Non-relativistic Limit of a Dirac particle Interacting with the Quantum Radiation Field

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Abstract
The non-relativistic (scaling) limit of a Hamiltonian of a Dirac particle interacting with the quantum radiation field yields a self-adjoint extension of the Pauli-Fierz Hamiltonian with spin 1/2 in non-relativistic quantum electrodynamics.

Keywords: quantum electrodynamics, Dirac operator, Dirac-Maxwell operator, Pauli-Fierz Hamiltonian, non-relativistic limit, scaling limit, Fock space, strongly anticommuting self-adjoint operators

1 Introduction

A Hamiltonian $H$ of a Dirac particle — a relativistic charged particle with spin 1/2 — interacting with the quantum radiation field is called a Dirac-Maxwell operator. In this note we report a result on the non-relativistic limit of $H$.

The Dirac-Maxwell operator $H$ is of the form $H = H_D + H_{\text{rad}} + H_I$, where $H_D$ is a Dirac operator describing the Dirac particle system only, $H_{\text{rad}}$ is the free Hamiltonian of the quantum radiation field (a quantum version of the Maxwell Hamiltonian in the Coulomb gauge) and $H_I$ is the interaction term between the Dirac particle and the quantum radiation field. As for the Dirac operator $H_D$, the non-relativistic limit has already been investigated and well understood ([10, Chapter 6] and references therein). We extend the methods used in the case of the Dirac operator $H_D$ to the case of $H$. This can be done in an abstract framework with further developments of the theory of scaling limits on strongly anticommuting self-adjoint operators [1]. The main result we report in this note is that the non-relativistic limit of $H$ yields a self-adjoint extension of the Pauli-Fierz Hamiltonian with spin 1/2 in non-relativistic quantum electrodynamics.

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2 The Dirac-Maxwell Operator and The Pauli-Fierz Hamiltonian

For a linear operator $T$ on a Hilbert space, we denote its domain by $D(T)$, and its adjoint by $T^*$ (provided that $T$ is densely defined). For two objects $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ such that products $a_j b_j$ $(j = 1, 2, 3)$ and their sum can be defined, we set $a \cdot b := \sum_{j=1}^{3} a_j b_j$.

We use the physical unit system in which $c$ (the speed of light) = 1 and $\hbar = 1$ ($\hbar := h/(2\pi)$; $h$ is the Planck constant).

2.1 The Dirac operator

Let $D_j$ $(j = 1, 2, 3)$ be the generalized partial differential operator in the variable $x_j$, the $j$-th component of $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, and $\nabla := (D_1, D_2, D_3)$.

We denote the mass and the charge of the Dirac particle by $m > 0$ and $q \in \mathbb{R} \setminus \{0\}$ respectively. We consider the situation where the Dirac particle is in a potential $V$ which is a Hermitian-matrix-valued Borel measurable function on $\mathbb{R}^3$. Then the Hamiltonian of the Dirac particle is given by the Dirac operator

$$H_D := \alpha \cdot (-i \nabla) + m\beta + V$$

acting in the Hilbert space

$$\mathcal{H}_D := \bigoplus^4 L^2(\mathbb{R}^3)$$

with domain $D(H_D) := [\bigoplus^4 H^1(\mathbb{R}^3)] \cap D(V)$ ($H^1(\mathbb{R}^3)$ is the Sobolev space of order 1), where $\alpha_j$ $(j = 1, 2, 3)$ and $\beta$ are $4 \times 4$ Hermitian matrices satisfying the anticommutation relations

\[
\{\alpha_j, \alpha_k\} = 2\delta_{jk}, \quad j, k = 1, 2, 3, \tag{2.3}
\]

\[
\{\alpha_j, \beta\} = 0, \quad \beta^2 = 1, \quad j = 1, 2, 3, \tag{2.4}
\]

$\{A, B\} := AB + BA$ and $\delta_{jk}$ is the Kronecker delta. We assume the following:

Hypothesis (A)

Each matrix element of $V$ is almost everywhere (a.e.) finite with respect to the three-dimensional Lebesgue measure $d\mathbf{x}$ and the subspace $\cap_{j=1}^{3}[D(D_j) \cap D(V)]$ is dense in $\mathcal{H}_D$.

Under this hypothesis, $H_D$ is a symmetric operator. For detailed analyses of the Dirac operator, see, e.g., [10].
2.2 The quantum radiation field

The Hilbert space of one-photon states in momentum representation is given by

$$\mathcal{H}_{\text{ph}} := L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3),$$

(2.5)

where \( \mathbb{R}^3 := \{ k = (k_1, k_2, k_3) | k_j \in \mathbb{R}, j = 1, 2, 3 \} \) physically means the momentum space of photons. Then a Hilbert space for the quantum radiation field in the Coulomb gauge is given by

$$\mathcal{F}_{\text{rad}} := \bigotimes_{n=0}^{\infty} \bigotimes_{s}^{n} \mathcal{H}_{\text{ph}},$$

(2.6)

the Boson Fock space over \( \mathcal{H}_{\text{ph}} \), where \( \bigotimes_{s}^{n} \mathcal{H}_{\text{ph}} \) denotes the \( n \)-fold symmetric tensor product of \( \mathcal{H}_{\text{ph}} \) and \( \bigotimes_{s}^{n} \mathcal{H}_{\text{ph}} := \mathbb{C} \). For basic facts on the theory of the Boson Fock space, we refer the reader to [8, §X.7].

We denote by \( a(F) \) \((F \in \mathcal{H}_{\text{ph}})\) the annihilation operator with test vector \( F \) on \( \mathcal{F}_{\text{rad}} \); its adjoint is given by

$$\big(a(F)^* \Psi\big)^{(n)} = \sqrt{n} S_n(F \otimes \Psi^{(n-1)}), \quad n \geq 0, \Psi = \{ \Psi^{(n)}\}_{n=0}^{\infty} \in D(a(F)^*),$$

where \( S_n \) is the symmetrization operator on \( \bigotimes_{s}^{n} \mathcal{H}_{\text{ph}} \) and \( \Psi^{-1} := 0 \).

For each \( f \in L^2(\mathbb{R}^3) \), we define

$$a^{(1)}(f) := a(f, 0), \quad a^{(2)}(f) := a(0, f).$$

(2.7)

The mapping \( f \rightarrow a^{(r)}(f^*) \) restricted to \( \mathcal{S}(\mathbb{R}^3) \) (the Schwartz space of rapidly decreasing \( C^\infty \)-functions on \( \mathbb{R}^3 \)) defines an operator-valued distribution \( (f^*) \) denotes the complex conjugate of \( f \). We denote its symbolical kernel by \( a^{(r)}(k) : a^{(r)}(f) = \int a^{(r)}(k)f(k)^*dk \).

We take a nonnegative Borel measurable function \( \omega \) on \( \mathbb{R}^3 \) to denote the one free photon energy. We assume that, for a.e. \( k \in \mathbb{R}^3 \) with respect to the Lebesgue measure on \( \mathbb{R}^3 \), \( 0 < \omega(k) < \infty \). Then the function \( \omega \) defines uniquely a multiplication operator on \( \mathcal{H}_{\text{ph}} \) which is nonnegative, self-adjoint and injective. We denote it by the same symbol \( \omega \). The free Hamiltonian of the quantum radiation field is then defined by

$$H_{\text{rad}} := d \Gamma(\omega),$$

(2.8)

the second quantization of \( \omega \) [7, p.302, Example 2] and [8, §X.7]. The operator \( H_{\text{rad}} \) is a nonnegative self-adjoint operator. The symbolical expression of \( H_{\text{rad}} \) is \( H_{\text{rad}} = \sum_{r=1}^{2} \int \omega(k)a^{(r)}(k)a^{(r)}(k)dk \).

Remark 2.1 Usually \( \omega \) is taken to be of the form \( \omega_{\text{phys}}(k) := |k|, \quad k \in \mathbb{R}^3 \), but, in this paper, for mathematical generality, we do not restrict ourselves to this case.

There exist \( \mathbb{R}^3 \)-valued Borel measurable functions \( e^{(r)} \) \((r = 1, 2)\) on \( \mathbb{R}^3 \) such that, for a.e. \( k \)

$$e^{(r)}(k) \cdot e^{(s)}(k) = \delta_{rs}, \quad e^{(r)}(k) \cdot k = 0, \quad r, s = 1, 2.$$

(2.9)

These vector-valued functions \( e^{(r)} \) are called the polarization vectors of a photon.
The time-zero quantum radiation field is given by $A(x) := (A_1(x), A_2(x), A_3(x))$ with
\[
A_j(x) := \sum_{r=1}^{2} \int dk \frac{e_j^{(r)}(k)}{\sqrt{2(2\pi)^3 \omega(k)}} \left\{ a^{(r)}(k)^* e^{-ik \cdot x} + a^{(r)}(k) e^{ik \cdot x} \right\}, \quad j = 1, 2, 3, \tag{2.10}
\]
in the sense of operator-valued distribution.

Let $\rho$ be a real tempered distribution on $\mathbb{R}^3$ such that
\[
\frac{\hat{\rho}}{\sqrt{\omega}}, \quad \frac{\hat{\rho}}{\omega} \in L^2(\mathbb{R}^3), \tag{2.11}
\]
where $\hat{\rho}$ denotes the Fourier transform of $\rho$. The quantum radiation field
\[
A^\rho := (A_1^\rho, A_2^\rho, A_3^\rho) \tag{2.12}
\]
with momentum cutoff $\hat{\rho}$ is defined by
\[
A_j^\rho(x) := \sum_{r=1}^{2} \int dk \frac{e_j^{(r)}(k)}{\sqrt{2\omega(k)}} \left\{ a^{(r)}(k)^* e^{-ik \cdot x} \hat{\rho}(k)^* + a^{(r)}(k) e^{ik \cdot x} \hat{\rho}(k) \right\}. \tag{2.13}
\]
Symbolically $A_j^\rho(x) = \int A_j(x - y) \rho(y) dy$.

### 2.3 The Dirac-Maxwell operator

The Hilbert space of state vectors for the coupled system of the Dirac particle and the quantum radiation field is taken to be
\[
\mathcal{F} := \mathcal{H}_\text{D} \otimes \mathcal{F}_{\text{rad}}. \tag{2.14}
\]
This Hilbert space can be identified as
\[
\mathcal{F} = L^2(\mathbb{R}^3; \oplus^4 \mathcal{F}_{\text{rad}}) = \int_{\mathbb{R}^3} \oplus^4 \mathcal{F}_{\text{rad}} dx \tag{2.15}
\]
the Hilbert space of $\oplus^4 \mathcal{F}_{\text{rad}}$-valued Lebesgue square integrable functions on $\mathbb{R}^3$ (the constant fibre direct integral with base space $(\mathbb{R}^3, dx)$ and fibre $\oplus^4 \mathcal{F}_{\text{rad}}$ [9, §XIII.6]). We freely use this identification. The total Hamiltonian of the coupled system — a Dirac-Maxwell operator — is defined by
\[
H := H_\text{D} + H_{\text{rad}} - q \alpha \cdot A^\rho = \alpha \cdot (-i \nabla - q A^\rho) + m\beta + V + H_{\text{rad}}. \tag{2.16}
\]
The (essential) self-adjointness of $H$ is discussed in [2].

### 2.4 The Pauli-Fierz Hamiltonian with spin 1/2

A Hamiltonian which describes a quantum system of non-relativistic charged particles interacting with the quantum radiation filed is called a Pauli-Fierz Hamiltonian [6]. Here
we consider a non-relativistic charged particle with mass \( m \), charge \( q \) and spin \( \frac{1}{2} \). Suppose that the particle is in an external electromagnetic vector potential \( A^{\text{ex}} = (A^1, \phi) \), where \( A^{\text{ex}} := (A_1^{\text{ex}}, A_2^{\text{ex}}, A_3^{\text{ex}}) : \mathbb{R}^3 \to \mathbb{R}^3 \) and \( \phi : \mathbb{R}^3 \to \mathbb{R} \) are Borel measurable and a.e. finite with respect to \( d\mathbf{x} \). Let

\[
\begin{align*}
\sigma_1 &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_2 &:= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_3 &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{align*}
\]

(2.17)

the Pauli spin matrices, and set

\[
\sigma := (\sigma_1, \sigma_2, \sigma_3).
\]

(2.18)

Then the Pauli-Fierz Hamiltonian of this quantum system is defined by

\[
H_{\text{PF}} := \frac{\{\sigma \cdot (-i \nabla - q A^e - q A^{\text{ex}})\}^2}{2m} + \phi + H_{\text{rad}}
\]

(2.19)

acting in the Hilbert space

\[
\mathcal{F}_{\text{PF}} := L^2(\mathbb{R}^3;\mathbb{C}^2) \otimes \mathcal{F}_{\text{rad}} = L^2(\mathbb{R}^3;\mathbb{C}^2) = \int_{\mathbb{R}^3} \mathcal{F}_{\text{rad}} d\mathbf{x}.
\]

(2.20)

### 3 Main Results

#### 3.1 A Dirac operator coupled to the quantum radiation field

We use the following representation of \( \alpha_j \) and \( \beta \) [10, p.3]:

\[
\begin{align*}
\alpha_j &:= \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, & \beta &:= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},
\end{align*}
\]

(3.1)

where \( I_2 \) is the \( 2 \times 2 \) identity matrix. Hence the eigenspaces \( \mathcal{H}^\pm_D \) of \( \beta \) with eigenvalue \( \pm 1 \) take the forms respectively

\[
\mathcal{H}^+_D = \left\{ \begin{pmatrix} f \\ g \\ 0 \end{pmatrix} \right| f, g \in L^2(\mathbb{R}^3) \right\}, \quad \mathcal{H}^-_D = \left\{ \begin{pmatrix} 0 \\ f \\ g \end{pmatrix} \right| f, g \in L^2(\mathbb{R}^3) \right\}.
\]

(3.2)

and we have

\[
\mathcal{H}_D = \mathcal{H}^+_D \oplus \mathcal{H}^-_D.
\]

(3.3)

Let \( P_\pm \) be the orthogonal projections onto \( \mathcal{H}^\pm_D \). Then we have

\[
V = V_0 + V_1
\]

(3.4)

with

\[
V_0 = P_+ V P_+ + P_- V P_-, \quad V_1 = P_+ V P_- + P_- V P_+.
\]

(3.5)
Note that

\[ [V_0, \beta] = 0, \quad \{V_1, \beta\} = 0, \]

where \([A, B] := AB - BA\). In operator-matrix form relative to the orthogonal decomposition (3.3), we have

\[
V_0 = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix},
\]

(3.6)

where \(U_{\pm}\) are \(2 \times 2\) Hermitian matrix-valued functions on \(\mathbb{R}^3\) and \(W\) is a \(2 \times 2\) complex matrix-valued function on \(\mathbb{R}^3\).

Let

\[
\varphi(V_1) := \alpha \cdot (-i \nabla - qA^\rho) + V_1
\]

(3.7)

Then, recalling that \(A_j^\rho\) is \(H_{\text{rad}}^{1/2}\)-bounded [2], we see that \(\varphi(V_1)\) is densely defined and symmetric with

\[
D(\varphi(V_1)) \supset \left( \cap_{j=1}^3 [D(D_j) \cap D(V)] \right) \otimes_{\text{alg}} D(H_{\text{rad}}^{1/2}),
\]

where \(\otimes_{\text{alg}}\) means algebraic tensor product.

By (3.3), we have the following orthogonal decomposition of \(\mathcal{F}\):

\[
\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-,
\]

(3.8)

where

\[
\mathcal{F}_\pm := \mathcal{H}_D^\pm \otimes \mathcal{F}_{\text{rad}} \cong \mathcal{F}_{\text{PF}}.
\]

(3.9)

Relative to this orthogonal decomposition, we can write

\[
\varphi(V_1) = \begin{pmatrix} 0 & D_W \cdot D_W^* \\ D_W^* & 0 \end{pmatrix}
\]

(3.10)

where

\[
D_W := \sigma \cdot (-i \nabla - qA^\rho) + W,
\]

(3.11)

\[
D_{W^*} := \sigma \cdot (-i \nabla - qA^\rho) + W^*
\]

(3.12)

acting in \(\mathcal{F}_{\text{PF}}\).

For a closable linear operator \(T\) on a Hilbert space, we denote its closure by \(\overline{T}\) unless otherwise stated.

Note that \(D_W\) is densely defined as an operator on \(\mathcal{F}_{\text{PF}}\) and \((D_W)^* \supset D_{W^*}\). Hence \((D_W)^*\) is densely defined. Thus \(D_W\) is closable. Based on this fact, we can define

\[
\tilde{\varphi}(V_1) := \begin{pmatrix} 0 & (D_W)^* \\ D_W & 0 \end{pmatrix}.
\]

(3.13)

Lemma 3.1 Under Hypothesis (A), \(\tilde{\varphi}(V_1)\) is a self-adjoint extension of \(\varphi(V_1)\).
3.2 A scaled Dirac-Maxwell operator

For a self-adjoint operator $A$, we denote the spectrum and the spectral measure of $A$ by $\sigma(A)$ and $E_A(\cdot)$ respectively. In the case where $A$ is bounded from below, we set

$$\mathcal{E}_0(A) := \inf \sigma(A), \quad A' := A - \mathcal{E}_0(A) \geq 0.$$  

Let $\Lambda : (0, \infty) \to (0, \infty)$ be a nondecreasing function such that $\Lambda(\kappa) \to \infty$ as $\kappa \to \infty$ and $A$ be a self-adjoint operator on a Hilbert space. Then, for each $\kappa > 0$, we define $A^{(\kappa)}$ by

$$A^{(\kappa)} := \begin{cases} E_{A'}([0, \Lambda(\kappa)]) A' E_{A'}([0, \Lambda(\kappa)]) + \mathcal{E}_0(A) & \text{if } A \text{ is bounded from below} \\ E_{|A|}([0, \Lambda(\kappa)]) |A| E_{|A|}([0, \Lambda(\kappa)]) & \text{if } A \text{ is nonnegative} \\ E_{|A|}([0, \Lambda(\kappa)]) |A| E_{|A|}([0, \Lambda(\kappa)]) & \text{if } A \text{ is not bounded from below} \end{cases}$$

Then $A^{(\kappa)}$ is a bounded self-adjoint operator with

$$\|A^{(\kappa)}\| \leq \Lambda(\kappa).$$

**Proposition 3.2** The following hold:

(i) For all $\psi \in D(A)$, s - $\lim_{\kappa \to \infty} A^{(\kappa)} \psi = A\psi$, where s - $\lim$ means strong limit.

(ii) For all $z \in \mathbb{C} \setminus \mathbb{R}$, s - $\lim_{\kappa \to \infty} (A^{(\kappa)} - z)^{-1} = (A - z)^{-1}$.

With this preliminary, we define for $\kappa > 0$ a scaled Dirac-Maxwell operator

$$H(\kappa) := \kappa \tilde{p}(V_1) + \kappa^2 m \beta - \kappa^2 m + V_{0,\kappa} + H^{(\kappa)}_{\text{rad}},$$

where

$$V_{0,\kappa} := \begin{pmatrix} U^{(\kappa)}_+ & 0 \\ 0 & U^{(\kappa)}_- \end{pmatrix}.$$  

Some remarks may be in order on this definition. The parameter $\kappa$ in $H(\kappa)$ means the speed of light concerning the Dirac particle only. The speed of light related to the external potential $V = V_0 + V_1$ and the quantum radiation field $A^\theta$ is absorbed in them respectively. The third term $-\kappa^2 m$ on the right hand side of (3.16) is a subtraction of the rest energy of the Dirac particle. Hence taking the scaling limit $\kappa \to \infty$ in $H(\kappa)$ in a suitable sense corresponds in fact to a partial non-relativistic limit of the quantum system under consideration.

If one considers the non-relativistic limit in a way similar to the usual Dirac operator $H_D$, then one may define

$$\widehat{H}(\kappa) := \kappa \tilde{p}(V_1) + \kappa^2 m \beta - \kappa^2 m + V_0 + H_{\text{rad}}$$

as a scaled Dirac-Maxwell operator, where no cutoffs on $V_0$ and $H_{\text{rad}}$ are made. In this form, however, we find that, besides the (essential) self-adjointness problem of $\widehat{H}(\kappa)$, the
methods used in the usual Dirac type operators ([10, Chapter 6] or those in [1]) seem not to work. This is because of the existence of the operator $H_{\text{rad}}$ in $\tilde{H}(\kappa)$ which is singular as a perturbation of $H_0(\kappa) := \kappa\tilde{\vartheta}(V_1) + \kappa^2m\beta - \kappa^2m + V_0$ (if one would try to apply the methods on scaling limits discussed in the cited literatures, then one would have to treat $H_{\text{rad}}$ as a perturbation of $H_0(\kappa)$). To avoid this difficulty, we replace $H_{\text{rad}}$ in $\overline{H}(\kappa)$ by a bounded self-adjoint operator which is obtained by cutting off $H_{\text{rad}}$. This is one of the basic ideas of the present paper. We apply the same idea to $V_0$ which also may be singular as a perturbation of $\kappa\tilde{\vartheta}(V_1) + \kappa^2m\beta - \kappa^2m$. In this way we arrive at Definition (3.16) of a scaled Dirac-Maxwell operator.

**Lemma 3.3** Under Hypothesis (A), $H(\kappa)$ is self-adjoint with $D(H(\kappa)) = D(\tilde{\vartheta}(V_1))$.

### 3.3 Self-adjoint extension of the Pauli-Fierz Hamiltonian

Essential self-adjointness of the the Pauli-Fierz Hamiltonian $H_{PF}$ given by (2.19) and its generalizations is discussed in [4, 5]. These papers show that, under additional conditions on $\hat{\vartheta}, \omega, A^\pi$ and $\phi$, the Pauli-Fierz Hamiltonians are essentially self-adjoint. In this note we define a self-adjoint extension of $H_{PF}$, which may not be known before.

We define

$$H_{PF}(\kappa; W, U_+) := \frac{(\overline{D}_W)^* D_W}{2m} + U_+^{(\kappa)} + H_{\text{rad}}^{(\kappa)}, \quad \kappa > 0$$

acting in $\mathcal{F}_{PF}$.

**Lemma 3.4** Under Hypotheses (A), $H_{PF}(\kappa; W, U_+)$ is self-adjoint and bounded from below.

A generalization of the Pauli-Fierz Hamiltonian $H_{PF}$ is defined by

$$H_{PF}(W, U_+) := \frac{D_W \cdot D_W}{2m} + U_+ + H_{\text{rad}}$$

acting in $\mathcal{F}_{PF}$.

We formulate additional conditions:

**Hypothesis (B)**

The function $U_+$ is bounded from below. In this case we set

$$u_0 := \mathcal{E}_0(U_+).$$

**Remark 3.1** Under Hypothesis (A), $D(H_{PF}(W, U_+))$ is not necessarily dense in $\mathcal{F}_{PF}$, but, $D(\tilde{D}_W) \cap D(U_+) \cap D(H_{\text{rad}})$ is dense in $\mathcal{F}_{PF}$. Hence $D(\tilde{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{\text{rad}}^{1/2})$
also is dense in $\mathcal{F}_{PF}$. Therefore we can define a densely defined symmetric form $s_{PF}$ as follows:

$$D(s_{PF}) := D(\bar{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{rad}^{1/2}) \text{ (form domain)},$$

$$s_{PF}(\Psi, \Phi) := \frac{1}{2m}(\bar{D}_W \Psi, \bar{D}_W \Phi) + (\Psi, U_+ \Phi) + (H_{rad}^{1/2} \Psi, H_{rad}^{1/2} \Phi),$$

$$\Psi, \Phi \in D(s_{PF}).$$

(3.21) (3.22) (3.23)

Assume Hypothesis (B) in addition to Hypothesis (A). Then it is easy to see that $s_{PF}$ is closed. Let $H_{PF}^{(f)}$ be the self-adjoint operator associated with $s_{PF}$. Then $H_{PF}^{(f)} \geq u_0$ and $H_{PF}^{(f)}$ is a self-adjoint extension of $H_{PF}(W, U_+)$. 

**Theorem 3.5** Under Hypotheses (A) and (B), there exists a self-adjoint extension of $H_{PF}(W, U_+)$ of $H_{PF}(W, U_+)$ which have the following properties:

(i) $\overline{H}_{PF}(W, U_+) \geq u_0$.

(ii) $D([\overline{H}_{PF}(W, U_+)]^{1/2}) \subset D(\bar{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{rad}^{1/2})$.

(iii) For all $z \in (C \setminus R) \cup \{\xi \in R | \xi < u_0\}$,

$$s - \lim_{\kappa \rightarrow \infty} (H_{PF}(\kappa; W, U_+) - z)^{-1} = (\overline{H}_{PF}(W, U_+) - z)^{-1},$$

where $s - \lim$ means strong limit.

(iv) For all $\xi < u_0$ and $\Psi \in D([\overline{H}_{PF}(W, U_+)]^{1/2})$,

$$s - \lim_{\kappa \rightarrow \infty} (H_{PF}(\kappa; W, U_+) - \xi)^{1/2} \Psi = (\overline{H}_{PF}(W, U_+) - \xi)^{1/2} \Psi.$$ 

**Remark 3.2** As for conditions for $\hat{\rho}$ and $\omega$ for Theorem 3.5 to hold, we only need condition (2.11); no additional conditions is necessary.

**Remark 3.3** In the same manner as in Theorem 3.5, we can define a self-adjoint extension of the Pauli-Fierz Hamiltonian without spin.

**Remark 3.4** Under Hypotheses (A), (B) and that $D(H_{PF}(W, U_+))$ is dense, $H_{PF}(W, U_+)$ is a symmetric operator bounded from below. Hence it has the Friedrichs extension $\overline{H}_{PF}(W, U_+)$. But it is not clear that, in the case where $H_{PF}(W, U_+)$ is not essentially self-adjoint, $\overline{H}_{PF}(W, U_+) = \bar{H}_{PF}(W, U_+)$ or $\overline{H}_{PF}(W, U_+) = H_{PF}^{(f)}$ (Remark 3.1) or both of them do not hold.

### 3.4 Main theorems

We now state main results on the non-relativistic limit of $H(\kappa)$. 

Theorem 3.6 Let Hypotheses (A) and (B) be satisfied. Suppose that
\[ \lim_{\kappa \to \infty} \frac{\Lambda(\kappa)^2}{\kappa} = 0. \] (3.24)

Then, all \( z \in \mathbb{C} \setminus \mathbb{R} \),
\[ s - \lim_{\kappa \to \infty} (H(\kappa) - z)^{-1} = \begin{pmatrix} (\overline{H}_{\text{PF}}(W, U_+) - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \] (3.25)

In the case where \( U_+ \) is not necessarily bounded from below, we have the following.

Theorem 3.7 Let Hypothesis (A) and (3.24) be satisfied. Suppose that \( H_{\text{PF}}(W, U_+) \) is essentially self-adjoint. Then, all \( z \in \mathbb{C} \setminus \mathbb{R} \),
\[ s - \lim_{\kappa \to \infty} (H(\kappa) - z)^{-1} = \begin{pmatrix} (\overline{H}_{\text{PF}}(W, U_+) - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \] (3.26)

Remark 3.5 Under additional conditions on \( \varrho, \omega, W \) and \( U_+ \), one can prove that \( H_{\text{PF}}(W, U_+) \) is essentially self-adjoint for all values of the coupling constant \( q \) [4, 5].

We now apply Theorems 3.6 and 3.7 to the case where \( V = V_{\text{em}} = \phi - q\alpha \cdot A^{\text{ex}} \), i.e., the case where \( W = -q\sigma \cdot A^{\text{ex}} \) and \( U_\pm = \phi I_2 \). We assume the following.

Hypothesis (C)

(C.1) The subspace \( \cap_{j=1}^{3} [D(D_j) \cap D(A^{\text{ex}}_j) \cap D(\phi)] \) is dense in \( L^2(\mathbb{R}^3) \).

(C.2) \( \phi \) is bounded from below. In this case we set \( \phi_0 := \inf \sigma(\phi) \).

Under Hypothesis (C), we have a self-adjoint operator
\[ \overline{H}_{\text{PF}} := \overline{H}_{\text{PF}}(-q\sigma \cdot A^{\text{ex}}, \phi), \] (3.27)
which is a self-adjoint extension of the original Pauli-Fierz Hamiltonian \( H_{\text{PF}} \) given by (2.19).

Let
\[ H_{\text{DM}}(\kappa) := \kappa \overline{\varphi}(-q\alpha \cdot A^{\text{ex}}) + \kappa^2 m\beta - \kappa^2 m + \phi(\kappa) + H^{(\kappa)}_{\text{rad}}, \] (3.28)
Then \( H_{\text{DM}}(\kappa) \) is the Dirac-Maxwell operator \( H(\kappa) \) with \( V_1 = -q\alpha \cdot A^{\text{ex}} \) and \( V_0 = \phi \)

Theorems 3.6 and 3.7 immediately yield the following results on the non-relativistic limit of \( H_{\text{DM}}(\kappa) \).

Corollary 3.8 Let Hypothesis (C) and (3.24) be satisfied. Then, for all \( z \in \mathbb{C} \setminus \mathbb{R} \),
\[ s - \lim_{\kappa \to \infty} (H_{\text{DM}}(\kappa) - z)^{-1} = \begin{pmatrix} (\overline{H}_{\text{PF}} - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \] (3.29)
Corollary 3.9 Assume (C.1) and (3.24). Suppose that $H_{PF}$ is essentially self-adjoint. Then, all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$s - \lim_{\kappa \to \infty} (H_{DM}(\kappa) - z)^{-1} = \begin{pmatrix} (H_{PF} - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

(3.30)

Thus a mathematically rigorous connection of relativistic QED to non-relativistic QED is established.

Proofs of these results are given in [3]. The method used is an extension of a theory [1] of scaling limits of strongly anticommuting self-adjoint operators.

References


