frac{C2^{-s'}}{2^{js}}, \quad j\ge j_0. \tag{6}$$

We estimate the left hand side of (6) as follows:

$$\begin{aligned}\left|\int f(x)2^j\overline{\psi(2^jx-k_j)}\,dx\right| &= \left|\sum_{l=2}^{\infty}\sum_{m=-\infty}^{\infty} c(l,m)\int \psi(2^lx-m)2^j\overline{\psi(2^jx-k_j)}\,dx\right| \\ &= c(j,k_j).\end{aligned} \tag{7}$$

By (6) and (7), $f\in\Gamma^s(x_0)$ implies

$$c(j,k_j)=\min(2^{-js_j\left(\frac{k_j}{2^j}\right)},2^{-\frac{j}{\log j}})\le\frac{C2^{-s'}}{2^{js}}, \quad j\ge j_0. \tag{8}$$

Observe that

$$\begin{aligned}\lim_{j\to\infty}\left|s_j\left(\frac{k_j}{2^j}\right)-s_j(x_0)\right| &\le \lim_{j\to\infty}\sup_{x\in\mathbf{R}}|s_j'(x)|\left(x_0-\frac{k_j}{2^j}\right) \\ &\le C_1\lim_{j\to\infty}\frac{j^2}{2^j} \\ &= 0.\end{aligned}$$

By (8), we have

$$\begin{aligned} s &\le \liminf_{j\to\infty} \max\left(s_j\left(\frac{k_j}{2^j}\right), \frac{1}{\log j}\right) \\ &= \liminf_{j\to\infty} s_j\left(\frac{k_j}{2^j}\right) \\ &= \liminf_{j\to\infty} s_j(x_0) + \lim_{j\to\infty}\left(s_j\left(\frac{k_j}{2^j}\right) - s_j(x_0)\right) \\ &= s(x_0). \end{aligned}$$

Therefore $\beta(f,x_0) \le s(x_0) = H(f,x_0)$. Since $H(f,x_0) \le \beta(f,x_0)$ is trivial, we have $H(f,x_0) = \beta(f,x_0) = s(x_0)$. ■

3 Use of Weierstrass Type Functions

In this section, we construct the Weierstrass type continuous function which has a prescribed cusp singularity at each point in **R**.

We begin with the following lemma.

Lemma 2. *Let $s \in [0,\infty]$, $l_0 \in \mathbf{Z}_+$ and $\{s_l\}_{l\in\mathbf{Z}_+} \subset \mathbf{R}$ be such that*

(a) $\liminf_{l\to\infty} s_l = s$,

(b) $s_l \ge \dfrac{1}{\sqrt{l+1}}, \quad l \ge l_0.$

Suppose $\lambda > 1$ and $\{\theta_l\}_{l\in\mathbf{Z}_+} \subset \mathbf{R}$ are chosen arbitrary.

(i) *If $m \in \mathbf{Z}_+$ and $\{\alpha_l\}_{l\in\mathbf{Z}_+}$ is a bounded sequence in $\mathbf{R}$ and if we define a continuous function f by*

$$f(x) = \sum_{l=0}^{\infty} \frac{\alpha_l l^m}{\lambda^{l s_l}} \sin(\lambda^l x + \theta_l), \quad x \in \mathbf{R},$$

then we have

$$H(f,x_0) \ge s$$

at each point x_0 in **R**.

(ii) *If we define a continuous function g by*

$$g(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{l s_l}} \sin(\lambda^l x + \theta_l), \quad x \in \mathbf{R},$$

then we have

$$H(g,x_0) = \beta(g,x_0) = s$$

at each point x_0 *in* $\mathbf{R}$.

Proof. (i) By (b), f is a continuous function on $\mathbf{R}$ and hence we have only to show (i) when $s > 0$.

Let $x_0 \in \mathbf{R}$ be fixed arbitrary.

First, we consider the case $0 < s \leq 1$. Let $\varepsilon \in (0, s)$ be arbitrary. By (a), we can choose $l_0 \in \mathbf{Z}_+$ such that $s_l > s - \frac{\varepsilon}{2}$ for $l \geq l_0$ and we put $f_1(x) = \sum_{l=l_0}^{\infty} \frac{\alpha_l l^m}{\lambda^{l s_l}} \sin(\lambda^l x + \theta_l)$. To show $H(f, x_0) \geq s - \varepsilon$, it suffices to show $f_1 \in C^{s-\varepsilon}(x_0)$ since $H(f - f_1, x_0) = \infty$ is obvious. Let x be a real number such that $|x - x_0| < \frac{1}{\lambda^{l_0}}$ and choose $N \in \mathbf{Z}_+$ such that $\frac{1}{\lambda^{N+1}} \leq |x - x_0| < \frac{1}{\lambda^N}$. Then we have

$$\begin{aligned} |f_1(x) - f_1(x_0)| &= \left| \sum_{l=l_0}^{\infty} \frac{\alpha_l l^m}{\lambda^{l s_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l)) \right| \\ &\leq \left| \sum_{l=l_0}^{N-1} \frac{\alpha_l l^m}{\lambda^{l s_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l)) \right| \\ &\quad + \left| \sum_{l=N}^{\infty} \frac{\alpha_l l^m}{\lambda^{l s_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l)) \right| \\ &= \mathrm{A}_1 + \mathrm{A}_2. \end{aligned} \tag{9}$$

Observe first that there exists a constant $M_1 \in (0, \infty)$ such that

$$|\alpha_l| l^m \leq M_1 \lambda^{\frac{l\varepsilon}{2}}, \quad l \geq l_0. \tag{10}$$

To estimate A_1 and A_2 we use (10) to obtain

$$\begin{aligned} \mathrm{A}_1 &\leq 2 \sum_{l=l_0}^{N-1} \frac{|\alpha_l| l^m}{\lambda^{l s_l}} \left| \cos\left(\frac{\lambda^l (x + x_0)}{2} + \theta_l \right) \sin\left(\frac{\lambda^l (x - x_0)}{2} \right) \right| \\ &\leq \sum_{l=l_0}^{N-1} |\alpha_l| l^m \lambda^{l(1-s_l)} |x - x_0| \\ &\leq M_1 \sum_{l=l_0}^{N-1} \lambda^{l(1-s+\varepsilon)} |x - x_0| \\ &= \frac{M_1 \lambda^{l_0(1-s+\varepsilon)} (\lambda^{(N-l_0)(1-s+\varepsilon)} - 1)}{\lambda^{1-s+\varepsilon} - 1} |x - x_0| \\ &\leq \frac{M_1 \lambda^{N(1-s+\varepsilon)}}{\lambda^{1-s+\varepsilon} - 1} |x - x_0| \\ &\leq \frac{M_1}{\lambda^{1-s+\varepsilon} - 1} |x - x_0|^{s-\varepsilon}, \end{aligned}$$

$$\mathrm{A}_2 \leq 2 \sum_{l=N}^{\infty} \frac{|\alpha_l| l^m}{\lambda^{l s_l}} \left| \cos\left(\frac{\lambda^l (x + x_0)}{2} + \theta_l \right) \sin\left(\frac{\lambda^l (x - x_0)}{2} \right) \right|$$

$$
\begin{aligned}
&\le 2\sum_{l=N}^{\infty}\frac{|\alpha_l| l^m}{\lambda^{l s_l}} \\
&\le 2M_1\sum_{l=N}^{\infty}\frac{1}{\lambda^{l(s-\varepsilon)}} \\
&= \frac{\frac{2M_1}{\lambda^{N(s-\varepsilon)}}}{1-\frac{1}{\lambda^{s-\varepsilon}}} \\
&\le \frac{2M_1\lambda^{2(s-\varepsilon)}}{\lambda^{s-\varepsilon}-1}|x-x_0|^{s-\varepsilon}.
\end{aligned}
$$

The estimates for A_1 and A_2 with (9) show that there exists a constant $M_2 \in (0,\infty)$ such that

$$
|f_1(x)-f_1(x_0)| \le M_2|x-x_0|^{s-\varepsilon}, \quad |x-x_0| < \frac{1}{\lambda^{l_0}}.
$$

Thus $H(f_1,x_0) \ge s-\varepsilon$ and hence $H(f,x_0) \ge s-\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $H(f,x_0) \ge s$.

Next, we consider the case $n < s \le n+1$ for some $n \in \mathbf{N}$. In this case, f is n-times continuously differentiable on $\mathbf{R}$ and we have

$$
f^{(n)}(x) = \sum_{l=0}^{\infty}\frac{\alpha_l l^m}{\lambda^{l(s_l-n)}}\sin\left(\lambda^l x+\theta_l+\frac{n\pi}{2}\right).
$$

Thus $H(f^{(n)},x_0) \ge s-n$ by an argument similar to the case where $0 < s \le 1$ and hence $H(f,x_0) \ge s$ holds even for $1 < s < \infty$.

Finally, we consider the case $s=\infty$. In this case, f is obviously infinitely differentiable at x_0 and hence $H(f,x_0)=\infty$.

(ii) $H(g,x_0) \ge s$ follows from (i), if we put $\alpha_l = 1$ for $l \in \mathbf{Z}_+$ and $m=0$ in (i).

For $\beta(g,x_0)$, let us assume g locally belongs to $\Gamma^\rho(x_0)$. Let ψ be a function in $\mathcal{S}_0(\mathbf{R})$ such that $\hat{\psi}(\xi) = 0$ if $|\xi-1| \ge \frac{\lambda-1}{\lambda}$ and $\hat{\psi}(1) = 2$. Then there exist two constants $M_3 \in (0,\infty)$ and $\eta \in (0,1]$ such that

$$
\left|\int g(x)\frac{1}{a}\psi\left(\frac{x-x_0}{a}\right)dx\right| \le M_3 a^\rho, \quad 0 < a \le \eta. \tag{11}
$$

Let j_0 be a non-negative integer such that $\frac{1}{\lambda^{j_0}} \le \eta$. For every $j \ge j_0$, we put $a_j = \frac{1}{\lambda^j}$. By (11), we have

$$
\left|\int g(x)\lambda^j\psi(\lambda^j(x-x_0))\,dx\right| \le \frac{M_3}{\lambda^{j\rho}}, \quad j \ge j_0. \tag{12}
$$

We estimate the left hand side of (12) as follows:

$$
\left|\int g(x)\lambda^j\psi(\lambda^j(x-x_0))\,dx\right| \quad = \quad \left|\int\sum_{l=0}^{\infty}\frac{1}{\lambda^{l s_l}}\sin(\lambda^{l-j}x+\lambda^l x_0+\theta_l)\psi(x)\,dx\right|
$$

$$
\begin{aligned}
&= \left| \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l}} \int \frac{e^{i(\lambda^{l-j}x+\lambda^l x_0+\theta_l)} - e^{-i(\lambda^{l-j}x+\lambda^l x_0+\theta_l)}}{2i} \psi(x)\,dx \right| \\
&= \left| \sum_{l=0}^{\infty} \frac{e^{i(\lambda^l x_0+\theta_l)}\hat{\psi}(-\lambda^{l-j}) - e^{-i(\lambda^l x_0+\theta_l)}\hat{\psi}(\lambda^{l-j})}{2i\lambda^{ls_l}} \right| \\
&= \frac{|\hat{\psi}(1)|}{2\lambda^{js_j}} \\
&= \frac{1}{\lambda^{js_j}}. \qquad (13)
\end{aligned}
$$

By (12) and (13), $g \in \Gamma^\rho(x_0)$ implies $\frac{1}{\lambda^{js_j}} \le \frac{M_3}{\lambda^{j\rho}}$ for every $j \ge j_0$ and hence $\rho \le \liminf_{j\to\infty} s_j = s \le H(g, x_0)$. Therefore $\beta(g, x_0) \le s \le H(g, x_0)$. Since $H(g, x_0) \le \beta(g, x_0)$ is trivial, we have $H(g, x_0) = \beta(g, x_0) = s$. ■

Theorem 2. *Let s be a function from* **R** *to* $[0, \infty]$, *which is the lower limit of a sequence of continuous functions and let* $\{s_l\}_{l\in\mathbf{Z}_+}$ *be a sequence of continuous functions satisfying part* (i), (ii) *and* (iii) *of Lemma 1.*

Suppose $\lambda > 1$ *and* $\{\theta_l\}_{l\in\mathbf{Z}_+} \subset \mathbf{R}$ *are chosen arbitrary. If we define a continuous function f by*

$$
f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l(x)}} \sin(\lambda^l x + \theta_l),
$$

then we have

$$
H(f, x_0) = \beta(f, x_0) = s(x_0)
$$

at each point x_0 in **R**.

Proof. First, we consider the case $n \le s(x_0) < n+1$ for some $n \in \mathbf{Z}_+$. Using the Taylor expansion we have

$$
\begin{aligned}
\frac{1}{\lambda^{ls_l(x)}} = \frac{1}{\lambda^{ls_l(x_0)}} &+ \sum_{j=1}^{n} \frac{1}{j!} \frac{d^j}{dx^j} \frac{1}{\lambda^{ls_l(x)}}\bigg|_{x=x_0} (x-x_0)^j \\
&+ \frac{1}{(n+1)!} \frac{d^{n+1}}{dx^{n+1}} \frac{1}{\lambda^{ls_l(x)}}\bigg|_{x=\xi_l} (x-x_0)^{n+1}, \qquad (14)
\end{aligned}
$$

where $\xi_l \in (\min(x, x_0), \max(x, x_0))$. It goes without saying that if $n = 0$ the second term in the right hand side of (14) does not appear. By (14), we can write

$$
f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l(x)}} \sin(\lambda^l x + \theta_l) = f_1(x) + f_2(x) + f_3(x), \qquad (15)
$$

$$f_1(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l(x_0)}} \sin(\lambda^l x + \theta_l), \tag{16}$$

$$f_2(x) = \sum_{l=0}^{\infty}\sum_{j=1}^{n} \frac{1}{j!}\frac{d^j}{dx^j}\frac{1}{\lambda^{ls_l(x)}}\bigg|_{x=x_0} \sin(\lambda^l x + \theta_l)(x-x_0)^j \tag{17}$$

and

$$f_3(x) = \frac{1}{(n+1)!}\sum_{l=0}^{\infty} \frac{d^{n+1}}{dx^{n+1}}\frac{1}{\lambda^{ls_l(x)}}\bigg|_{x=\xi_l} \sin(\lambda^l x + \theta_l)(x-x_0)^{n+1}, \tag{18}$$

where $\xi_l \in (\min(x, x_0), \max(x, x_0))$.

By part (ii) of Lemma 2, $H(f_1, x_0) = \beta(f_1, x_0) = s(x_0)$ follows at once. f_2 does not appear if $n = 0$, and if $n \geq 1$ we have

$$f_2(x) = \sum_{l=0}^{\infty}\sum_{j=1}^{n}\sum_{k=1}^{j}\sum_{(*)_j} \frac{1}{j!}\frac{(-\log\lambda)^k l^k \alpha_{j,i_1,\ldots,i_k} s_l^{(i_1)}(x_0)\ldots s_l^{(i_k)}(x_0)}{\lambda^{ls_l(x_0)}} \cdot \sin(\lambda^l x + \theta_l)(x-x_0)^j, \tag{19}$$

where $\sum_{(*)_j}$ mean the summation under the condition $i_1 + \cdots + i_k = j$ with $i_1 \leq \cdots \leq i_k$ and $\{\alpha_{j,i_1,\ldots,i_k}\}$ are positive integers satisfying $\sum_{(*)_j} \alpha_{j,i_1,\ldots,i_k} \leq (k+1)^j$. By (19), part (iii) of Lemma 1 and part (i) of Lemma 2, we can deduce that $H(f_2, x_0) \geq s(x_0) + 1$. For f_3, we have

$$f_3(x) = \frac{1}{(n+1)!}\sum_{l=0}^{\infty}\sum_{k=1}^{n+1}\sum_{(*)_{n+1}} \frac{(-\log\lambda)^k l^k \alpha_{n+1,i_1,\ldots,i_k} s_l^{(i_1)}(\xi_l)\ldots s_l^{(i_k)}(\xi_l)}{\lambda^{ls_l(\xi_l)}} \cdot \sin(\lambda^l x + \theta_l)(x-x_0)^{n+1}, \tag{20}$$

where $\sum_{(*)_{n+1}}$ mean the summation under the condition $i_1 + \cdots + i_k = n+1$ with $i_1 \leq \cdots \leq i_k$ and $\{\alpha_{n+1,i_1,\ldots,i_k}\}$ are positive integers satisfying $\sum_{(*)_{n+1}} \alpha_{n+1,i_1,\ldots,i_k} \leq (k+1)^{n+1}$. By (20) and part (iii) of Lemma 1, we can deduce that $H(f_3, x_0) \geq n+1$. By the estimates for f_1, f_2 and f_3, and (15), we can conclude that $H(f, x_0) = \beta(f, x_0) = s(x_0)$.

Next, we consider the case $s(x_0) = \infty$. Let n be a positive integer and let $f = f_1 + f_2 + f_3$, where f_1, f_2 and f_3 are defined by (16), (17) and (18), respectively. But in this case, we have $H(f_1, x_0) = H(f_2, x_0) = \infty$ and $H(f_3, x_0) \geq n+1$ by part (iii) of Lemma 1 and part (i) of Lemma 2, since $\liminf_{l\to\infty} s_l(x_0) = \infty$. By the estimates for f_1, f_2 and f_3, and (15), we have $H(f, x_0) \geq n+1$. Since n is arbitrary, we can conclude that $H(f, x_0) = \beta(f, x_0) = s(x_0)$ even for $s(x_0) = \infty$. ■

In the case where s is a continuous function, we have the following result.

Theorem 3. *Let s be a continuous function from* $\mathbf{R}$ *to* $(0,\infty)$ *such that*

$$s(x_0) < H(s, x_0)$$

at each point x_0 in $\mathbf{R}$. *Suppose $\lambda > 1$ and $\{\theta_l\}_{l\in\mathbf{Z}_+} \subset \mathbf{R}$ are chosen arbitrary. If we define a continuous function f by*

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls(x)}} \sin(\lambda^l x + \theta_l),$$

then we have

$$H(f, x_0) = \beta(f, x_0) = s(x_0)$$

at each point x_0 in $\mathbf{R}$.

Proof. Let $x_0 \in \mathbf{R}$ be fixed arbitrary and let x be a real number such that $|x-x_0| < 1$. Then we have

$$\begin{aligned} f(x) &= \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls(x_0)}} \sin(\lambda^l x + \theta_l) + \sum_{l=0}^{\infty} \left(\frac{1}{\lambda^{ls(x)}} - \frac{1}{\lambda^{ls(x_0)}}\right) \sin(\lambda^l x + \theta_l) \\ &= f_1(x) + f_2(x). \end{aligned} \tag{21}$$

By part (ii) of Lemma 2, $H(f_1, x_0) = \beta(f_1, x_0) = s(x_0)$ follows at once. Let ε be a positive number such that $s(x_0) + \varepsilon < H(s, x_0)$ and $s(x_0) + \varepsilon \notin \mathbf{N}$. Then $s \in C^{s(x_0)+\varepsilon}(x_0)$ and there exist a polynomial P of degree at most $[s(x_0) + \varepsilon]$, two constants $C \in (0,\infty)$ and $\delta \in (0,1)$ such that

$$s(x) = s(x_0) + P(x - x_0) + Q(x - x_0)$$

and

$$|Q(x - x_0)| \le C|x - x_0|^{s(x_0)+\varepsilon}, \quad |x - x_0| \le \delta.$$

To estimate f_2, using the mean value theorem, we write

$$\frac{1}{\lambda^{ls(x)}} - \frac{1}{\lambda^{ls(x_0)}} = \frac{(-\log\lambda)l(s(x) - s(x_0))}{\lambda^{l\tau_l}},$$

where $\tau_l \in [\min(s(x), s(x_0)), \max(s(x), s(x_0))]$. Then we have

$$\left| f_2(x) - \left((-\log\lambda) \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\tau_l}} \sin(\lambda^l x + \theta_l) \right) P(x - x_0) \right|$$

$$= (\log\lambda)\left|\sum_{l=0}^{\infty}\frac{l}{\lambda^{l\gamma}}\sin(\lambda^l x+\theta_l)\right||Q(x-x_0)|$$
$$\leq C(\log\lambda)\sum_{l=0}^{\infty}\frac{l}{\lambda^{l\gamma}}|x-x_0|^{s(x_0)+\varepsilon}.$$

Hence $H(f_2,x_0)\geq s(x_0)+\varepsilon$. By the estimates for f_1 and f_2, and (21), we can conclude that $H(f,x_0)=\beta(f,x_0)=s(x_0)$. ■

Corollary 1. *Each point in* $\mathbf{R}$ *is a cusp singularity of the Weierstrass functions.*

Proof. Let $\mathcal{W}_c$ and $\mathcal{W}_s$ be the Weierstrass functions (for the definitions of $\mathcal{W}_c$ and $\mathcal{W}_s$, see (1) and (2)). If we put $\lambda=b$, $s(x)=\frac{\log(\frac{1}{a})}{\log b}$ and $\theta_l=\frac{\pi}{2}$ for $l\in\mathbf{Z}_+$ or $\theta_l=0$ for $l\in\mathbf{Z}_+$, then we have $H(\mathcal{W}_c,x)=\beta(\mathcal{W}_c,x)=\frac{\log(\frac{1}{a})}{\log b}=H(\mathcal{W}_s,x)=\beta(\mathcal{W}_s,x)$ at each point x in $\mathbf{R}$ from Theorem 3. ■

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