On Ideals in $H^\infty$ Whose Closures are Intersections of Maximal Ideals

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§1. Introduction

Let $H^\infty$ be the Banach algebra of bounded analytic functions on the open unit disk $D$. We denote by $M(H^\infty)$ the set of non-zero multiplicative linear functionals of $H^\infty$ endowed with the weak*-topology of the dual space of $H^\infty$. Identifying a point in $D$ with its point evaluation, we think as $D \subset M(H^\infty)$. For $\varphi \in M(H^\infty)$, put $\text{Ker} \varphi = \{ f \in H^\infty; \varphi(f) = 0 \}$. Then $\text{Ker} \varphi$ is a maximal ideal in $H^\infty$, and for a maximal ideal $I$ in $H^\infty$ there exists $\psi \in M(H^\infty)$ such that $I = \text{Ker} \psi$. Usually $M(H^\infty)$ is called the maximal ideal space of $H^\infty$. For $f \in H^\infty$, the function $\hat{f}(\varphi) = \varphi(f)$ on $M(H^\infty)$ is called the Gelfand transform of $f$. We identify $f$ with $\hat{f}$, so that we think of $H^\infty$ the closed subalgebra of continuous functions on $M(H^\infty)$. Let $L^\infty$ be the Banach algebra of bounded measurable functions on $\partial D$. We denote by $M(L^\infty)$ the maximal ideal space of $L^\infty$. We may think that $M(L^\infty) \subset M(H^\infty)$ and $M(L^\infty)$ is the Shilov boundary of $H^\infty$, that is, the smallest closed subset of $M(H^\infty)$ on which every function in $H^\infty$ attains its maximal modulus. A nice reference on this subject is [3].

For $f \in H^\infty$, there exists a radial limit $f(e^{i\theta})$ for almost everywhere. Let $h$ be a bounded measurable function on $\partial D$ such that $\int_0^{2\pi} \log |h| \, d\theta / 2\pi > -\infty$. Put

$$ f(z) = \exp \left( \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |h(e^{i\theta})| \, d\theta / 2\pi \right), \quad z \in D. $$

A function of this form is called outer, and $|f(e^{i\theta})| = |h(e^{i\theta})|$ almost everywhere. A function $u \in H^\infty$ is called inner if $|u(e^{i\theta})| = 1$ a.e. on $\partial D$. For a sequence $\{z_n\}_n$ in $D$ with $\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$, there corresponds a Blaschke product

$$ b(z) = \prod_{n=1}^{\infty} \frac{z - z_n}{1 - \overline{z}_n z}, \quad z \in D. $$

A Blaschke product is called interpolating if for every bounded sequence of complex numbers $\{a_n\}_n$ there exists $h \in H^\infty$ such that $h(z_n) = a_n$ for every $n$. For a non-negative bounded singular measure $\mu, \mu \neq 0$, on $\partial D$, let

$$ \psi_\mu(z) = \exp \left( - \int_{\partial D} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu \right), \quad z \in D. $$
Then $\psi_\mu$ is inner and called a singular function. It is well known that every function in $H^\infty$ is factored as an inner function times an outer function, and an inner function is factored as a Blaschke product times a singular function.

For a subset $E$ of $M(H^\infty)$, let $I(E) = \cap \{\text{Ker } \varphi; \varphi \in E\}$ be the intersection of maximal ideals associated with points in $E$. For $f \in H^\infty$, let $Z(f) = \{\varphi \in M(H^\infty); \varphi(f) = 0\}$ be the zero set of $f$. In this paper, we mean that an ideal is a non-zero proper ideal in $H^\infty$. For an ideal $I$ in $H^\infty$, put $Z(I) = \cap \{Z(f); f \in I\}$, then $I \subset I(Z(I))$. An ideal $I$ is called prime if for any $f, g \in H^\infty$ with $fg \in I$, then $f \in I$ or $g \in I$. There are many studies of prime ideals in $H^\infty$, see [4, 14, 15, 16]. Recently, Gorkin and Mortini [6, Theorem 1] proved that a closed prime ideal $I$ is an intersection of maximal ideals, that is, $I = I(Z(I))$. And they pointed out that if $I$ is a (non-closed) prime ideal such that $Z(I) \cap M(L^\infty) = \emptyset$, then the closure of $I$ is an intersection of maximal ideals, that is, $\overline{I} = I(Z(I))$.

Let $E$ be a closed subset of $M(H^\infty) \setminus D$ such that $E \cap M(L^\infty) = \emptyset$. Let $J = J(E)$ be the ideal of $H^\infty$ which consists of functions in $H^\infty$ vanishing on some open subsets $U$ of $M(H^\infty) \setminus D$ such that $E \subset U$. In [7, Theorem 4.2], Gorkin and Mortini also showed that $\overline{J} = I(Z(J))$. It is a very interesting problem to determine the class of ideals $I$ satisfying $\overline{I} = I(Z(I))$. But it seems difficult to give a complete characterization of these ideals.

In Section 2, we introduce the following condition on ideals $I$ in $H^\infty$ to study ideals $I$ satisfying $\overline{I} = I(Z(I))$. We prove that if an ideal $I$ of $H^\infty$ satisfies condition (a), then $\overline{I} = I(Z(I))$. We also give some examples of ideals $I$ satisfying condition (a).

In Section 3, we study an ideal $I(f)$ of $H^\infty$ which is generated by a noninvertible outer function $f$. There exist noninvertible outer functions $f$ and $g$ satisfying $\overline{I(f)} = I(Z(I(f)))$ and $\overline{I(g)} \neq I(Z(I(g)))$. As an application of the theorem given in Section 2, we characterize noninvertible outer functions $f$ satisfying $\overline{I(f)} = I(Z(I(f)))$.

(a) For any $0 < \sigma < 1$ and a subset $E$ of $D$ such that $Z(I) \cap \text{cl } E = \emptyset$, there exists $h \in I$ such that $\|h\|_\infty \leq 1$ and $|h| \geq \sigma$ on $E$, where $\text{cl } E$ is the weak*-closure of $E$ in $M(H^\infty)$.

2. Closure of ideals

We introduce the following condition on ideals $I$ in $H^\infty$.

(a) For any $0 < \sigma < 1$ and a subset $E$ of $D$ such that $Z(I) \cap \text{cl } E = \emptyset$, there exists $h \in I$ such that $\|h\|_\infty \leq 1$ and $|h| \geq \sigma$ on $E$, where $\text{cl } E$ is the weak*-closure of $E$ in $M(H^\infty)$.

The main theorem of this paper is the following.
THEOREM 2.1. Let $I$ be an ideal in $H^\infty$ satisfying condition (α). Then $\overline{I} = I(Z(I))$.

Generally the converse of Theorem 2.1 does not hold, but it holds for some ideals. Let $G$ be the set of point $\varphi$ in $M(H^\infty)$ such that $\varphi(b) = 0$ for some interpolating Blaschke product $b$. By Hoffman's work [11], $G$ is an open subset of $M(H^\infty)$ and for each $\varphi \in G$ there exists a continuous one to one map $L_{\varphi}$ from $D$ into $M(H^\infty)$ such that $L_{\varphi}(0) = \varphi$ and $f \circ L_{\varphi} \in H^\infty$ for every $f \in H^\infty$. Put $P(\varphi) = L_{\varphi}(D)$, and this set is called the Gleason part containing $\varphi$. Then we have

PROPOSITION 2.1. Let $I$ be an ideal in $H^\infty$ such that $P(\varphi) \subset Z(I)$ for every $\varphi \in Z(I) \cap G$. Then $\overline{I} = I(Z(I))$ if and only if $I$ satisfies condition (α).

By the proof of Theorem 2.1 and Proposition 2.1, we have

COROLLARY 2.1. Let $I$ be an ideal in $H^\infty$ algebraically generated by countable functions. Suppose that $P(\varphi) \subset Z(I)$ for every $\varphi \in Z(I) \cap G$. Then $I(Z(I))$ is a closed ideal generated by countable functions.

Examples of ideals satisfying condition (α) are given in the following.

PROPOSITION 2.2. The following ideals $I$ in $H^\infty$ satisfy condition (α).

(i) $I$ is a prime ideal in $H^\infty$ which does not contain any interpolating Blaschke product.

(ii) Let $f$ be a function in $H^\infty$ which does not vanish on $D$. Let $I$ be the ideal in $H^\infty$ algebraically generated by functions $f^{1/n}, n = 1, 2, \ldots$.

(iii) Let $E$ be a closed subset of $M(H^\infty) \setminus D$ such that $E \cap M(L^\infty) = \emptyset$. Let $I$ be the ideal of functions in $H^\infty$ which vanish on some open subsets $U$ of $M(H^\infty) \setminus D$ such that $E \subset U$.

(iv) Let $S$ be a set of non-negative bounded singular measures $\mu, \mu \neq 0$, on $\partial D$. Suppose that $S$ satisfies the following conditions.

(a) For $\mu, \nu \in S$, there exists $\lambda \in S$ such that $\lambda \leq \mu \wedge \nu$, where $\mu \wedge \nu$ is the greatest lower bound of $\mu$ and $\nu$,

(b) For every $\mu \in S$ and a positive integer $n$, there exists $\lambda \in S$ such that $n\lambda \leq \mu$.

Let $I$ be the ideal algebraically generated by singular functions $\psi_{\mu}, \mu \in S$.

By Theorem 2.1 and Proposition 2.2, we have

COROLLARY 2.2. Let $f$ be a function in $H^\infty$ which does not vanish on $D$. Let $I$ be the ideal in $H^\infty$ algebraically generated by functions $f^{1/n}, n = 1, 2, \ldots$ Then
\[ I = I(Z(I)). \]

**Corollary 2.3 [7, Theorem 4.2].** Let \( E \) be a closed subset of \( M(H^\infty) \setminus D \) such that \( E \cap M(L^\infty) = \emptyset \). Let \( I \) be the ideal of functions in \( H^\infty \) which vanish on some open subsets \( U \) of \( M(H^\infty) \setminus D \) such that \( E \subset U \). Then \( \overline{I} = I(Z(I)) \).

We also have the following.

**Corollary 2.4.** Let \( I \) be a prime ideal in \( H^\infty \). Then \( \overline{I} = I(Z(I)) \).

In [6], to prove that \( I = I(Z(I)) \) for a closed prime ideal \( I \) Gorkin and Mortini used the following formula given by Guillory and Sarason [9, pp.177-178]. Let \( R \) be an open subset of \( D \) such that \( \partial R \cap D \) is a system of rectifiable curves. Then

\[
\int_{\partial D} \frac{F}{u} \, dz = \int_{\partial R \cap D} \frac{F}{u} \, dz
\]

(2.1)

for \( F \in H^\infty \) and an inner function \( u \) satisfying \( |u(z)| < \beta \) for \( z \in R \) and \( |u(z)| \geq \alpha \) for \( z \in D \setminus R, 0 < \alpha < \beta < 1 \). Formula (2.1) is used in several papers, see [8, 12, 13]. When \( u \) is not inner, equation (2.1) does not holds.

To prove Theorem 2.1, we need another formula similar to (2.1). The following theorem is interesting in its own right.

**Theorem 2.2.** Let \( f \in H^\infty, \|f\|_\infty = 1, \) and \( 0 < \varepsilon < 1/2 < \sigma < 1 \). Let \( R \) be an open subset of \( D \) such that \( \partial R \cap D \) is a system of rectifiable curves satisfying

(i) \( |f(z)| < \varepsilon \) for \( z \in R \).

We assign the usual orientation on \( \partial R \). Put \( \Gamma = \partial R \cap D \). Let \( h \in H^\infty \) such that \( \|h\|_\infty = 1 \),

(ii) \( 0 < 1/2 \leq |h(z)| \) for \( z \in D \setminus R \),

(iii) \( |h(e^{i\theta})| \geq \sigma \) for almost every \( e^{i\theta} \in \partial D \) with \( |f(e^{i\theta})| > \varepsilon \).

Then

\[
\left| \int_{\Gamma} \frac{F}{h} \, dz - \int_{\partial D} fF\overline{h} \, dz \right| \leq 4(\varepsilon + 1 - \sigma)\|F\|_1
\]

for every \( F \in H^\infty \), where \( \|F\|_1 = \int_0^{2\pi} |F(e^{i\theta})| \, d\theta / 2\pi \).

As an application of Theorem 2.2, we shall prove Theorem 2.1. Our theorems owe to the deep theorems due to Bourgain [2] and Suárez [18, 19].

Let \( g(z) = (1 - z)/2 \). Then \( g \) is an outer function and is not invertible in \( H^\infty \). Let \( I = gH^\infty \) be the ideal generated by \( g \). Then it is not difficult to see that for \( h \in I \),

\[
\|h - hg(\sum_{k=0}^{n-1} \left(\frac{1+z}{2}\right)^k)\|_\infty = \|h - h(1 - \left(\frac{1+z}{2}\right)^n)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.
\]
Hence $I = I(Z(I))$. One might ask whether $I = I(Z(I))$ for an ideal $I$ generated by a single outer function in $H^\infty$ which is not invertible in $H^\infty$. To answer this question, we need to recall Jensen's equality. For a point $\varphi \in M(H^{\infty})$, there is a probability measure $\mu_{\varphi}$ on $M(L^{\infty})$ such that $\int_{M(L^{\infty})} f \, d\mu_{\varphi} = \varphi(f)$ for every $f \in H^{\infty}$. We denote by $\text{supp} \mu_{\varphi}$ the closed support set of $\mu_{\varphi}$. Then the following Jensen inequality holds

$$\log |\varphi(f)| \leq \int_{M(L^{\infty})} \log |f| \, d\mu_{\varphi}, \quad f \in H^{\infty}.$$  

When it holds that

$$\log |\varphi(f)| = \int_{M(L^{\infty})} \log |f| \, d\mu_{\varphi},$$

we say that $f$ satisfies Jensen's equality for $\varphi \in M(H^{\infty})$. It is well known that every invertible functions in $H^\infty$ satisfies Jensen's equality for every point in $M(H^\infty)$, see [10, Chapter 10]. Our third theorem is

**Theorem 2.3.** Let $f$ be an outer function in $H^\infty$ which is not invertible in $H^\infty$. Let $I = H^\infty f$ be the ideal generated by $f$. Then $I = I(Z(I))$ if and only if $f$ satisfies Jensen's equality for every point $m$ in $M(H^{\infty})$ with $m(f) \neq 0$.

Axler and Shields [1, Proposition 5] showed that a function $f$ in $H^\infty$ satisfying $\text{Re} f > 0$ on $D$ satisfies Jensen's equality for every point in $M(H^\infty)$. For an inner function $q$, the function $q + 1$ satisfies this condition. Put $QA = H^\infty \cap H^\infty + C$, where $C$ is the space of continuous functions on $\partial D$ and $H^\infty + C$ is the set of complex conjugates of functions in $H^\infty + C$. In [20], Wolff proved that for every $f \in L^{\infty}$ there exists an outer function $h \in QA$ such that $hf \in H^{\infty} + C$. When $f \not\in H^{\infty} + C$, the function $h$ is not invertible in $H^{\infty}$. So that there are many outer functions in $QA$ which are not invertible in $H^{\infty}$. In [17], Sarason proved that if $f \in H^{\infty}$, then $f \in QA$ if and only if $f_{\text{supp} \mu_{\varphi}}$ is constant for every $\varphi \in M(H^{\infty}) \setminus D$. Hence QA outer functions satisfy Jensen's equality for every $\varphi \in M(H^{\infty})$. We have following corollaries as applications of Theorem 2.3.

**Corollary 2.5.** Let $I = fH^{\infty}$ be an ideal in $H^{\infty}$ generated by a function $f$ which is not invertible in $H^{\infty}$ and $\text{Re} f > 0$ on $D$. Then $I = I(Z(I))$.

**Corollary 2.6.** Let $I = fH^{\infty}$ be an ideal in $H^{\infty}$ generated by an outer function in $QA$ which is not invertible in $H^{\infty}$. Then $I = I(Z(I))$.

**References**


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