

A System of Stochastic Differential Equations within NETFD — Can *Quantumness* be an Origin of Dissipation? —

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Most of the mathematical approaches for quantum Langevin equation are based on the non-commutativity of the random force operators. Non-commutative random force operators are introduced in order to guarantee that the equal-time commutation relation for the stochastic annihilation and creation operators preserves in time. If it is true, it means that the origin of dissipation is of quantum mechanical. However, physically, it is hard to believe it. By making use of the unified canonical operator formalism for the system of the quantum stochastic differential equations within Non-Equilibrium Thermo Field Dynamics, it is shown that it is not true in general.

I. INTRODUCTION

The studies of the Langevin equation for quantum systems were started in connection with the development of laser [1–3], and are still continuing in order to develop a satisfactory formulation [4–9] (see comments in [10]). Most of the mathematical approaches for quantum Langevin equation are based on the non-commutativity of the random force operators. For dissipative systems, for example, we have equations for the operators $\langle a(t) \rangle$ and $\langle a^\dagger(t) \rangle$ averaged with respect to random force operators of the forms

$$\frac{d}{dt}\langle a(t) \rangle = -i\omega\langle a(t) \rangle - \kappa\langle a(t) \rangle, \quad (1)$$

$$\frac{d}{dt}\langle a^\dagger(t) \rangle = i\omega\langle a^\dagger(t) \rangle - \kappa\langle a^\dagger(t) \rangle, \quad (2)$$

with the initial condition

$$\langle a(0) \rangle = a, \quad \langle a^\dagger(0) \rangle = a^\dagger, \quad (3)$$

where a and a^\dagger satisfy the canonical commutation relation

$$[a, a^\dagger] = 1. \quad (4)$$

The equal-time commutation relation for these operators decays in time:

$$[\langle a(t) \rangle, \langle a^\dagger(t) \rangle] = e^{-2\kappa t}. \quad (5)$$

Random force operators $df(t)$ and $df^\dagger(t)$ are introduced in order to rescue this situation. If the random force operators in the Langevin equations

$$da(t) = -i\omega a(t)dt - \kappa a(t)dt + \sqrt{2\kappa} df(t), \quad (6)$$

$$da^\dagger(t) = i\omega a^\dagger(t)dt - \kappa a^\dagger(t)dt + \sqrt{2\kappa} df^\dagger(t), \quad (7)$$

satisfy

$$[df(t), df^\dagger(t)] = dt, \quad (8)$$

the equal-time commutation relation for the stochastic operators $a(t)$ and $a^\dagger(t)$ preserves in time:

$$d([\langle a(t) \rangle, \langle a^\dagger(t) \rangle]) = 0, \quad (9)$$

meaning that

$$[\langle a(t) \rangle, \langle a^\dagger(t) \rangle] = 1, \quad (10)$$

The above argument is of zero temperature related only to the zero-point fluctuation. However, it has been extended to include the situations for finite temperature. Then, we have a crucial question. Should we interpret that the origin of thermal dissipation is quantum mechanical? In this paper, we will investigate this question with the help of the system of the stochastic differential equations within Non-Equilibrium Thermo Field Dynamics (NETFD) [11–38].

NETFD is a *canonical operator formalism* of quantum systems in far-from-equilibrium state which provides us with a unified formulation for dissipative systems by the method similar to the usual quantum field theory that accommodates the concept of the dual structure in the interpretation of nature, i.e. in terms of the *operator algebra* and the *representation space*. The representation space of NETFD (named *thermal space*) is composed of the direct product of two Hilbert spaces, the one for *non-tilde* fields and the other for *tilde* fields. It can be said that NETFD is a framework which gives a foundation of Green's function formalisms, such as Schwinger's closed-time path method, Keldysh-method, and so on [39–41], in terms of dissipative quantum field operators within the representation space constructed on an unstable vacuum.

In the extension to take account of the quantum stochastic processes [23–28], NETFD again allowed us to construct a unified canonical theory of quantum stochastic operators. The stochastic Liouville equations both of the Ito and of the Stratonovich types were introduced in the Schrödinger representation. Whereas, the Langevin equations both of the Ito and of the Stratonovich types were constructed as the Heisenberg equation of motion with the help of the time-evolution generator of corresponding stochastic Liouville equations (Fig. 1). The Ito formula was generalized for quantum systems.

NETFD has been applied to various systems, e.g. the dynamical rearrangement of thermal vacuum in superconductor [29], spin relaxation [30], various transient phenomena in quantum optics [31–35], nonlinear damped harmonic oscillator [36], the tracks in the cloud chamber (a non-demolition continuous measurement) [37], microscopic derivation of the quantum Kramers equation [38].

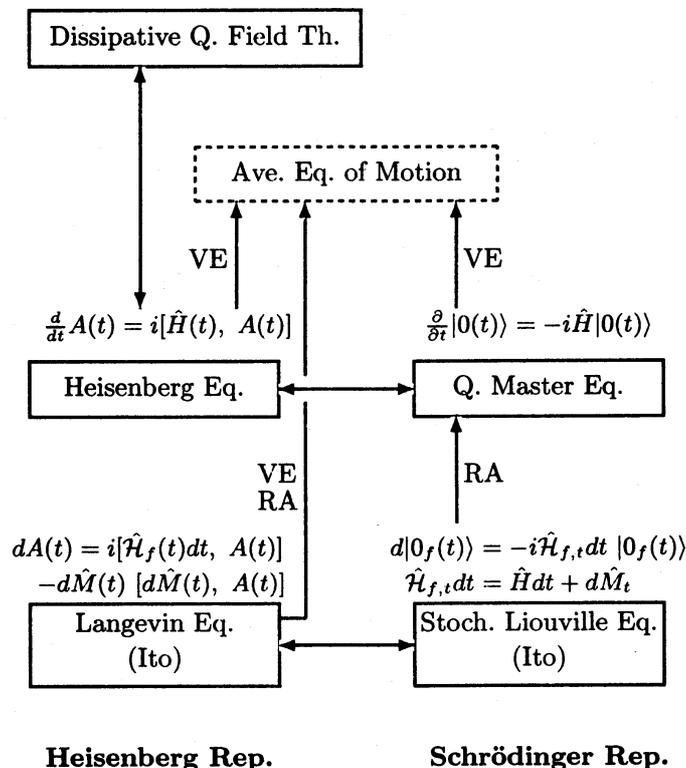


FIG. 1. System of the Stochastic Differential Equations within Non-Equilibrium Thermo Field Dynamics. RA stands for the random average. VE stands for the vacuum expectation.

In the next section, the framework of NETFD will be briefly explained. In section III, two systems of the stochastic differential equations will be introduced, one with *non-unitary* time-evolution generator and the other with *unitary* time-evolution generator. Both systems are constructed to be consistent with the same quantum master equation. The key is the existence of the fluctuation-dissipation theorem between the multiple of martingale operators and the *imaginary* part of hat-Hamiltonian. In section IV, two systems will be applied to the model of damped harmonic oscillator interacting with irrelevant random force system by the linear dissipative coupling. It will be shown that both systems are consistently applicable. The existence of the non-commutative random force operators is essential for the system with unitary time-evolution generator. In section V, two systems will be applied to the model of damped harmonic oscillator interacting with irrelevant system by the position-position interaction. It will be shown that the system with unitary time-evolution generator cannot produce the framework which is consistent with the master equation, since there appear only commutative random force operators in martingale. Section VI will be devoted to summary and discussion.

II. FRAMEWORK OF NETFD

The dynamics of physical systems is described, within NETFD, by the Schrödinger equation for the thermal ket-vacuum $|0(t)\rangle$:

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle. \quad (11)$$

The time-evolution generator \hat{H} is an tildian operator satisfying

$$(i\hat{H})^\sim = i\hat{H}. \quad (12)$$

The *tilde conjugation* \sim is defined by

$$(A_1 A_2)^\sim = \tilde{A}_1 \tilde{A}_2, \quad (13)$$

$$(c_1 A_1 + c_2 A_2)^\sim = c_1^* \tilde{A}_1 + c_2^* \tilde{A}_2, \quad (14)$$

$$(\tilde{A})^\sim = A, \quad (15)$$

$$(A^\dagger)^\sim = \tilde{A}^\dagger, \quad (16)$$

where c_1 and c_2 are c -numbers. The tilde and non-tilde operators at an equal time are mutually commutative:

$$[A, \tilde{B}] = 0. \quad (17)$$

The thermal bra-vacuum $\langle 1|$ is the eigen-vector of the hat-Hamiltonian \hat{H} with zero eigen-value:

$$\langle 1|\hat{H} = 0. \quad (18)$$

This guarantees the conservation of the inner product between the bra and ket vacuums in time:

$$\langle 1|0(t)\rangle = 1. \quad (19)$$

Let us assume that the thermal vacuums satisfy

$$\langle 1|^\sim = \langle 1|, \quad |0(t_0)\rangle^\sim = |0(t_0)\rangle, \quad (20)$$

at a certain time $t = t_0$. Then, (12) guarantees that they are satisfied for all the time:

$$\langle 1|^\sim = \langle 1|, \quad |0(t)\rangle^\sim = |0(t)\rangle. \quad (21)$$

The tilde operator and the non-tilde operator are related by the thermal state condition for the bra vacuum:

$$\langle 1|\tilde{A} = \langle 1|A^\dagger, \quad (22)$$

which reduces the numbers of the degrees of freedom to the original ones. The numbers of the degrees of freedom were doubled by the introduction of tilde operators.

The observable operator A should be an Hermitian operator consisting only of non-tilde operators.

III. TWO SYSTEMS OF STOCHASTIC DIFFERENTIAL EQUATIONS

A. Quantum Master Equation

Let us consider the system of quantum stochastic differential equations which is constructed to be consistent with the quantum master equation (the quantum Fokker-Planck equation)

$$\frac{\partial}{\partial t} |0(t)\rangle = -i\hat{H}|0(t)\rangle, \quad (23)$$

with the hat-Hamiltonian

$$\hat{H} = \hat{H}_S + i\hat{\Pi}, \quad (24)$$

where

$$\hat{H}_S = H_S - \tilde{H}_S, \quad (25)$$

with H_S being the Hamiltonian of a relevant system. It is easily seen that \hat{H}_S satisfies

$$\langle 1 | \hat{H}_S = 0. \quad (26)$$

It is assumed that the *imaginary* part $\hat{\Pi}$ of the hat-Hamiltonian can be divided into two parts, i.e., the relaxational part $\hat{\Pi}_R$ and the diffusive part $\hat{\Pi}_D$:

$$\hat{\Pi} = \hat{\Pi}_R + \hat{\Pi}_D, \quad (27)$$

and each of them satisfies

$$\langle 1 | \hat{\Pi}_R = 0, \quad \langle 1 | \hat{\Pi}_D = 0. \quad (28)$$

Introducing the time-evolution operator $\hat{V}(t)$ by

$$\frac{d}{dt} \hat{V}(t) = -i\hat{H}\hat{V}(t), \quad (29)$$

with the initial condition $\hat{V}(0) = 1$, we can define the Heisenberg operator

$$A(t) = \hat{V}^{-1}(t)A\hat{V}(t), \quad (30)$$

which satisfies the Heisenberg equation

$$\frac{d}{dt} A(t) = i[\hat{H}(t), A(t)], \quad (31)$$

for dissipative systems.

The equation of motion for the averaged quantity $\langle 1 | A(t) | 0 \rangle$ is derived by means of the Heisenberg equation (31) by taking its vacuum expectation:

$$\frac{d}{dt} \langle 1 | A(t) | 0 \rangle = i \langle 1 | [\hat{H}(t), A(t)] | 0 \rangle. \quad (32)$$

The same equation can be also derived with the help of the master equation (23) as

$$\frac{d}{dt} \langle 1 | A | 0(t) \rangle = -i \langle 1 | A \hat{H} | 0(t) \rangle. \quad (33)$$

We would like to emphasize here that the existence of the Heisenberg equation of motion (31) for coarse grained operators is one of the notable features of NETFD. This enabled us to construct a *canonical formalism of the dissipative quantum field theory*, where the coarse grained operators $a(t)$ etc. in the Heisenberg representation preserve the equal-time canonical commutation relation

$$[a(t), a^\dagger(t)] = 1, \quad [\tilde{a}(t), \tilde{a}^\dagger(t)] = 1. \quad (34)$$

Note that we have an equation of motion for a vector $\langle 1|A(t)$:

$$\begin{aligned} \frac{d}{dt}\langle 1|A(t) &= i\langle 1|[\hat{H}(t), A(t)] \\ &= i\langle 1|[H_S(t), A(t)] \\ &\quad -\kappa \{ \langle 1|[A(t), a^\dagger(t)]a(t) + \langle 1|a^\dagger(t)[a(t), A(t)] \} + 2\kappa\bar{n}\langle 1|[a(t), [A(t), a^\dagger(t)]] \end{aligned} \quad (35)$$

in terms of only non-tilde operators with the help of the condition (71). Applying the ket-vacuum $|0\rangle$ to (35), we obtain the equation of motion for the averaged quantity (32).

B. Non-Unitary Time-Evolution

The system of stochastic differential equations with *non-unitary* time-evolution is constructed by the following general procedures.

The stochastic Liouville equation

$$d|0_f(t)\rangle\rangle = -i\hat{\mathcal{H}}_{f,t}dt|0_f(t)\rangle\rangle, \quad (36)$$

of the Ito type is specified with the stochastic hat-Hamiltonian

$$\begin{aligned} \hat{\mathcal{H}}_{f,t}dt &= \hat{H}dt + d\hat{M}_t \\ &= \hat{H}_Sdt + i\hat{I}dt + d\hat{M}_t, \end{aligned} \quad (37)$$

where $\hat{I} = \hat{I}_R + \hat{I}_D$ is the same that appeared in the master equation (23). The martingale operator $d\hat{M}_t$ annihilates the bra-vacuum $\langle 1|$ of the relevant system:

$$\langle 1|d\hat{M}_t = 0, \quad (38)$$

which means that the stochastic Liouville equation (36) preserves its probability just within the relevant system. This feature is the same as the one within the system of stochastic differential equations for classical systems. The martingale operator satisfies the fluctuation-dissipation theorem of the second kind:

$$d\hat{M}_td\hat{M}_t = -2\hat{I}_Ddt, \quad (39)$$

which should be interpreted as a weak relation.*

Applying $\langle 1|$ to (36), we have an equation for

$$|0(t)\rangle = \langle 1|0_f(t)\rangle\rangle, \quad (40)$$

which is nothing but the quantum master equation (23).

Introducing the stochastic time-evolution operator

$$d\hat{V}_f(t) = -i\hat{\mathcal{H}}_{f,t}dt\hat{V}_f(t), \quad (41)$$

we can define the stochastic Heisenberg operator

$$A(t) = \hat{V}_f^{-1}(t)A\hat{V}_f(t), \quad (42)$$

which satisfies the stochastic Heisenberg equation (the Langevin equation)

*It is similar to the classical cases where the fluctuation-dissipation theorem of the second kind is specified within the stochastic limit.

$$dA(t) = i[\hat{\mathcal{H}}_f(t)dt, A(t)] - d\hat{M}(t) [d\hat{M}(t), A(t)], \quad (43)$$

of the Ito type. Here, we introduced the martingale operator in the Heisenberg representation by

$$d\hat{M}(t) = d\left(\hat{V}_f^{-1}(t)M_t\hat{V}_f(t)\right). \quad (44)$$

Note that

$$d\hat{M}(t) = d'\hat{M}(t), \quad (45)$$

with

$$d'\hat{M}(t) = \hat{V}_f^{-1}(t)dM_t\hat{V}_f(t). \quad (46)$$

Making use of the relation of the Ito-Stratonovich stochastic calculus (see Appendix B), we can derive from (36) the stochastic Liouville equation

$$d|0_f(t)\rangle\rangle = -i\hat{H}_{f,t}dt \circ |0_f(t)\rangle\rangle, \quad (47)$$

of the Stratonovich type, where the symbol \circ indicates the Stratonovich stochastic multiplication. The stochastic hat-Hamiltonian

$$\hat{H}_{f,t}dt = \hat{H}_Sdt + i\hat{\Pi}_Rdt + d\hat{M}_t, \quad (48)$$

contains only the relaxational part $\hat{\Pi}_R$.

With this hat-Hamiltonian, we can write down the stochastic Heisenberg equation

$$dA(t) = i[\hat{H}_f(t)dt, A(t)], \quad (49)$$

of the Stratonovich type. Note that it does not have the term producing diffusive time-evolution, which is the same characteristics that appeared in the system of classical stochastic differential equations.

C. Unitary Time-Evolution

The system of stochastic differential equations with *unitary* time-evolution is constructed by the following general procedures.

The stochastic Liouville equation

$$d|0_f(t)\rangle\rangle = -i\hat{H}_{f,t}^Udt \circ |0_f(t)\rangle\rangle, \quad (50)$$

of the Stratonovich type is specified with the stochastic hat-Hamiltonian

$$\hat{H}_{f,t}^Udt = \hat{H}_Sdt + d\hat{M}_t^U, \quad (51)$$

with the Hermitian martingale operator

$$\left(d\hat{M}_t^U\right)^\dagger = d\hat{M}_t^U. \quad (52)$$

Note that (50) does not satisfy generally the conservation of probability just within the relevant system, i.e.,

$$\langle 1|d\hat{M}_t^U \neq 0, \quad (53)$$

but it does within whole the system, the relevant and irrelevant systems, i.e.,

$$\langle\langle 1|d\hat{M}_t^U = 0. \quad (54)$$

Here, $\langle\langle 1| = \langle 1|\langle|$ with $\langle|$ being the bra-vacuum of the quantum Brownian motion (see Appendix C). The martingale operator satisfies the fluctuation-dissipation theorem

$$d\hat{M}_t^U d\hat{M}_t^U = -2\hat{\Pi} dt, \quad (55)$$

of the second kind.

Introducing the unitary stochastic time-evolution operator $\hat{U}_f(t)$ by

$$d\hat{U}_f(t) = -i\hat{H}_{f,t}^U dt \circ \hat{U}_f(t), \quad (56)$$

with the initial condition $\hat{U}_f(0) = 1$, we can define the stochastic Heisenberg operator

$$A(t) = \hat{U}_f^{-1}(t) A \hat{U}_f(t), \quad (57)$$

which satisfies the stochastic Heisenberg equation (the Langevin equation)

$$dA(t) = i[\hat{H}_f^U(t) dt \circ A(t)], \quad (58)$$

of the Stratonovich type. Note that the time-evolution generator $\hat{U}_f(t)$ is a unitary operator:

$$\hat{U}_f^\dagger(t) = \hat{U}_f^{-1}(t). \quad (59)$$

By making use of the relation between the Ito and Stratonovich stochastic calculus, we can derive from (50) the stochastic Liouville equation

$$d|0_f(t)\rangle\rangle = -i\hat{\mathcal{H}}_{f,t}^U dt |0_f(t)\rangle\rangle, \quad (60)$$

of the Ito type with the stochastic hat-Hamiltonian

$$\hat{\mathcal{H}}_{f,t}^U = \hat{H} dt + d\hat{M}_t^U. \quad (61)$$

Applying $\langle\langle |$ to (60), we see easily that it reduces to the quantum master equation (23).

Within the Ito calculus, the time-evolution operator $\hat{U}_f(t)$ satisfies

$$d\hat{U}_f(t) = -i\hat{\mathcal{H}}_{f,t}^U dt \hat{U}_f(t), \quad (62)$$

with the initial condition $\hat{U}_f(0) = 1$. The stochastic Heisenberg operator $A(t)$ defined by (57) satisfies the stochastic Heisenberg equation

$$dA(t) = i[\hat{\mathcal{H}}_f^U(t) dt, A(t)] - d\hat{M}_t^U(t) [d\hat{M}_t^U(t), A(t)], \quad (63)$$

of the Ito type. Here, we introduced the martingale operator in the Heisenberg representation by

$$d\hat{M}_t^U(t) = d\left(\hat{U}_f^{-1}(t) M_t^U \hat{U}_f(t)\right). \quad (64)$$

Note that [37]

$$d\hat{M}_t^U(t) = d' \hat{M}_t^U(t), \quad (65)$$

with

$$d' \hat{M}_t^U(t) = \hat{U}_f^{-1}(t) dM_t^U \hat{U}_f(t). \quad (66)$$

IV. APPLICATION TO QUANTUM DAMPED HARMONIC OSCILLATOR

A. Quantum Master Equation

The hat-Hamiltonian of the semi-free field is bi-linear in $(a, \bar{a}, a^\dagger, \bar{a}^\dagger)$, and is invariant under the phase transformation $a \rightarrow ae^{i\theta}$:

$$\hat{H} = g_1 a^\dagger a + g_2 \bar{a}^\dagger \bar{a} + g_3 a \bar{a} + g_4 a^\dagger \bar{a}^\dagger + g_0, \quad (67)$$

where g 's are time-dependent c -number complex functions.

The operators a , \tilde{a}^\dagger , etc. satisfy the canonical commutation relation:[†]

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}, \mathbf{k}'}, \quad [\tilde{a}_{\mathbf{k}}, \tilde{a}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}, \mathbf{k}'}. \quad (68)$$

The tilde and non-tilde operators are mutually commutative. Throughout this paper, we do not label explicitly the operators a , \tilde{a}^\dagger , etc. with a subscript \mathbf{k} for specifying a momentum and/or other degrees of freedom. However, remember that we are dealing with a *dissipative quantum field*.

The tildian nature $(i\hat{H})^\sim = i\hat{H}$ makes (67) tildian:

$$\hat{H} = \omega(a^\dagger a - \tilde{a}^\dagger \tilde{a}) + i\hat{\Pi}, \quad (69)$$

with

$$\hat{\Pi} = c_1(a^\dagger a + \tilde{a}^\dagger \tilde{a}) + c_2 a \tilde{a} + c_3 a^\dagger \tilde{a}^\dagger + c_4, \quad (70)$$

where $\omega = \Re g_1 = -\Re g_2$, $c_1 = \Im g_1 = \Im g_2$, $c_2 = \Im g_3$, $c_3 = \Im g_4$ and $c_4 = \Im g_0$.

With the help of (22) for $A = a$:

$$\langle 1 | \tilde{a} = \langle 1 | a^\dagger, \quad (71)$$

the property $\langle 1 | \hat{H} = 0$ gives us the relations

$$2c_1 + c_2 + c_3 = 0, \quad c_3 + c_4 = 0. \quad (72)$$

Then, (70) reduces to

$$\hat{\Pi} = c_1(a^\dagger a + \tilde{a}^\dagger \tilde{a}) - c_2 a \tilde{a} - (2c_1 + c_2) a^\dagger \tilde{a}^\dagger + (2c_1 + c_2). \quad (73)$$

Let us write down here the Heisenberg equations for a and a^\dagger :

$$\frac{d}{dt} a(t) = -i\omega a(t) + c_1 a(t) - (2c_1 + c_2) \tilde{a}^\ddagger(t), \quad (74)$$

$$\frac{d}{dt} a^\ddagger(t) = i\omega a^\ddagger(t) - c_1 a^\ddagger(t) - c_2 \tilde{a}(t). \quad (75)$$

Since the semi-free hat-Hamiltonian \hat{H} is not necessarily Hermite, we introduced the symbol \ddagger in order to distinguish it from the Hermite conjugation \dagger . However in the following, we will use \dagger instead of \ddagger , for simplicity, unless it is confusing. By making use of the Heisenberg equations (74) and (75), we obtain the equation of motion for a vector $\langle 1 | a^\dagger(t) a(t) \rangle$ in the form

$$\frac{d}{dt} \langle 1 | a^\dagger(t) a(t) \rangle = -2\kappa \langle 1 | a^\dagger(t) a(t) \rangle + i\Sigma^< \langle 1 |, \quad (76)$$

where we introduced κ and $\Sigma^<$ respectively by

$$\kappa = c_1 + c_2, \quad (77)$$

$$\Sigma^< = i(2c_1 + c_2). \quad (78)$$

In deriving (76), we used the thermal state condition (22) in order to eliminate tilde operators.

Applying the thermal ket vacuum $|0\rangle$ to (76), we obtain the equation of motion for the *one-particle distribution function*

$$n(t) = \langle 1 | a^\ddagger(t) a(t) | 0 \rangle = \langle 1 | a^\dagger a | 0(t) \rangle, \quad (79)$$

[†]Throughout this paper, we confine ourselves to the case of boson fields, for simplicity. The extension to the case of fermion fields are rather straightforward.

$$\frac{d}{dt}n(t) = -2\kappa n(t) + i\Sigma^<. \quad (80)$$

The equation (80) is the Boltzmann equation of the system. The function $\Sigma^<$ is given when the interaction hat-Hamiltonian is specified.

The initial ket-vacuum $|0\rangle = |0(t=0)\rangle$ is specified by

$$a|0\rangle = f\tilde{a}^\dagger|0\rangle, \quad (81)$$

with a real quantity f . Here, we are neglecting the *initial correlation* [42]. The initial condition of the one-particle distribution function $n = n(t=0)$ can be derived by treating $\langle 1|a\tilde{a}|0\rangle$ as follows. In the first place,

$$\begin{aligned} \langle 1|a\tilde{a}|0\rangle &= \langle 1|afa^\dagger|0\rangle \\ &= f(\langle 1|a^\dagger a|0\rangle + \langle 1|0\rangle) \\ &= f(n+1), \end{aligned} \quad (82)$$

where we used the tilde conjugate of (81) for the first equality, and the canonical commutation relation (68) for the second. On the other hand,

$$\begin{aligned} \langle 1|a\tilde{a}|0\rangle &= \langle 1|\tilde{a}a|0\rangle \\ &= \langle 1|a^\dagger a|0\rangle \\ &= n. \end{aligned} \quad (83)$$

Here, for the first equality, we used (17), i.e., the commutativity between the tilde and non-tilde operators, and, for the second equality, (71). Equating (82) and (83), we see that

$$n = \frac{f}{1-f}, \quad \left(f = \frac{n}{1+n}\right). \quad (84)$$

If it is assumed that there is only one stationary state, we can refer the stationary state as a thermal equilibrium state. We will assign the thermal equilibrium state to be specified by the Planck distribution function with temperature T :

$$n(t \rightarrow \infty) = \bar{n} = \frac{1}{e^{\omega/T} - 1}. \quad (85)$$

Then, we have from (80)

$$i\Sigma^< = 2\kappa\bar{n}. \quad (86)$$

In this case, the Boltzmann equation (80) reduces to

$$\frac{d}{dt}n(t) = -2\kappa(n(t) - \bar{n}). \quad (87)$$

Solving (77) and (78) with respect to c_1 and c_2 , and substituting (86) for $\Sigma^<$ into (70), we finally arrive at the most general form of the semi-free hat-Hamiltonian \hat{H} corresponding to the stationary process [13]:

$$\hat{H} = \hat{H}_S + i\hat{\Pi}, \quad (88)$$

where

$$\hat{H}_S = H_S - \tilde{H}_S, \quad H_S = \omega a^\dagger a, \quad (89)$$

$$\hat{\Pi} = -\kappa[(1+2\bar{n})(a^\dagger a + \tilde{a}^\dagger \tilde{a}) - 2(1+\bar{n})a\tilde{a} - 2\bar{n}a^\dagger \tilde{a}^\dagger] - 2\kappa\bar{n}. \quad (90)$$

This hat-Hamiltonian is the same expression as the one derived by means of the principle of correspondence [11,12] when NETFD was constructed first by referring to the projection operator formalism of the damping theory [43–45]. The hat-Hamiltonian (88) with (89) and (90) describes time-evolution of the system of a damped harmonic oscillator.

A detailed investigation of the system is given in Appendix A. Here we just write down an attractive expression which leads us to a new concept, named *spontaneous creation of dissipation*. The expression is given by

$$|0(t)\rangle = \exp \left[- \int d^3k \langle 1 | \gamma_{k,t} \tilde{\gamma}_{k,t} | 0 \rangle \gamma_k^\dagger \tilde{\gamma}_k^\dagger \right] |0\rangle \quad (91)$$

where

$$\langle 1 | \gamma_{k,t} \tilde{\gamma}_{k,t} | 0 \rangle = -n_k(t) + n_k(0) \quad (92)$$

is the *order parameter* for dissipative time-evolution of the unstable vacuum. The annihilation and creation operators

$$\gamma_{k,t}^{\mu=1} = \gamma_{k,t}, \quad \gamma_{k,t}^{\mu=2} = \tilde{\gamma}_k^\dagger, \quad (93)$$

$$\tilde{\gamma}_{k,t}^{\mu=1} = \gamma_k^\dagger, \quad \tilde{\gamma}_{k,t}^{\mu=2} = -\tilde{\gamma}_{k,t}, \quad (94)$$

are defined by

$$\gamma_{k,t}^\mu = B_k(t)^{\mu\nu} a_k^\nu, \quad \tilde{\gamma}_{k,t}^\mu = \bar{a}_k^\nu B_k^{-1}(t)^{\nu\mu}, \quad (95)$$

with the *time-dependent Bogoliubov transformation*:

$$B_k(t)^{\mu\nu} = \begin{pmatrix} 1 + n_k(t) & -n_k(t) \\ -1 & 1 \end{pmatrix}. \quad (96)$$

They satisfy the canonical commutation relation

$$[\gamma_{k,t}^\mu, \tilde{\gamma}_{k',t}^\nu] = \delta_{k,k'} \delta^{\mu\nu}, \quad (97)$$

and annihilate the bra- and ket-vacuums at time t :

$$\gamma_{k,t} |0(t)\rangle = 0, \quad \langle 1 | \tilde{\gamma}_k^\dagger = 0. \quad (98)$$

B. Non-Unitary Time-Evolution

Confining ourselves to the case where the interaction hat-Hamiltonian between the relevant system and the irrelevant system of Brownian motion is bi-linear in $(a, a^\dagger, \text{ and their tilde conjugates})$ and $(dB_t, dB_t^\dagger, \text{ and their tilde conjugates})$, and is invariant under the phase transformation $a \rightarrow ae^{i\theta}$, and $dB_t \rightarrow dB_t e^{i\theta}$, the martingale operator satisfying (38) is given by

$$: d\hat{M}_t : = i \left(\gamma^\dagger dW_t + \tilde{\gamma}^\dagger d\tilde{W}_t \right), \quad (99)$$

where we introduced the random force operator

$$dW_t = \sqrt{2\kappa} \left(\mu dB_t + \nu d\tilde{B}_t^\dagger \right), \quad (100)$$

and the annihilation and creation operator

$$\gamma_\nu = \mu a + \nu \tilde{a}^\dagger, \quad \gamma^\dagger = a^\dagger - \tilde{a}, \quad (101)$$

of the relevant system with $\mu + \nu = 1$, which satisfy the commutation relation

$$[\gamma_\nu, \gamma^\dagger] = 1, \quad (102)$$

and annihilate the relevant bra-vacuum:

$$\langle 1 | \gamma^\dagger = 0, \quad \langle 1 | \tilde{\gamma}^\dagger = 0. \quad (103)$$

Note that the normal ordering $:\cdots:$ in (99) is defined with respect to the annihilation and the creation operators. Making use of the annihilation and the creation operators, we can rewrite $\hat{\Pi}_R$ and $\hat{\Pi}_D$ consisting of $\hat{\Pi}$ introduced in (90) as

$$\hat{\Pi}_R = -\kappa \left(\gamma^\dagger \gamma_\nu + \tilde{\gamma}^\dagger \tilde{\gamma}_\nu \right), \quad (104)$$

$$\hat{\Pi}_D = 2\kappa (\bar{n} + \nu) \gamma^\dagger \tilde{\gamma}^\dagger. \quad (105)$$

The Langevin equations for $a(t)$ and $a^\dagger(t)$ are given by

$$da(t) = -i\omega a(t)dt - \kappa[(\mu - \nu)a(t) + 2\nu\bar{a}^\dagger(t)]dt + dW_t, \quad (106)$$

$$da^\dagger(t) = i\omega a^\dagger(t)dt - \kappa[2\mu\bar{a}(t) - (\mu - \nu)a^\dagger(t)]dt + d\tilde{W}_t, \quad (107)$$

where we used the facts

$$dW(t) = dW_t, \quad d\tilde{W}(t) = d\tilde{W}_t. \quad (108)$$

If we put $\nu = 1/2$, then $\mu = 1/2$, (106) and (107) reduce, respectively, to

$$da(t) = -i\omega a(t)dt - \kappa\bar{a}^\dagger(t)dt + dW_t, \quad (109)$$

$$da^\dagger(t) = i\omega a^\dagger(t)dt - \kappa\bar{a}(t)dt + d\tilde{W}_t. \quad (110)$$

Although dW_t and $d\tilde{W}_t$ are commutative, we have the conservation of the equal-time canonical commutation relation

$$d([a(t), a^\dagger(t)]) = 0. \quad (111)$$

Applying the bra-vacuum $\langle\langle 1 |$ to (106) and (107), we have, for any value of ν ,

$$d\langle\langle 1 | a(t) = -i\omega\langle\langle 1 | a(t)dt - \kappa\langle\langle 1 | a(t)dt + \sqrt{2\kappa}\langle\langle 1 | dB_t, \quad (112)$$

$$d\langle\langle 1 | a^\dagger(t) = i\omega\langle\langle 1 | a^\dagger(t)dt - \kappa\langle\langle 1 | a^\dagger(t)dt + \sqrt{2\kappa}\langle\langle 1 | d\tilde{B}_t^\dagger. \quad (113)$$

Note that these Langevin equations for the vector $\langle\langle 1 | a(t)$ and $\langle\langle 1 | a^\dagger(t)$ have, respectively, the same structure as (6) and (7).

C. Unitary Time-Evolution

The unitary martingale operator satisfying (52) is given by

$$\begin{aligned} :d\hat{M}_t^U: &= i\sqrt{2\kappa} : \left(a^\dagger dB_t - dB_t^\dagger a + \text{t.c.} \right) : \\ &= i \left(\gamma^\dagger dW_t + \tilde{\gamma}^\dagger d\tilde{W}_t \right) - i \left(dW_t^\dagger \gamma_\nu + d\tilde{W}_t^\dagger \tilde{\gamma}_\nu \right), \end{aligned} \quad (114)$$

with t.c indicating tilde conjugate. Note that there is no cross term between tilde and non-tilde operators. Here, we introduced new random force operators

$$dW_t^\dagger = \sqrt{2\kappa} \left(dB_t^\dagger - d\tilde{B}_t \right), \quad (115)$$

which annihilates the bra-vacuum $\langle\langle |$ of the irrelevant system:

$$\langle\langle | dW_t^\dagger = 0, \quad \langle\langle | d\tilde{W}_t^\dagger = 0, \quad (116)$$

and satisfies the commutation relation

$$[dW_t, dW_t^\dagger] = 2\kappa. \quad (117)$$

The expression (114) is consistent with the microscopic Hamiltonian of the linear dissipative coupling.

The martingale operator (114) satisfies the fluctuation-dissipation theorem (55) with (90). Therefore, we conclude that there exists the system of stochastic differential equations with unitary time-evolution operator consistent with the quantum master equation (23) with the hat-Hamiltonian (88).

The Langevin equations for $a(t)$ and $a^\dagger(t)$ are given by

$$da(t) = -i\omega a(t)dt + \sqrt{2\kappa} dB(t), \quad (118)$$

$$da^\dagger(t) = i\omega a^\dagger(t)dt + \sqrt{2\kappa} dB^\dagger(t), \quad (119)$$

where the operators $dB(t)$ and $dB^\dagger(t)$ in the Heisenberg representation are defined by

$$\begin{aligned} \sqrt{2\kappa} dB(t) &= \hat{U}_f^{-1}(t) \circ \sqrt{2\kappa} dB_t \circ \hat{U}_f(t) \\ &= \sqrt{2\kappa} dB_t - \kappa a(t)dt, \end{aligned} \quad (120)$$

$$\begin{aligned} \sqrt{2\kappa} dB^\dagger(t) &= \hat{U}_f^{-1}(t) \circ \sqrt{2\kappa} dB_t^\dagger \circ \hat{U}_f(t) \\ &= \sqrt{2\kappa} dB_t^\dagger - \kappa a^\dagger(t)dt. \end{aligned} \quad (121)$$

In deriving (120) and (121), we used the properties

$$\begin{aligned} [\hat{U}_f(t) \circ \sqrt{2\kappa} dB_t] &= [\hat{U}_f(t), \sqrt{2\kappa} dB_t] + \frac{1}{2}[d\hat{U}_f(t), \sqrt{2\kappa} dB_t] \\ &= \kappa a \hat{U}_f(t)dt, \end{aligned} \quad (122)$$

$$[\hat{U}_f(t) \circ \sqrt{2\kappa} dB_t^\dagger] = \kappa a^\dagger \hat{U}_f(t)dt. \quad (123)$$

and

$$[\hat{U}_f(t), dB_t] = [\hat{U}_f(t), dB_t^\dagger] = 0, \quad (124)$$

which comes from the characteristics of the Ito multiplication:

$$\langle |\hat{U}_f(t)dB_t| \rangle = \langle |\hat{U}_f(t)dB_t^\dagger| \rangle = 0. \quad (125)$$

Substituting (120) and (121), and applying (118) and (119) reduce to (112) and (113) having the same structures as (6) and (7), respectively.

V. APPLICATION TO QUANTUM KRAMERS EQUATION

A. Quantum Master Equation

Let us find out the general structure of hat-Hamiltonian which is bilinear in $(x, p, \tilde{x}, \tilde{p})$. x and p satisfies the canonical commutation relation

$$[x, p] = i. \quad (126)$$

Accordingly, \tilde{x} and \tilde{p} satisfies

$$[\tilde{x}, \tilde{p}] = -i. \quad (127)$$

The conditions, $(i\hat{H})^\sim = i\hat{H}$, and $\langle 1|\hat{H} = 0$ give us the general expression

$$\hat{H} = \hat{H}_S + i\hat{\Pi}, \quad (128)$$

$$\hat{H}_S = H_S - \tilde{H}_S, \quad H_S = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2, \quad (129)$$

$$\hat{\Pi} = \hat{\Pi}_R + \hat{\Pi}_D, \quad (130)$$

with

$$\begin{aligned} \hat{\Pi}_R &= -i\frac{1}{2}\kappa(x - \tilde{x})(p + \tilde{p}), \\ \hat{\Pi}_D &= -\frac{1}{2}\kappa m\omega(1 + 2\bar{n})(x - \tilde{x})^2. \end{aligned} \quad (131)$$

Here, we neglected the diffusion in x -space. The Schrödinger equation

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \quad (132)$$

gives the *quantum Kramers equation* [46].

The Heisenberg equation for the dissipative system is given by

$$\begin{aligned} \frac{d}{dt}x(t) &= i[\hat{H}(t), x(t)] \\ &= \frac{1}{m}p(t) + \frac{1}{2}\kappa\{x(t) - \tilde{x}(t)\}, \end{aligned} \quad (133)$$

$$\frac{d}{dt}p(t) = -m\omega^2x(t) - \frac{1}{2}\kappa\{p(t) + \tilde{p}(t)\} + i\kappa m\omega(1 + 2\bar{n})\{x(t) - \tilde{x}(t)\}. \quad (134)$$

Applying the bra-vacuum $\langle 1|$ of the relevant system, we have the equations for the vectors:

$$\begin{aligned} \frac{d}{dt}\langle 1|x(t) \rangle &= \frac{1}{m}\langle 1|p(t) \rangle, \\ \frac{d}{dt}\langle 1|p(t) \rangle &= -m\omega^2\langle 1|x(t) \rangle - \kappa\langle 1|p(t) \rangle. \end{aligned} \quad (135)$$

B. Non-Unitary Time-Evolution

The stochastic Liouville equation within the Ito calculus becomes

$$d|0_f(t)\rangle\rangle = -i\hat{\mathcal{H}}_{f,t}dt|0_f(t)\rangle\rangle, \quad (136)$$

with the stochastic hat-Hamiltonian

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}dt + d\hat{M}_t. \quad (137)$$

Here, the martingale operator $d\hat{M}_t$ satisfying (38) is defined by

$$d\hat{M}_t = (x - \tilde{x})\left(dX_t + d\tilde{X}_t\right), \quad (138)$$

with

$$dX_t = \frac{\sqrt{\kappa m\omega}}{2}\left(dB_t + d\tilde{B}_t^\dagger\right), \quad (139)$$

where $dB_t, d\tilde{B}_t^\dagger$ and their tilde conjugates are the operators representing quantum Brownian motion (see Appendix C). The generalized fluctuation-dissipation theorem is given by

$$d\hat{M}_t d\hat{M}_t = -2\hat{\Pi}_D dt. \quad (140)$$

Taking a random average, the stochastic Liouville equation (136) reduces to the quantum master equation (132) with $|0(t)\rangle = \langle |0_f(t)\rangle\rangle$.

The stochastic Heisenberg equation (the Langevin equation) for this hat-Hamiltonian is given by

$$\begin{aligned} dx(t) &= i[\hat{\mathcal{H}}_f(t)dt, x(t)] - d\hat{M}(t) [d\hat{M}(t), x(t)] \\ &= \frac{1}{m}p(t)dt + \frac{1}{2}\kappa \{x(t) - \tilde{x}(t)\} dt, \end{aligned} \quad (141)$$

$$dp(t) = -m\omega^2 x(t)dt - \frac{1}{2}\kappa \{p(t) + \tilde{p}(t)\} dt - (dX_t + d\tilde{X}_t), \quad (142)$$

where we used the properties

$$dX(t) = dX_t, \quad d\tilde{X}(t) = d\tilde{X}_t. \quad (143)$$

Applying the bra vacuum $\langle\langle 1|$ to (141) and (142), we have the Langevin equations for vectors

$$d\langle\langle 1|x(t) = \frac{1}{m}\langle\langle 1|p(t)dt, \quad (144)$$

$$d\langle\langle 1|p(t) = -m\omega^2\langle\langle 1|x(t)dt - \kappa\langle\langle 1|p(t)dt - 2\langle\langle 1|dX_t. \quad (145)$$

The averaged equation of motion is given by applying $|0\rangle$ to (144) and (145) in the forms

$$\frac{d}{dt}\langle\langle x(t)\rangle\rangle = \frac{1}{m}\langle\langle p(t)\rangle\rangle, \quad (146)$$

$$\frac{d}{dt}\langle\langle p(t)\rangle\rangle = -m\omega^2\langle\langle x(t)\rangle\rangle - \kappa\langle\langle p(t)\rangle\rangle, \quad (147)$$

where $\langle\langle \dots \rangle\rangle = \langle 1|(\dots)|1\rangle$. The vacuums $\langle |$ and $| \rangle$ are introduced in Appendix C. These averaged equations can be also derived from (133) and (134) by taking the average $\langle\langle \dots \rangle\rangle$.

C. Unitary Time-Evolution

The martingale operator representing position-position interaction may be given by

$$dM_t^U = xdX_t - \tilde{x}d\tilde{X}_t. \quad (148)$$

We did not include the crossing terms between tilde and non-tilde operators to be consistent with the microscopic interaction Hamiltonian.

The fluctuation-dissipation theorem for this martingale operator is given by

$$dM_t^U dM_t^U = -2\hat{\Pi}^U dt, \quad (149)$$

with

$$\hat{\Pi}^U = -\frac{\kappa m\omega}{8}(1 + 2\bar{n})(x - \tilde{x})^2. \quad (150)$$

Then, the Ito stochastic hat-Hamiltonian becomes

$$\hat{\mathcal{H}}_{f,t}^U dt = \hat{H}^U dt + dM_t^U, \quad (151)$$

where

$$\hat{H}^U = \hat{H}_S + i\hat{\Pi}^U. \quad (152)$$

is the hat-Hamiltonian for the master equation. The master equation is different from (23).

The stochastic Heisenberg equations (the Langevin equations) for $x(t)$ and $p(t)$ become

$$dx(t) = \frac{1}{m}p(t)dt, \quad (153)$$

$$dp(t) = -m\omega^2 x(t)dt - dX_t, \quad (154)$$

where we used the fact

$$dX(t) = dX_t. \quad (155)$$

Applying $\langle\langle 1|$ to (153) and (154), we have the Langevin equations for the vectors $\langle\langle 1|x(t)$ and $\langle\langle 1|p(t)$ in the forms

$$d\langle\langle 1|x(t) = \frac{1}{m}\langle\langle 1|p(t)dt, \quad (156)$$

$$d\langle\langle 1|p(t) = -m\omega^2\langle\langle 1|x(t)dt - \langle\langle 1|X_t, \quad (157)$$

which are different from (144) and (145).

VI. SUMMARY AND DISCUSSION

Within the system of non-unitary time-evolution generator (non-unitary system), the time-evolution generator $V_f(t)$ is constituted by the commutative random force operators dW_t and $d\bar{W}_t$. Therefore, the random force operators $dW(t)$ and $dX(t)$ in the Heisenberg representation is, respectively, equal to dW_t and dX_t in the Schrödinger representation, i.e.,

$$dW(t) = dW_t, \quad dX(t) = dX_t. \quad (158)$$

In the application of the system of unitary time-evolution generator (unitary system) to the damped harmonic oscillator where the martingale operator is constituted by non-commutative random force operators manifesting the linear dissipative coupling between the relevant and irrelevant sub-systems, the random force operators in the Heisenberg representation are related to those in the Schrödinger representation by

$$dW(t) = dW_t - \kappa\gamma_\nu(t)dt, \quad (159)$$

$$dW^\dagger(t) = dW_t^\dagger - \kappa\gamma^\dagger(t)dt. \quad (160)$$

The second terms show up because of the non-commutativity. The appearance of these terms is essential in order to make the unitary system consistent with corresponding master equation.

On the contrary, in the application of the unitary system to the quantum Kramers equation where the martingale operator is constituted only by commutative random force operators manifesting the position-position coupling between the relevant and irrelevant sub-systems, the random force operators in the Heisenberg representation is equal to those in the Schrödinger representation, i.e.,

$$dX(t) = dX_t. \quad (161)$$

Therefore, the unitary system cannot be consistent with corresponding master equation.

The above applications tells us that the origin of dissipation cannot be quantum mechanical. In spite of this unsatisfactory nature of the unitary system, it is attractive since hat-Hamiltonian for microscopic system is Hermitian and there is no mixing terms between tilde and non-tilde operators. The hat-Hamiltonian should have the structure

$$\hat{H} = H - \tilde{H}, \quad H^\dagger = H, \quad (162)$$

for microscopic systems. In fact, we succeeded to extract the correct stochastic hat-Hamiltonian for the stochastic Kramers equation by an appropriate coarse graining of operators (the *stochastic mapping*) in time and corresponding renormalization of physical quantities [38]. The simple limit [49] does not give us the correct Kramers equation. This something touchy situation should be investigated based on the unified system of stochastic differential equations shown in this paper. It will be reported in the future publications.

ACKNOWLEDGMENTS

The authors would like to thank Dr. N. Arimitsu, Dr. T. Saito, Dr. A. Tanaka, Mr. T. Imagire, Mr. T. Indei and Mr. Y. Endo for their collaboration with fruitful discussions.

APPENDIX A: NEW ASPECT FOR A DAMPED HARMONIC OSCILLATOR

1. Thermal Doublet

Let us introducing the thermal doublet notation by

$$a(t)^{\mu=1} = a(t), \quad a(t)^{\mu=2} = \bar{a}^\dagger(t), \quad (\text{A1})$$

$$\bar{a}(t)^{\mu=1} = a^\dagger(t), \quad \bar{a}(t)^{\mu=2} = -\bar{a}(t). \quad (\text{A2})$$

Then, the canonical commutation relation can be written as

$$[a(t)^\mu, \bar{a}(t)^\nu] = \delta^{\mu\nu}. \quad (\text{A3})$$

Note that

$$a(t)^\mu = \hat{V}^{-1}(t)a^\mu\hat{V}(t), \quad \bar{a}(t)^\mu = \hat{V}^{-1}(t)\bar{a}^\mu\hat{V}(t). \quad (\text{A4})$$

Making use of the thermal doublet notation, the hat-Hamiltonian (88) reduces to

$$\hat{H} = \omega\bar{a}^\mu a^\mu + i\hat{\Pi} + \omega, \quad (\text{A5})$$

$$\hat{\Pi} = -\kappa\bar{a}^\mu A^{\mu\nu} a^\nu + \kappa, \quad (\text{A6})$$

with

$$A^{\mu\nu} = \begin{pmatrix} 1 + 2\bar{n} & -2\bar{n} \\ 2(1 + \bar{n}) & -(1 + 2\bar{n}) \end{pmatrix}. \quad (\text{A7})$$

The Heisenberg equations for the semi-free particle become

$$\begin{aligned} \frac{d}{dt}a(t)^\mu &= i[\hat{H}(t), a(t)^\mu] \\ &= -i[\omega\delta^{\mu\nu} - i\kappa A^{\mu\nu}]a(t)^\nu, \\ \frac{d}{dt}\bar{a}(t)^\mu &= i[\hat{H}(t), \bar{a}(t)^\mu] \\ &= \bar{a}(t)^\nu i[\omega\delta^{\nu\mu} - i\kappa A^{\nu\mu}]. \end{aligned} \quad (\text{A8})$$

2. Annihilation and Creation Operators

Let us introduce the annihilation and creation operators,

$$\gamma(t)^{\mu=1} = \gamma(t), \quad \gamma(t)^{\mu=2} = \tilde{\gamma}^\dagger(t), \quad (\text{A9})$$

$$\bar{\gamma}(t)^{\mu=1} = \gamma^\dagger(t), \quad \bar{\gamma}(t)^{\mu=2} = -\tilde{\gamma}(t), \quad (\text{A10})$$

by

$$\gamma(t)^\mu = B(t)^{\mu\nu}a(t)^\nu, \quad \bar{\gamma}(t)^\mu = \bar{a}(t)^\nu B^{-1}(t)^{\nu\mu}, \quad (\text{A11})$$

with the *time-dependent Bogoliubov transformation*:

$$B(t)^{\mu\nu} = \begin{pmatrix} 1 + n(t) & -n(t) \\ -1 & 1 \end{pmatrix}, \quad (\text{A12})$$

where $n(t)$ is the one-particle distribution function satisfying the Boltzmann equation (87).

The annihilation and creation operators satisfy the canonical commutation relation

$$[\gamma(t)^\mu, \bar{\gamma}(t)^\nu] = \delta^{\mu\nu}, \quad (\text{A13})$$

and annihilate the bra- and ket-vacuums at the initial time:

$$\gamma(t)|0\rangle = 0, \quad \langle 1|\tilde{\gamma}^\dagger(t) = 0. \quad (\text{A14})$$

The equation of motion for the thermal doublet $\gamma(t)^\mu$ is derived as

$$\begin{aligned} \frac{d}{dt}\gamma(t)^\mu &= \frac{dB(t)^{\mu\nu}}{dt}a(t)^\nu + B(t)^{\mu\nu}\frac{d}{dt}a(t)^\nu \\ &= \left[\frac{dB(t)}{dt}B^{-1}(t)\right]^{\mu\nu}\gamma(t)^\nu - i[B(t)(\omega 1 - i\kappa A)B^{-1}(t)]^{\mu\nu}\gamma(t)^\nu \\ &= -i[\omega\delta^{\mu\nu} - i\kappa\tau_3^{\mu\nu}]\gamma(t)^\nu, \end{aligned} \quad (\text{A15})$$

where the matrix $\tau_3^{\mu\nu}$ is defined by

$$\tau_3^{11} = -\tau_3^{22} = 1, \quad \tau_3^{12} = \tau_3^{21} = 0. \quad (\text{A16})$$

For the third equality, we used the relations

$$\frac{dB(t)}{dt} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \frac{dn(t)}{dt}, \quad (\text{A17})$$

$$\frac{dB(t)}{dt}B^{-1}(t) = -\frac{n(t)}{dt}\tau_+, \quad (\text{A18})$$

$$B(t)AB^{-1}(t) = \tau_3 + 2[n(t) - \bar{n}]\tau_+, \quad (\text{A19})$$

where

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (\text{A20})$$

The Boltzmann equation (87) has been used also.

The solution of (A15) is given by

$$\gamma(t)^\mu = \exp\{-i(\omega\delta^{\mu\nu} - i\kappa\tau_3^{\mu\nu})(t - t')\}\gamma(t')^\nu. \quad (\text{A21})$$

This expression gives

$$\begin{aligned} \langle 1|\gamma(t)\gamma^\dagger(t')|0\rangle &= e^{-i(\omega - i\kappa)(t - t')}\langle 1|\gamma(t')\gamma^\dagger(t')|0\rangle \\ &= e^{-i(\omega - i\kappa)(t - t')}, \end{aligned} \quad (\text{A22})$$

$$\begin{aligned} \langle 1|\tilde{\gamma}(t')\tilde{\gamma}^\dagger(t)|0\rangle &= \langle 1|\gamma(t')\gamma^\dagger(t)|0\rangle \sim \\ &= e^{-i(\omega + i\kappa)(t - t')}. \end{aligned} \quad (\text{A23})$$

3. Two-Point Function (Propagator)

The time-ordered two-point function $G(t, t')^{\mu\nu}$ has the form

$$\begin{aligned} G(t, t')^{\mu\nu} &= -i\langle 1|T[a(t)^\mu \bar{a}(t')^\nu]|0\rangle \\ &= [B^{-1}(t)\mathcal{G}(t, t')B(t')]^{\mu\nu}, \end{aligned} \quad (\text{A24})$$

where

$$\begin{aligned} \mathcal{G}(t, t')^{\mu\nu} &= -i\langle 1|T[\gamma(t)^\mu \tilde{\gamma}(t')^\nu]|0\rangle \\ &= \begin{pmatrix} G^R(t, t') & 0 \\ 0 & G^A(t, t') \end{pmatrix}, \end{aligned} \quad (\text{A25})$$

$$G^R(t, t') = -i\theta(t - t')e^{-i(\omega - i\kappa)(t - t')}, \quad (\text{A26})$$

$$G^A(t, t') = i\theta(t' - t)e^{-i(\omega + i\kappa)(t - t')}. \quad (\text{A27})$$

In deriving the above expression, we used the elements of the solution (A21) with some algebraic manipulations. For example,

$$\begin{aligned} \mathcal{G}(t, t')^{11} &= -i\langle 1|T[\gamma(t)\gamma(t')^\dagger]|0\rangle \\ &= -i\left\{\theta(t - t')\langle 1|\gamma(t)\gamma^\dagger(t')|0\rangle + \theta(t' - t)\langle 1|\gamma^\dagger(t')\gamma(t)|0\rangle\right\} \\ &= -i\theta(t - t')e^{-i(\omega - i\kappa)(t - t')} \\ &= G^R(t, t'). \end{aligned} \quad (\text{A28})$$

In the third equality, we used (A22) and (A23).

4. Miscellaneous

The representation space (the thermal space) of NETFD is the vector space spanned by the set of bra and ket state vectors which are generated, respectively, by cyclic operations of the annihilation operators $\gamma(t)$ and $\tilde{\gamma}(t)$ on $|1\rangle$, and of the creation operators $\gamma^\dagger(t)$ and $\tilde{\gamma}^\dagger(t)$ on $|0\rangle$.

The normal product is defined by means of the annihilation and the creation operators, i.e. $\gamma^\dagger(t)$, $\tilde{\gamma}^\dagger(t)$ stand to the left of $\gamma(t)$, $\tilde{\gamma}(t)$. The process, rewriting physical operators in terms of the annihilation and creation operators, leads to a Wick-type formula, which in turn leads to Feynman-type diagrams for multi-point functions in the renormalized interaction representation. The internal line in the Feynman-type diagrams is the unperturbed two-point function (A24).

5. Condensation of Particle Pairs

Introducing the annihilation and creation operators in the Schrödinger representation

$$\gamma^{\mu=1} = \gamma_t, \quad \gamma^{\mu=2} = \tilde{\gamma}^\dagger, \quad (\text{A29})$$

$$\tilde{\gamma}^{\mu=1} = \gamma^\dagger, \quad \tilde{\gamma}^{\mu=2} = -\tilde{\gamma}_t, \quad (\text{A30})$$

by the relation

$$\gamma(t)^\mu = \hat{V}^{-1}(t)\gamma_t^\mu\hat{V}(t), \quad \tilde{\gamma}(t)^\mu = \hat{V}^{-1}(t)\tilde{\gamma}_t^\mu\hat{V}(t), \quad (\text{A31})$$

with $\hat{V}(t)$ being specified by (29), we can rewrite the hat-Hamiltonian (88) as

$$\hat{H} = \omega\left(\gamma^\dagger\gamma_t - \tilde{\gamma}^\dagger\tilde{\gamma}_t\right) - i\hat{\Pi}, \quad (\text{A32})$$

with

$$\hat{\Pi} = -\kappa\left(\gamma^\dagger\gamma_t + \tilde{\gamma}^\dagger\tilde{\gamma}_t + 2[n(t) - \bar{n}]\gamma^\dagger\tilde{\gamma}^\dagger\right). \quad (\text{A33})$$

It is easily derived by means of the doublet notation (A5).

Substituting (A32) into the quantum master equation (88), we have

$$\begin{aligned} \frac{\partial}{\partial t}|0(t)\rangle &= -2\kappa[n(t) - \bar{n}]\gamma^\dagger\tilde{\gamma}^\dagger|0(t)\rangle \\ &= \frac{dn(t)}{dt}\gamma^\dagger\tilde{\gamma}^\dagger|0(t)\rangle. \end{aligned} \quad (\text{A34})$$

It is solved to give

$$\begin{aligned}
|0(t)\rangle &= \exp \left[\int_0^t dt' \frac{dn(t')}{dt'} \gamma^\ddagger \tilde{\gamma}^\ddagger \right] |0\rangle \\
&= \exp \left[[n(t) - n(0)] \gamma^\ddagger \tilde{\gamma}^\ddagger \right] |0\rangle.
\end{aligned} \tag{A35}$$

This expression tells us that the time-evolution of the unstable vacuum is realized by the condensation of $\gamma_k^\ddagger \tilde{\gamma}_k^\ddagger$ -pairs into the vacuum. The attractive expression (A35), which was obtained first in [29], led us to the notion of a mechanism named the *spontaneous creation of dissipation* [14,15,50–52]. The corresponding *order parameter* is given by

$$\langle 1 | \gamma_t \tilde{\gamma}_t | 0 \rangle = -n(t) + n(0) \tag{A36}$$

where we used the relation

$$\gamma_t = \gamma_{t=0} - [n(t) - n(0)] \tilde{\gamma}^\ddagger. \tag{A37}$$

We can obtain the results (A35) and (A36) only by algebraic manipulations. This technical convenience of the operator algebra in NETFD, which is very much similar to that of the usual quantum field theory, enables us to treat open systems in far-from-equilibrium state simpler and more transparent [30–35].

It also shows that the vacuum is the functional of the one-particle distribution function $n_k(t)$. The dependence of the thermal vacuum on $n_k(t)$ is given by

$$\frac{\delta}{\delta n_k(t)} |0(t)\rangle = \gamma_k^\ddagger \tilde{\gamma}_k^\ddagger |0(t)\rangle. \tag{A38}$$

We see that the vacuum $|0(t)\rangle$ represents the state where exists the macroscopic object described by the one-particle distribution function $n_k(t)$. The master equation (23) can be rewritten as

$$\left\{ \frac{\partial}{\partial t} + \int d^3k \frac{dn_k(t)}{dt} \frac{\delta}{\delta n_k(t)} \right\} |0(t)\rangle = 0. \tag{A39}$$

This shows that the reference vacuum, in this case, is migrating in the super-representation space spanned by the one-particle distribution function $\{n_k(t)\}$ with the *velocity* $\{dn_k(t)/dt\}$ as a conserved quantity.

It is easy to see from the normal product form (A32) of \hat{H} that it satisfies (18), since the annihilation and creation operators satisfy

$$\gamma_t |0(t)\rangle = 0, \quad \langle 1 | \tilde{\gamma}^\ddagger = 0. \tag{A40}$$

The hat-Hamiltonian (88) can be also written in the form

$$\hat{H} = \omega (d^\dagger d - \bar{d}^\dagger \bar{d}) - i\kappa (d^\dagger d + \bar{d}^\dagger \bar{d}), \tag{A41}$$

where $d^{\mu=1} = d$, $d^{\mu=2} = \bar{d}^\dagger$ and $\bar{d}^{\mu=1} = d^\dagger$, $\bar{d}^{\mu=2} = -\bar{d}$ are defined by

$$d^\mu = \bar{B}^{\mu\nu} a^\nu, \quad \bar{d}^\mu = \bar{a}^\nu \bar{B}^{-1\nu\mu}, \tag{A42}$$

with

$$\bar{B}^{\mu\nu} = \begin{pmatrix} 1 + \bar{n} & -\bar{n} \\ -1 & 1 \end{pmatrix}. \tag{A43}$$

The ket-thermal vacuum, $|0\rangle = |0(0)\rangle$, is specified by (81) which can be expressed in terms of d and \bar{d}^\dagger , which are introduced in (A42) below, as

$$d|0\rangle = (n - \bar{n}) \bar{d}^\dagger |0\rangle. \tag{A44}$$

It is easy to see from the diagonalized form (A41) of \hat{H} that

$$d(t) = \hat{V}^{-1}(t) d \hat{V}(t) = d e^{-(i\omega + \kappa)t}, \tag{A45}$$

$$\bar{d}^\dagger(t) = \hat{V}^{-1}(t) \bar{d}^\dagger \hat{V}(t) = \bar{d}^\dagger e^{-(i\omega - \kappa)t}. \tag{A46}$$

The difference between the operators which diagonalize \hat{H} and the ones which make \hat{H} in the form of normal product is one of the features of NETFD, and shows the point that the formalism is quite different from usual quantum mechanics and quantum field theory. This is a manifestation of the fact that the hat-Hamiltonian is a time-evolution generator for irreversible processes. In thermal equilibrium state, $n(t) = \bar{n}$, they coincide.

6. Irreversibility

Let us check here the irreversibility of the system. The entropy of the system is given by

$$\mathcal{S}(t) = - \{n(t) \ln n(t) - [1 + n(t)] \ln [1 + n(t)]\}, \quad (\text{A47})$$

whereas the heat change of the system is given by

$$d'Q = \omega dn. \quad (\text{A48})$$

Thermodynamics tells us that

$$d\mathcal{S} = d\mathcal{S}_e + d\mathcal{S}_i, \quad d\mathcal{S}_e = d'Q/T_R, \quad (\text{A49})$$

$$d\mathcal{S}_i \geq 0. \quad (\text{A50})$$

The latter inequality (A50) is the second law of thermodynamics. Putting (A47) and (A48) into (A49), for $d\mathcal{S}$ and $d\mathcal{S}_e$, respectively, we have a relation for the entropy production rate [53]

$$\frac{d\mathcal{S}_i}{dt} = \frac{d\mathcal{S}}{dt} - \frac{d\mathcal{S}_e}{dt} = 2\kappa [n(t) - \bar{n}] \ln \frac{n(t)[1 + \bar{n}]}{\bar{n}[1 + n(t)]} \geq 0. \quad (\text{A51})$$

It is easy to check that the expression on the right-hand side of the second equality satisfies the last inequality which is consistent with (A50). The equality realizes either for the thermal equilibrium state, $n(t) = \bar{n}$, or for the quasi-stationary process, $\kappa \rightarrow 0$.

APPENDIX B: ITO AND STRATONOVICH MULTIPLICATIONS

The definitions of the Ito [47] and the Stratonovich [48] multiplications are given, respectively, by

$$X^{(H)}(t) \cdot dY^{(H)}(t) = X^{(H)}(t) [Y^{(H)}(t + dt) - Y^{(H)}(t)], \quad (\text{B1})$$

$$dX^{(H)}(t) \cdot Y^{(H)}(t) = [X^{(H)}(t + dt) - X^{(H)}(t)] Y^{(H)}(t), \quad (\text{B2})$$

and

$$X^{(H)}(t) \circ dY^{(H)}(t) = \frac{X^{(H)}(t + dt) + X^{(H)}(t)}{2} [Y^{(H)}(t + dt) - Y^{(H)}(t)], \quad (\text{B3})$$

$$dX^{(H)}(t) \circ Y^{(H)}(t) = [X^{(H)}(t + dt) - X^{(H)}(t)] \frac{Y^{(H)}(t + dt) + Y^{(H)}(t)}{2}, \quad (\text{B4})$$

for arbitrary stochastic operators $X^{(H)}(t)$ and $Y^{(H)}(t)$ in the Heisenberg representation. From (B1), (B2) and (B3), (B4), we have the formulae which connect the Ito and the Stratonovich products in the differential form

$$X^{(H)}(t) \circ dY^{(H)}(t) = X^{(H)}(t) dY^{(H)}(t) + \frac{1}{2} dX^{(H)}(t) \cdot dY^{(H)}(t), \quad (\text{B5})$$

$$dX^{(H)}(t) \circ Y^{(H)}(t) = dX^{(H)}(t) \cdot Y^{(H)}(t) + \frac{1}{2} dX^{(H)}(t) \cdot dY^{(H)}(t). \quad (\text{B6})$$

The connection formulae for the stochastic operators in the Schrödinger representation are given, in the same form as (B5) and (B6), by

$$X^{(S)}(t) \circ dY^{(S)}(t) = X^{(S)}(t) dY^{(S)}(t) + \frac{1}{2} dX^{(S)}(t) \cdot dY^{(S)}(t), \quad (\text{B7})$$

$$dX^{(S)}(t) \circ Y^{(S)}(t) = dX^{(S)}(t) \cdot Y^{(S)}(t) + \frac{1}{2} dX^{(S)}(t) \cdot dY^{(S)}(t), \quad (\text{B8})$$

where the operators $X^{(S)}(t)$ and $dX^{(S)}(t)$ in the Schrödinger representation are introduced respectively through $X^{(H)}(t) = \hat{V}_f^{-1}(t) X^{(S)}(t) \hat{V}_f(t)$ and $dX^{(H)}(t) = \hat{V}_f^{-1}(t) dX^{(S)}(t) \hat{V}_f(t)$.

APPENDIX C: QUANTUM BROWNIAN MOTION

Let us introduce the annihilation and creation operators b_t, b_t^\dagger and their tilde conjugates satisfying the canonical commutation relation:

$$[b_t, b_{t'}^\dagger] = \delta(t - t'), \quad [\tilde{b}_t, \tilde{b}_{t'}^\dagger] = \delta(t - t'). \quad (\text{C1})$$

The vacuums $| \rangle$ and $| \rangle$ are defined by

$$b_t | \rangle = 0, \quad \tilde{b}_t | \rangle = 0, \quad (|b_t^\dagger \rangle = (|\tilde{b}_t^\dagger \rangle. \quad (\text{C2})$$

The argument t represents time.

Introducing the operators

$$B_t = \int_0^{t-dt} dB_{t'} = \int_0^t dt' b_{t'}, \quad (\text{C3})$$

$$B_t^\dagger = \int_0^{t-dt} dB_{t'}^\dagger = \int_0^t dt' b_{t'}^\dagger, \quad (\text{C4})$$

and their tilde conjugates for $t \geq 0$, we see that they satisfy $B(0) = 0, B^\dagger(0) = 0$,

$$[B_s, B_t^\dagger] = \min(s, t), \quad (\text{C5})$$

and their tilde conjugates, and that they annihilate the vacuum $| \rangle$ with the thermal state condition for $| \rangle$:

$$dB_t | \rangle = 0, \quad d\tilde{B}_t | \rangle = 0, \quad (|dB_t^\dagger \rangle = (|d\tilde{B}_t^\dagger \rangle. \quad (\text{C6})$$

These operators represent the quantum Brownian motion.

Let us introduce a set of new operators by the relation

$$dC_t^\mu = \bar{B}^{\mu\nu} dB_t^\nu, \quad (\text{C7})$$

with the Bogoliubov transformation defined by

$$\bar{B}^{\mu\nu} = \begin{pmatrix} 1 + \bar{n} & -\bar{n} \\ -1 & 1 \end{pmatrix}, \quad (\text{C8})$$

where \bar{n} is the Planck distribution function. We introduced the thermal doublet:

$$dB_t^{\mu=1} = dB_t, \quad dB_t^{\mu=2} = d\tilde{B}_t^\dagger, \quad (\text{C9})$$

$$d\tilde{B}_t^{\mu=1} = dB_t^\dagger, \quad d\tilde{B}_t^{\mu=2} = -d\tilde{B}_t, \quad (\text{C10})$$

and the similar doublet notations for dC_t^μ and $d\tilde{C}_t^\mu$. The new operators annihilate the new vacuum $\langle |$, and have the thermal state condition for $| \rangle$:

$$dC_t | \rangle = 0, \quad d\tilde{C}_t | \rangle = 0, \quad (\langle dC_t^\dagger = \langle |d\tilde{C}_t^\dagger. \quad (\text{C11})$$

We will use the representation space constructed on the vacuums $\langle |$ and $| \rangle$. Then, we have, for example,

$$\langle |dB_t | \rangle = \langle |dB_t^\dagger | \rangle = 0, \quad (\text{C12})$$

$$\langle |dB_t^\dagger dB_t | \rangle = \bar{n} dt, \quad \langle |dB_t dB_t^\dagger | \rangle = (1 + \bar{n}) dt. \quad (\text{C13})$$

They can be written

$$dB_t^\dagger dB_t = dB_t d\tilde{B}_t = \bar{n} dt, \quad (\text{C14})$$

$$dB_t dB_t^\dagger = dB_t^\dagger d\tilde{B}_t^\dagger = (1 + \bar{n}) dt, \quad (\text{C15})$$

as weak relations.

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