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Non-relativistic Limit of a Dirac particle Interacting with the Quantum Radiation Field

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Abstract
The non-relativistic (scaling) limit of a Hamiltonian of a Dirac particle interacting with the quantum radiation field yields a self-adjoint extension of the Pauli-Fierz Hamiltonian with spin 1/2 in non-relativistic quantum electrodynamics.

Keywords: quantum electrodynamics, Dirac operator, Dirac-Maxwell operator, Pauli-Fierz Hamiltonian, non-relativistic limit, scalig limit, Fock space, strongly anticommuting self-adjoint operators

1 Introduction

A Hamiltonian H of a Dirac particle — a relativistic charged particle with spin 1/2 — interacting with the quantum radiation field is called a *Dirac-Maxwell operator*. In this note we report a result on the non-relativistic limit of H.

The Dirac-Maxwell operator H is of the form $H = H_{\mathrm{D}} + H_{\mathrm{rad}} + H_I$, where H_{D} is a Dirac opeartor describing the Dirac particle system only, H_{rad} is the free Hamiltonian of the quantum radiation field (a quantum version of the Maxwell Hamiltonian in the Coulomb gauge) and H_I is the interaction term beween the Dirac particle and the quantum radiation field. As for the Dirac operator H_{D}, the non-relativistic limit has already been investigated and well understood ([10, Chapter 6] and references therein). We extend the methods used in the case of the Dirac operator H_{D} to the case of H. This can be done in an abstract framework with further developments of the theory of scaling limits on strongly anticommuting self-adjoint operators [1]. The main result we report in this note is that the non-relativistic limit of H yields a self-adjoint extension of the Pauli-Fierz Hamiltonian with spin 1/2 in non-relativistic quantum electrodynamics.

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2 The Dirac-Maxwell Operator and The Pauli-Fierz Hamiltonian

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$, and its adjoint by T^* (provided that T is densely defined). For two objects $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ such that products $a_j b_j$ $(j = 1, 2, 3)$ and their sum can be defined, we set $\boldsymbol{a} \cdot \boldsymbol{b} := \sum_{j=1}^{3} a_j b_j$.

We use the physical unit system in which c(the speed of light)$= 1$ and $\hbar = 1$ ($\hbar := h/(2\pi)$; h is the Planck constant).

2.1 The Dirac operator

Let D_j $(j = 1, 2, 3)$ be the generalized partial differential operator in the variable x_j, the j-th component of $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, and $\nabla := (D_1, D_2, D_3)$.

We denote the mass and the charge of the Dirac particle by $m > 0$ and $q \in \mathbf{R} \setminus \{0\}$ respectively. We consider the situation where the Dirac particle is in a potential V which is a *Hermitian-matrix-valued Borel measurable function* on $\mathbf{R}^3$. Then the Hamiltonian of the Dirac particle is given by the Dirac operator

$$H_{\mathrm{D}} := \boldsymbol{\alpha} \cdot (-i\nabla) + m\beta + V \tag{2.1}$$

acting in the Hilbert space

$$\mathcal{H}_{\mathrm{D}} := \oplus^4 L^2(\mathbf{R}^3) \tag{2.2}$$

with domain $D(H_{\mathrm{D}}) := [\oplus^4 H^1(\mathbf{R}^3)] \cap D(V)$ ($H^1(\mathbf{R}^3)$ is the Sobolev space of order 1), where α_j $(j = 1, 2, 3)$ and β are 4×4 Hermitian matrices satisfying the anticommutation relations

$$\{\alpha_j,\, \alpha_k\} = 2\delta_{jk}, \quad j, k = 1, 2, 3, \tag{2.3}$$

$$\{\alpha_j,\, \beta\} = 0, \quad \beta^2 = 1, \quad j = 1, 2, 3, \tag{2.4}$$

$\{A, B\} := AB + BA$ and δ_{jk} is the Kronecker delta. We assume the following:

Hypothesis (A)

Each matrix element of V is almost everywhere (a.e.) finite with respect to the three-dimensional Lebesgue measure $d\boldsymbol{x}$ and the subspace $\cap_{j=1}^{3} [D(D_j) \cap D(V)]$ is dense in $\mathcal{H}_{\mathrm{D}}$.

Under this hypothesis, H_{D} is a symmetric operator. For detailed analyses of the Dirac operator, see, e.g., [10].

2.2 The quantum radiation field

The Hilbert space of one-photon states in momentum representation is given by

$$\mathcal{H}_{\rm ph} := L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^3), \tag{2.5}$$

where $\mathbf{R}^3 := \{\boldsymbol{k} = (k_1, k_2, k_3) | k_j \in \mathbf{R},\ j = 1,2,3\}$ physically means the momentum space of photons. Then a Hilbert space for the quantum radiation field in the Coulomb gauge is given by

$$\mathcal{F}_{\rm rad} := \oplus_{n=0}^{\infty} \otimes_{\rm s}^{n} \mathcal{H}_{\rm ph}, \tag{2.6}$$

the Boson Fock space over $\mathcal{H}_{\rm ph}$, where $\otimes_{\rm s}^n \mathcal{H}_{\rm ph}$ denotes the n-fold symmetric tensor product of $\mathcal{H}_{\rm ph}$ and $\otimes_{\rm s}^0 \mathcal{H}_{\rm ph} := \mathbf{C}$. For basic facts on the theory of the Boson Fock space, we refer the reader to [8, §X.7].

We denote by $a(F)$ ($F \in \mathcal{H}_{\rm ph}$) the annihilation operator with test vector F on $\mathcal{F}_{\rm rad}$; its adjoint is given by

$$(a(F)^*\Psi)^{(n)} = \sqrt{n} S_n (F \otimes \Psi^{(n-1)}), \quad n \geq 0, \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in D(a(F)^*),$$

where S_n is the symmetrization operator on $\otimes^n \mathcal{H}_{\rm ph}$ and $\Psi^{-1} := 0$.

For each $f \in L^2(\mathbf{R}^3)$, we define

$$a^{(1)}(f) := a(f,0), \quad a^{(2)}(f) := a(0,f). \tag{2.7}$$

The mapping : $f \to a^{(r)}(f^*)$ restricted to $\mathcal{S}(\mathbf{R}^3)$ (the Schwartz space of rapidly decreasing C^∞-functions on $\mathbf{R}^3$) defines an operator-valued distribution (f^* denotes the complex conjugate of f). We denote its symbolical kernel by $a^{(r)}(\boldsymbol{k})$: $a^{(r)}(f) = \int a^{(r)}(\boldsymbol{k}) f(\boldsymbol{k})^* d\boldsymbol{k}$.

We take a nonnegative Borel measurable function ω on $\mathbf{R}^3$ to denote the one free photon energy. We assume that, for a.e. $\boldsymbol{k} \in \mathbf{R}^3$ with respect to the Lebesgue measure on $\mathbf{R}^3$, $0 < \omega(\boldsymbol{k}) < \infty$. Then the function ω defines uniquely a multiplication operator on $\mathcal{H}_{\rm ph}$ which is nonnegative, self-adjoint and injective. We denote it by the same symbol ω. The free Hamiltonian of the quantum radiation field is then defined by

$$H_{\rm rad} := d\Gamma(\omega), \tag{2.8}$$

the second quantization of ω [7, p.302, Example 2] and [8, §X.7]. The operator $H_{\rm rad}$ is a nonnegative self-adjoint operator. The symbolical expression of $H_{\rm rad}$ is $H_{\rm rad} = \sum_{r=1}^2 \int \omega(\boldsymbol{k}) a^{(r)}(\boldsymbol{k})^* a^{(r)}(\boldsymbol{k}) d\boldsymbol{k}$.

Remark 2.1 Usually ω is taken to be of the form $\omega_{\rm phys}(\boldsymbol{k}) := |\boldsymbol{k}|, \quad \boldsymbol{k} \in \mathbf{R}^3$, but, in this paper, for mathematical generality, we do not restrict ourselves to this case.

There exist $\mathbf{R}^3$-valued Borel measurable functions $\mathbf{e}^{(r)}$ ($r = 1,2$) on $\mathbf{R}^3$ such that, for a.e. $\boldsymbol{k}$

$$\mathbf{e}^{(r)}(\boldsymbol{k}) \cdot \mathbf{e}^{(s)}(\boldsymbol{k}) = \delta_{rs}, \quad \mathbf{e}^{(r)}(\boldsymbol{k}) \cdot \boldsymbol{k} = 0, \quad r,s = 1,2. \tag{2.9}$$

These vector-valued functions $\mathbf{e}^{(r)}$ are called the polarization vectors of a photon.

The time-zero quantum radiation field is given by $\boldsymbol{A}(\boldsymbol{x}) := (A_1(\boldsymbol{x}), A_2(\boldsymbol{x}), A_3(\boldsymbol{x}))$ with

$$A_j(\boldsymbol{x}) := \sum_{r=1}^{2} \int d\boldsymbol{k} \frac{e_j^{(r)}(\boldsymbol{k})}{\sqrt{2(2\pi)^3\omega(\boldsymbol{k})}} \left\{ a^{(r)}(\boldsymbol{k})^* e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} + a^{(r)}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \right\}, \quad j = 1,2,3, \tag{2.10}$$

in the sense of operator-valued distribution.

Let ϱ be a real tempered distribution on $\mathbf{R}^3$ such that

$$\frac{\hat{\varrho}}{\sqrt{\omega}}, \quad \frac{\hat{\varrho}}{\omega} \in L^2(\mathbf{R}^3), \tag{2.11}$$

where $\hat{\varrho}$ denotes the Fourier transform of ϱ. The quantum radiation field

$$\boldsymbol{A}^{\varrho} := (A_1^{\varrho}, A_2^{\varrho}, A_3^{\varrho}) \tag{2.12}$$

with momentum cutoff $\hat{\varrho}$ is defined by

$$A_j^{\varrho}(\boldsymbol{x}) := \sum_{r=1}^{2} \int d\boldsymbol{k} \frac{e_j^{(r)}(\boldsymbol{k})}{\sqrt{2\omega(\boldsymbol{k})}} \left\{ a^{(r)}(\boldsymbol{k})^* e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\varrho}(\boldsymbol{k})^* + a^{(r)}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\varrho}(\boldsymbol{k}) \right\}. \tag{2.13}$$

Symbolically $A_j^{\varrho}(\boldsymbol{x}) = \int A_j(\boldsymbol{x} - \boldsymbol{y})\varrho(\boldsymbol{y})d\boldsymbol{y}$.

2.3 The Dirac-Maxwell operator

The Hilbert space of state vectors for the coupled system of the Dirac particle and the quantum radiation field is taken to be

$$\mathcal{F} := \mathcal{H}_{\mathrm{D}} \otimes \mathcal{F}_{\mathrm{rad}}. \tag{2.14}$$

This Hilbert space can be identified as

$$\mathcal{F} = L^2(\mathbf{R}^3; \oplus^4\mathcal{F}_{\mathrm{rad}}) = \int_{\mathbf{R}^3}^{\oplus} \oplus^4\mathcal{F}_{\mathrm{rad}} d\boldsymbol{x} \tag{2.15}$$

the Hilbert space of $\oplus^4\mathcal{F}_{\mathrm{rad}}$-valued Lebesgue square integrable functions on $\mathbf{R}^3$ (the constant fibre direct integral with base space $(\mathbf{R}^3, d\boldsymbol{x})$ and fibre $\oplus^4\mathcal{F}_{\mathrm{rad}}$ [9, §XIII.6]). We freely use this identification. The total Hamiltonian of the coupled system — a *Dirac-Maxwell operator* — is defined by

$$H := H_{\mathrm{D}} + H_{\mathrm{rad}} - q\boldsymbol{\alpha}\cdot\boldsymbol{A}^{\varrho} = \boldsymbol{\alpha}\cdot(-i\nabla - q\boldsymbol{A}^{\varrho}) + m\beta + V + H_{\mathrm{rad}}. \tag{2.16}$$

The (essential) self-adjointness of H is discussed in [2].

2.4 The Pauli-Fierz Hamiltonian with spin 1/2

A Hamiltonian which describes a quantum system of non-relativistic charged particles interacting with the quantum radiation filed is called a Pauli-Fierz Hamiltonian [6]. Here we consider a non-relativistic charged particle with mass m, charge q and spin 1/2. Suppose that the particle is in an external electromagnetic vector potential $A^{\mathrm{ex}} = (\boldsymbol{A}^{\mathrm{ex}}, \phi)$, where $\boldsymbol{A}^{\mathrm{ex}} := (A_1^{\mathrm{ex}}, A_2^{\mathrm{ex}}, A_3^{\mathrm{ex}}) : \mathbf{R}^3 \to \mathbf{R}^3$ and $\phi : \mathbf{R}^3 \to \mathbf{R}$ are Borel measurable and a.e. finite with respect to $d\boldsymbol{x}$. Let

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.17}$$

the Pauli spin matrices, and set

$$\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3). \tag{2.18}$$

Then the Pauli-Fierz Hamiltonian of this quantum system is defined by

$$H_{\mathrm{PF}} := \frac{\{\boldsymbol{\sigma} \cdot (-i\nabla - q\boldsymbol{A}^{\varrho} - q\boldsymbol{A}^{\mathrm{ex}})\}^2}{2m} + \phi + H_{\mathrm{rad}} \tag{2.19}$$

acting in the Hilbert space

$$\mathcal{F}_{\mathrm{PF}} := L^2(\mathbf{R}^3; \mathbf{C}^2) \otimes \mathcal{F}_{\mathrm{rad}} = L^2(\mathbf{R}^3; \oplus^2 \mathcal{F}_{\mathrm{rad}}) = \int_{\mathbf{R}^3}^{\oplus} \oplus^2 \mathcal{F}_{\mathrm{rad}} d\boldsymbol{x}. \tag{2.20}$$

3 Main Results

3.1 A Dirac operator coupled to the quantum radiation field

We use the following representation of α_j and β [10, p.3]:

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{3.1}$$

where I_2 is the 2×2 identity matrix. Hence the eigenspaces $\mathcal{H}_{\mathrm{D}}^{\pm}$ of β with eigenvalue ± 1 take the forms respectively

$$\mathcal{H}_D^+ = \left\{ \begin{pmatrix} f \\ g \\ 0 \\ 0 \end{pmatrix} \middle| f, g \in L^2(\mathbf{R}^3) \right\}, \quad \mathcal{H}_D^- = \left\{ \begin{pmatrix} 0 \\ 0 \\ f \\ g \end{pmatrix} \middle| f, g \in L^2(\mathbf{R}^3) \right\}. \tag{3.2}$$

and we have

$$\mathcal{H}_{\mathrm{D}} = \mathcal{H}_{\mathrm{D}}^+ \oplus \mathcal{H}_{\mathrm{D}}^-. \tag{3.3}$$

Let $P_\pm$ be the orthogonal projections onto $\mathcal{H}_{\mathrm{D}}^{\pm}$. Then we have

$$V = V_0 + V_1 \tag{3.4}$$

with

$$V_0 = P_+ V P_+ + P_- V P_-, \quad V_1 = P_+ V P_- + P_- V P_+. \tag{3.5}$$

Note that

$$[V_0, \beta] = 0, \quad \{V_1, \beta\} = 0,$$

where $[A, B] := AB - BA$. In operator-matrix form relative to the orthogonal decomposition (3.3), we have

$$V_0 = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix}, \tag{3.6}$$

where $U_\pm$ are 2×2 Hermitian matrix-valued functions on $\mathbf{R}^3$ and W is a 2×2 complex matrix-valued function on $\mathbf{R}^3$.

Let

$$\not{D}(V_1) := \boldsymbol{\alpha} \cdot (-i\nabla - q\boldsymbol{A}^\varrho) + V_1 \tag{3.7}$$

Then, recalling that A_j^ϱ is $H_{\mathrm{rad}}^{1/2}$-bounded [2], we see that $\not{D}(V_1)$ is densely defined and symmetric with $D(\not{D}(V_1)) \supset \left(\cap_{j=1}^3 [D(D_j) \cap D(V)]\right) \otimes_{\mathrm{alg}} D(H_{\mathrm{rad}}^{1/2})$, where $\otimes_{\mathrm{alg}}$ means algebraic tensor product.

By (3.3), we have the following orthogonal decomposition of $\mathcal{F}$:

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-, \tag{3.8}$$

where

$$\mathcal{F}_\pm := \mathcal{H}_{\mathrm{D}}^\pm \otimes \mathcal{F}_{\mathrm{rad}} \cong \mathcal{F}_{\mathrm{PF}}. \tag{3.9}$$

Relative to this orthogonal decomposition, we can write

$$\not{D}(V_1) = \begin{pmatrix} 0 & D_{W^*} \\ D_W & 0 \end{pmatrix}, \tag{3.10}$$

where

$$\begin{aligned} D_W &:= \boldsymbol{\sigma} \cdot (-i\nabla - q\boldsymbol{A}^\varrho) + W, & (3.11) \\ D_{W^*} &:= \boldsymbol{\sigma} \cdot (-i\nabla - q\boldsymbol{A}^\varrho) + W^* & (3.12) \end{aligned}$$

acting in $\mathcal{F}_{\mathrm{PF}}$.

For a closable linear operator T on a Hilbert space, we denote its closure by $\bar{T}$ unless otherwise stated.

Note that D_W is densely defined as an operator on $\mathcal{F}_{\mathrm{PF}}$ and $(D_W)^* \supset D_{W^*}$. Hence $(D_W)^*$ is densely defined. Thus D_W is closable. Based on this fact, we can define

$$\tilde{\not{D}}(V_1) := \begin{pmatrix} 0 & \left(\overline{D}_W\right)^* \\ \overline{D}_W & 0 \end{pmatrix}. \tag{3.13}$$

Lemma 3.1 *Under Hypothesis (A), $\tilde{\not{D}}(V_1)$ is a self-adjoint extension of $\not{D}(V_1)$.*

3.2 A scaled Dirac-Maxwell operator

For a self-adjoint operator A, we denote the spectrum and the spectral measure of A by $\sigma(A)$ and $E_A(\cdot)$ respectively. In the case where A is bounded from below, we set

$$\mathcal{E}_0(A) := \inf \sigma(A), \quad A' := A - \mathcal{E}_0(A) \geq 0.$$

Let $\Lambda : (0,\infty) \to (0,\infty)$ be a nondecreasing function such that $\Lambda(\kappa) \to \infty$ as $\kappa \to \infty$ and A be a self-adjoint operator on a Hilbert space. Then, for each $\kappa > 0$, we define $A^{(\kappa)}$ by

$$A^{(\kappa)} := \begin{cases} E_{A'}([0,\Lambda(\kappa)])A'E_{A'}([0,\Lambda(\kappa)]) + \mathcal{E}_0(A) & \text{if } A \text{ is bounded from below and } \mathcal{E}_0(A) < 0 \\ E_{|A|}([0,\Lambda(\kappa)])AE_{|A|}([0,\Lambda(\kappa)]) & \text{if } A \text{ is nonnegative or } A \text{ is not bounded from below} \end{cases} \tag{3.14}$$

Then $A^{(\kappa)}$ is a bounded self-adjoint operator with

$$\|A^{(\kappa)}\| \leq \Lambda(\kappa). \tag{3.15}$$

Proposition 3.2 *The following hold:*

(i) *For all* $\psi \in D(A)$, s - $\lim_{\kappa\to\infty} A^{(\kappa)}\psi = A\psi$, *where* s - lim *means strong limit.*

(ii) *For all* $z \in \mathbf{C} \setminus \mathbf{R}$, s - $\lim_{\kappa\to\infty}(A^{(\kappa)} - z)^{-1} = (A - z)^{-1}$.

With this preliminary, we define for $\kappa > 0$ a scaled Dirac-Maxwell operator

$$H(\kappa) := \kappa\tilde{\not{D}}(V_1) + \kappa^2 m\beta - \kappa^2 m + V_{0,\kappa} + H_{\text{rad}}^{(\kappa)}, \tag{3.16}$$

where

$$V_{0,\kappa} := \begin{pmatrix} U_+^{(\kappa)} & 0 \\ 0 & U_-^{(\kappa)} \end{pmatrix}. \tag{3.17}$$

Some remarks may be in order on this definition. The parameter κ in $H(\kappa)$ means the speed of light *concerning the Dirac particle only.* The speed of light related to the external potential $V = V_0 + V_1$ and the quantum radiation field $\boldsymbol{A}^\varrho$ is absorbed in them respectively. The third term $-\kappa^2 m$ on the right hand side of (3.16) is a subtraction of the rest energy of the Dirac particle. Hence taking the scaling limit $\kappa \to \infty$ in $H(\kappa)$ in a suitable sense corresponds in fact to a *partial* non-relativistic limit of the quantum system under consideration.

If one considers the non-relativistic limit in a way similar to the usual Dirac operator H_{D}, then one may define

$$\widehat{H}(\kappa) := \kappa\tilde{\not{D}}(V_1) + \kappa^2 m\beta - \kappa^2 m + V_0 + H_{\text{rad}} \tag{3.18}$$

as a scaled Dirac-Maxwell operator, where no cutoffs on V_0 and H_{rad} are made. In this form, however, we find that, besides the (essential) self-adjointness problem of $\widehat{H}(\kappa)$, the methods used in the usual Dirac type operators ([10, Chapter 6] or those in [1]) seem not to work. This is because of the existence of the operator $H_{\rm rad}$ in $\widehat{H}(\kappa)$ which is singular as a perturbation of $H_0(\kappa) := \kappa\widetilde{\not{D}}(V_1) + \kappa^2 m\beta - \kappa^2 m + V_0$ (if one would try to apply the methods on sacaling limits discussed in the cited literatures, then one would have to treat $H_{\rm rad}$ as a perturbation of $H_0(\kappa)$). To avoid this difficulty, we replace $H_{\rm rad}$ in $\widehat{H}(\kappa)$ by a bounded self-adjoint operator which is obtained by cutting off $H_{\rm rad}$. This is one of the basic ideas of the present paper. We apply the same idea to V_0 which also may be singular as a perturbation of $\kappa\widetilde{\not{D}}(V_1) + \kappa^2 m\beta - \kappa^2 m$. In this way we arrive at Definition (3.16) of a scaled Dirac-Maxwell operator.

Lemma 3.3 *Under Hypothesis (A), $H(\kappa)$ is self-adjoint with $D(H(\kappa)) = D(\widetilde{\not{D}}(V_1))$.*

3.3 Self-adjoint extension of the Pauli-Fierz Hamiltonian

Essential self-adjointness of the the Pauli-Fierz Hamiltonian $H_{\rm PF}$ given by (2.19) and its generalizations is discussed in [4, 5]. These papers show that, under additional conditions on $\hat{\varrho}, \omega, \boldsymbol{A}^{\rm ex}$ and ϕ, the Pauli-Fierz Hamiltonians are essentially self-adjoint. In this note we define a self-adjoint extension of $H_{\rm PF}$, which may not be known before.

We define

$$H_{\rm PF}(\kappa; W, U_+) := \frac{\left(\overline{D}_W\right)^* \overline{D}_W}{2m} + U_+^{(\kappa)} + H_{\rm rad}^{(\kappa)}, \quad \kappa > 0 \tag{3.19}$$

acting in $\mathcal{F}_{\rm PF}$.

Lemma 3.4 *Under Hypotheses (A), $H_{\rm PF}(\kappa; W, U_+)$ is self-adjoint and bounded from below.*

A generalization of the Pauli-Fierz Hamiltonian $H_{\rm PF}$ is defined by

$$H_{\rm PF}(W, U_+) := \frac{D_{W^*} D_W}{2m} + U_+ + H_{\rm rad} \tag{3.20}$$

acting in $\mathcal{F}_{\rm PF}$.

We formulate additional conditions:

Hypothesis (B)

The function U_+ is bounded from below. In this case we set

$$u_0 := \mathcal{E}_0(U_+).$$

Remark 3.1 Under Hypothesis (A), $D(H_{\rm PF}(W, U_+))$ is not necessarily dense in $\mathcal{F}_{\rm PF}$, but, $D(\bar{D}_W) \cap D(U_+) \cap D(H_{\rm rad})$ is dense in $\mathcal{F}_{\rm PF}$. Hence $D(\bar{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{\rm rad}^{1/2})$

also is dense in $\mathcal{F}_{\rm PF}$. Therefore we can define a densely defined symmetric form $\mathfrak{s}_{\rm PF}$ as follows:

$$D(\mathfrak{s}_{\rm PF}) := D(\bar{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{\rm rad}^{1/2}) \text{ (form domain)}, \tag{3.21}$$

$$\mathfrak{s}_{\rm PF}(\Psi, \Phi) := \frac{1}{2m}(\bar{D}_W\Psi, \bar{D}_W\Phi) + (\Psi, U_+\Phi) + (H_{\rm rad}^{1/2}\Psi, H_{\rm rad}^{1/2}\Phi), \tag{3.22}$$

$$\Psi, \Phi \in D(\mathfrak{s}_{\rm PF}). \tag{3.23}$$

Assume Hypothesis (B) in addition to Hypothesis (A). Then it is easy to see that $\mathfrak{s}_{\rm PF}$ is closed. Let $H_{\rm PF}^{(\rm f)}$ be the self-adjoint operator associated with $\mathfrak{s}_{\rm PF}$. Then $H_{\rm PF}^{(\rm f)} \geq u_0$ and $H_{\rm PF}^{(\rm f)}$ is a self-adjoint extension of $H_{\rm PF}(W, U_+)$.

Theorem 3.5 *Under Hypotheses (A) and (B), there exists a self-adjoint extension of $\widetilde{H}_{\rm PF}(W, U_+)$ of $H_{\rm PF}(W, U_+)$ which have the following properties:*

(i) $\widetilde{H}_{\rm PF}(W, U_+) \geq u_0$.

(ii) $D(|\widetilde{H}_{\rm PF}(W, U_+)|^{1/2}) \subset D(\overline{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{\rm rad}^{1/2})$

(iii) *For all $z \in (\mathbf{C} \setminus \mathbf{R}) \cup \{\xi \in \mathbf{R} | \xi < u_0\}$,*

$$\text{s - }\lim_{\kappa \to \infty}(H_{\rm PF}(\kappa; W, U_+) - z)^{-1} = (\widetilde{H}_{\rm PF}(W, U_+) - z)^{-1},$$

where s - lim *means strong limit.*

(iv) *For all $\xi < u_0$ and $\Psi \in D(|\widetilde{H}_{\rm PF}(W, U_+)|^{1/2})$,*

$$\text{s - }\lim_{\kappa \to \infty}(H_{\rm PF}(\kappa; W, U_+) - \xi)^{1/2}\Psi = (\widetilde{H}_{\rm PF}(W, U_+) - \xi)^{1/2}\Psi.$$

Remark 3.2 As for conditions for $\hat{\rho}$ and ω for Theorem 3.5 to hold, we only need condition (2.11); no additional conditions is necessary.

Remark 3.3 In the same manner as in Theorem 3.5, we can define a self-adjoint extension of the Pauli-Fierz Hamiltonian without spin.

Remark 3.4 Under Hypotheses (A), (B) and that $D(H_{\rm PF}(W, U_+))$ is dense, $H_{\rm PF}(W, U_+)$ is a symmetric operator bounded from below. Hence it has the Friedrichs extension $\widehat{H}_{\rm PF}(W, U_+)$. But it is not clear that, in the case where $H_{\rm PF}(W, U_+)$ is not essentially self-adjoint, $\widetilde{H}_{\rm PF}(W, U_+) = \widehat{H}_{\rm PF}(W, U_+)$ or $\widetilde{H}_{\rm PF}(W, U_+) = H_{\rm PF}^{(\rm f)}$ (Remark 3.1) or both of them do not hold.

3.4 Main theorems

We now state main results on the non-relativistic limit of $H(\kappa)$.

Theorem 3.6 *Let Hypotheses (A) and (B) be satisfied. Suppose that*

$$\lim_{\kappa\to\infty}\frac{\Lambda(\kappa)^2}{\kappa}=0. \tag{3.24}$$

Then, all $z\in\mathbf{C}\setminus\mathbf{R}$,

$$\text{s - }\lim_{\kappa\to\infty}(H(\kappa)-z)^{-1}=\begin{pmatrix}\left(\widetilde{H}_{\mathrm{PF}}(W,U_+)-z\right)^{-1} & 0\\ 0 & 0\end{pmatrix}. \tag{3.25}$$

In the case where U_+ is not necessarily bounded from below, we have the following.

Theorem 3.7 *Let Hypothesis (A) and (3.24) be satisfied. Suppose that $H_{\mathrm{PF}}(W,U_+)$ is essentially self-adjoint. Then, all $z\in\mathbf{C}\setminus\mathbf{R}$,*

$$\text{s - }\lim_{\kappa\to\infty}(H(\kappa)-z)^{-1}=\begin{pmatrix}\left(\overline{H_{\mathrm{PF}}(W,U_+)}-z\right)^{-1} & 0\\ 0 & 0\end{pmatrix}. \tag{3.26}$$

Remark 3.5 Under additional conditions on ϱ,ω,W and U_+, one can prove that $H_{\mathrm{PF}}(W,U_+)$ is essentially self-adjoint for all values of the coupling constant q [4, 5].

We now apply Theorems 3.6 and 3.7 to the case where $V=V_{\mathrm{em}}=\phi-q\boldsymbol{\alpha}\cdot\boldsymbol{A}^{\mathrm{ex}}$, i.e., the case where $W=-q\boldsymbol{\sigma}\cdot\boldsymbol{A}^{\mathrm{ex}}$ and $U_\pm=\phi I_2$. We assume the following.

Hypothesis (C)

(C.1) The subspace $\cap_{j=1}^3[D(D_j)\cap D(A_j^{\mathrm{ex}})\cap D(\phi)]$ is dense in $L^2(\mathbf{R}^3)$.

(C.2) ϕ is bounded from below. In this case we set $\phi_0:=\inf\sigma(\phi)$.

Under Hypothesis (C), we have a self-adjoint opeartor

$$\widetilde{H}_{\mathrm{PF}}:=\widetilde{H}_{\mathrm{PF}}(-q\boldsymbol{\sigma}\cdot\boldsymbol{A}^{\mathrm{ex}},\phi), \tag{3.27}$$

which is a self-adjoint extension of the original Pauli-Fierz Hamiltonian H_{PF} given by (2.19).

Let

$$H_{\mathrm{DM}}(\kappa):=\kappa\not{D}(-q\boldsymbol{\alpha}\cdot\boldsymbol{A}^{\mathrm{ex}})+\kappa^2m\beta-\kappa^2m+\phi^{(\kappa)}+H_{\mathrm{rad}}^{(\kappa)}, \tag{3.28}$$

Then $H_{\mathrm{DM}}(\kappa)$ is the Dirac-Maxwell operator $H(\kappa)$ with $V_1=-q\boldsymbol{\alpha}\cdot\boldsymbol{A}^{\mathrm{ex}}$ and $V_0=\phi$.

Theorems 3.6 and 3.7 immediately yield the following results on the non-relativistic limit of $H_{\mathrm{DM}}(\kappa)$.

Corollary 3.8 *Let Hypothesis (C) and (3.24) be satisfied. Then, for all $z\in\mathbf{C}\setminus\mathbf{R}$,*

$$\text{s - }\lim_{\kappa\to\infty}(H_{\mathrm{DM}}(\kappa)-z)^{-1}=\begin{pmatrix}\left(\widetilde{H}_{\mathrm{PF}}-z\right)^{-1} & 0\\ 0 & 0\end{pmatrix}. \tag{3.29}$$

Corollary 3.9 *Assume (C.1) and (3.24). Suppose that $H_{\rm PF}$ is essentially self-adjoint. Then, all $z \in \mathbf{C} \setminus \mathbf{R}$,*

$$\mathrm{s}\text{ - }\lim_{\kappa\to\infty}(H_{\rm DM}(\kappa)-z)^{-1} = \begin{pmatrix} \left(\overline{H}_{\rm PF}-z\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.30}$$

Thus a mathematically rigorous connection of relativistic QED to non-relativistic QED is established.

Proofs of these results are given in [3]. The method used is an extension of a theory [1] of scaling limits of strongly anticommuting self-adjoint operators.

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