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Centralization of positive definite functions, Thoma characters, weak containment topology for the infinite symmetric group

平 井 武 (Takeshi HIRAI)
22-8 Nakazaichi-Cho, Iwakura, Sakyo, Kyoto

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Introduction.

In this paper, we study positive definite functions on a countable discrete group, especially on the infinite symmetric group $\mathfrak{S}_\infty$. We further study their relations to the topology in the space of unitary representations of G.

Let G be such a group and K be a finite group acting on G in such a way that, for every $k \in K$, $G \ni g \mapsto k(g) \in G$ is an automorphism. Then, for a function f on G, we put

$$f^K(g) := \frac{1}{|K|} \sum_{k \in K} f(k(g)) \qquad (g \in G)$$

and call it a centralization of f with respect to K. Here we treat mainly the case where K is a subgroup of G and its action is through the inner automorphism.

Take an increasing sequence of finite subgroups $G_n \nearrow G$ $(n = 1, 2, \ldots)$. For a positive definite function f on G we consider a series of centralized functions $f_n = f^{G_n}$ on G. If this series converges pointwise to a function on G, then $\lim_{n\to\infty} f_n$ is a positive definite invariant function (or class function). Relations of positive definite invariant functions to factor representations of G is given in [Th1].

Our problems treated here for the group $G = \mathfrak{S}_\infty$ are the following.

(1) For special interesting positive definite functions f given in [Bo], [BS], determine $\lim_{n\to\infty} f_n$.

(2) For irreducible unitary representations given in [Th2], and also for non-irreducible induced representations of $\mathfrak{S}_\infty$, take some of their matrix elements f and calculate the limits $\lim_{n\to\infty} f_n$ which heavily depend on the choice of increasing sequences of finite subgroups $G_n \nearrow G$.

(3) Translate the results in (1) and (2) into certain results in the weak containment topology of the space of unitary representations.

(4) Analyse relations of the results in (2) to the problem of determining Thoma characters in [Th2], and also to the problem of irreducible decompositions of factor representations in [Ob2].

1 Centralizations of positive definite functions

The infinite symmetric group consists of all finite permutations on the set of natural numbers N, and is denoted by $\mathfrak{S}_\infty$. The symmetric group $\mathfrak{S}_N$ is imbedded in it as the permutation group of the set $I_N := \{1, 2, \ldots, N\} \subset \mathbf{N}$.

A function $F(g)$ on $G = \mathfrak{S}_\infty$ is called *central* if $F(\sigma g \sigma^{-1}) = F(g)$ $(g, \sigma \in G)$. For a function f on G and a finite subgroup $G' \subset G$, we define a *centralization* of f on G' as

$$f^{G'}(g) := \frac{1}{|G'|} \sum_{\sigma \in G'} f(\sigma g \sigma^{-1}). \tag{1}$$

Taking an increasing sequence of finite subgroups $G_N \nearrow G$, we consider a series f^{G_N} of centralizations of f on G_N and study its pointwise convergence limit.

In particular, when we take a series $\mathfrak{S}_N \nearrow \mathfrak{S}_\infty = G$, we put

$$f_N(g) := f^{\mathfrak{S}_N}(g) = \frac{1}{|\mathfrak{S}_N|} \sum_{\sigma \in \mathfrak{S}_N} f(\sigma g \sigma^{-1}). \tag{2}$$

Note that for $N' > N$, we have $f_{N'} = (f_N)_{N'}$, but usually

$$f_{N'}|_{\mathfrak{S}_N} \neq f_N|_{\mathfrak{S}_N}.$$

Consider special kinds of positive definite functions on $G = \mathfrak{S}_\infty$ given as

$$\begin{aligned} f(g) &:= r^{|g|} \qquad (-1 \le r \le 1,\ g \in G), & (3)\\ f'(g) &:= q^{\|g\|} \qquad (0 \le q \le 1,\ g \in G), & (4)\\ f''(g) &:= \mathrm{sgn}(g) \cdot q^{\|g\|} \qquad (0 \le q \le 1,\ g \in G), & (5) \end{aligned}$$

where $|g|$ denotes the usual length of a permutation of g, and $\|g\|$ denotes the block length of g, which is by definition the number of different simple permutations appearing in a reduced expression of g (cf. [Bo] for (3), and [BS] for (4)).

Problem (M. Bożejko): *Let π_f, $\pi_{f'}$ and $\pi_{f''}$ be cyclic unitary representations of $G = \mathfrak{S}_\infty$ corresponding to the positive definite functions in* (3), (4), *and* (5) *by GNS construction. Then, are π_f, $\pi_{f'}$ and $\pi_{f''}$ irreducible ? If not, give irreducible decompositions of them.*

We give here a partial answer to this question as follows.

Theorem 1. *Let $|r| < 1$. Then for the positive definite function f in* (3) *its centralization f_N converges pointwise to the delta function δ_e on $G = \mathfrak{S}_\infty$ as N tends to ∞:*

$$f_N(e) = 1; \quad f_N(g) \to 0 \text{ for } g \neq e \quad (N \to \infty), \tag{6}$$

where e denotes the neutral element of G.

Theorem 2. *Let $0 < q < 1$. Then, for the positive definite function f' in* (4) *and f'' in* (5), *their centralizations f'_N and f''_N converge pointwise to the delta function δ_e on $G = \mathfrak{S}_\infty$ as N tends to ∞: for $F = f'$ or f'',*

$$F_N(e) = 1; \quad F_N(g) \to 0 \text{ for } g \neq e \quad (N \to \infty). \tag{7}$$

The delta function δ_e is a positive definite function associated to the regular representation λ_G of G which corresponds to a cyclic vector $v_0 = \delta_e \in L_2(G)$: $\delta_e(g) = (\lambda_G(g)v_0, v_0)$, and also is the character of this representation which is known to be a factor representation of type II_1.

Concerning to the definition of weak containment of unitary representations, we refer [Di, §18]. Then, we get the following theorem as a direct consequence of Theorems 1 and 2.

Theorem 3. *Each of the representations π_f, $\pi_{f'}$ and $\pi_{f''}$ contains weakly the regular representation λ_G of G.*

2 Lengths of permutations, sums of power series

Take $g \neq e$ from G, and decompose it into a product of mutually disjoint cycles (= cyclic permutations) as

$$g = g_1 g_2 \cdots g_m, \quad g_j = (i_{j1} \ i_{j2} \ \ldots \ i_{j\ell_j}). \tag{8}$$

We call ℓ_j the *length* of the cycle g_j, and put $n_\ell(g) = \big|\{\, j \,;\, \ell_j = \ell\}\big|$ the number of cycles g_j with length ℓ. For $\sigma \in G$, put $h = \sigma g \sigma^{-1}$, then

$$h = \sigma g \sigma^{-1} = h_1 h_2 \cdots h_m, \quad h_j = (\sigma(i_{j1}) \ \sigma(i_{j2}) \ \ldots \ \sigma(i_{j\ell_j})). \tag{9}$$

Thus we should evaluate the length $|h|$ from below to get an evaluation of $r^{|h|}$ from above.

To do so, let us introduce some notations. Take an element $h \in G, h \neq e$, and express it in a product of mutually disjoint cycles as

$$h = h_1 h_2 \cdots h_m. \tag{10}$$

Let us denote by $\mathrm{supp}(h)$ the set of numbers i for which $h(i) \neq i$, then $\mathrm{supp}(h) = \sqcup_{j=1}^m \mathrm{supp}(h_j)$. Assume a cycle h_j is given as $h_j = (a_{j1} \ a_{j2} \ \dots \ a_{j\ell_j})$. Then, $\mathrm{supp}(h_j) = \{ a_{j1}, a_{j2}, \dots, a_{j\ell_j} \}$. Put

$$a_j^- := \min_{1 \leq k \leq \ell_j} a_{jk}, \qquad a_j^+ := \max_{1 \leq k \leq \ell_j} a_{jk}, \tag{11}$$

and define an interval $[h_j] \subset I_N$ as $[h_j] := [a_j^-, a_j^+]$ and denote by $|[h_j]|$ its width $a_j^+ - a_j^-$, which is different from $|\mathrm{supp}(h_j)| = \ell_j$, the order of the set $\mathrm{supp}(h_j)$. Note that the number of different possible cycles h_j with the same $\mathrm{supp}(h_j)$ is equal to $(\ell_j - 1)!$.

Lemma 4. (i) *For an element $h \in G = \mathfrak{S}_\infty, h \neq e$, let $h = h_1 h_2 \cdots h_m$, in* (10) *be its decomposition into disjoint cycles. Then,*

$$|h| \geq \sum_{1 \leq j \leq m} 2\,|[h_j]| - (2m - 1/2)\,|\mathrm{supp}(h)|. \tag{12}$$

(ii) *For $g \in G, g \neq e$, let $g = g_1 g_2 \cdots g_m$ in* (8) *be its decomposition into disjoint cycles. Then, for $\sigma \in G$, we have*

$$|\sigma g \sigma^{-1}| \geq \sum_{1 \leq j \leq m} 2\,|[\sigma g_j \sigma^{-1}]| - (2m - 1/2)\,|\mathrm{supp}(g)|. \tag{13}$$

Lemma 5. *Let ρ be a real number such that $0 < \rho < 1$. Then, for a fixed non-negative integer $s \geq 0$,*

$$\sum_{s \leq p < \infty} \binom{p}{s} \rho^p = \frac{\rho^s}{(1 - \rho)^{s+1}}. \tag{14}$$

We omit the proofs of these lemmas.

3 Proof of Theorem 1

It is enough to consider $\hat{f}(g) = |f(g)| = |r|^{|g|}$. Put $\rho = |r|^2$, then,

$$\begin{aligned} \hat{f}_N(g) &= \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \hat{f}(\sigma g \sigma^{-1}) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} |r|^{|\sigma g \sigma^{-1}|} \\ &\leq \frac{|r|^{-(2m-1/2)|\mathrm{supp}(g)|}}{N!} \sum_{\sigma \in \mathfrak{S}_N} \prod_{1 \leq j \leq m} \rho^{|[\sigma g_j \sigma^{-1}]|} \quad \text{(by Lemma 4).} \end{aligned}$$

Fix two numbers $1 \le b_j^- < b_j^+ \le N$, and consider possible cycles h_j of length ℓ_j for which

$$[h_j] = B_j, \qquad B_j := [b_j^-, b_j^+] \subset I_N. \tag{15}$$

Then, the number of such cycles is equal to $(\ell_j - 1)! \times \{$the number of different choices of $(\ell_j - 2)$ integers from the interval $(b_j^-, b_j^+)\}$:

$$(\ell_j - 1)! \times \binom{b_j^+ - b_j^- - 1}{\ell_j - 2}. \tag{16}$$

Let $S(\,(g_j, B_j)_{1\le j\le m})$ be the subset of $\mathfrak{S}_N$ of all such σ that satisfies

$$[h_j] = B_j \quad \text{for} \quad h_j = \sigma g_j \sigma^{-1} \qquad (1 \le j \le m), \tag{17}$$

and put $s(\,(g_j, B_j)_{1\le j\le m}) = |S(\,(g_j, B_j)_{1\le j\le m})|$. Then,

$$\frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N}\prod_{1\le j\le m}\rho^{|[\sigma g_j\sigma^{-1}]|} = \frac{1}{N!}\sum s(\,(g_j, B_j)_{1\le j\le m})\prod_{1\le j\le m}\rho^{|B_j|}, \tag{18}$$

where the summation runs over all systems of m intervals $\{\, B_j\,; 1 \le j \le m\,\}$ in I_N. Since the family of m subsets $\mathrm{supp}(\sigma g_j \sigma^{-1})$ of I_N are mutually disjoint, a possible system $\{\, B_j \,\}$ should satisfy certain conditions, for example, their extremities are all different. For any non possible one, we put $s(\,(g_j, B_j)_{1\le j\le m}) = 0$.

We want to evaluate from above the number $s(\,(g_j, B_j)_{1\le j\le m})$. We note the following fact. Assume N sufficiently large so that $A := \mathrm{supp}(g) \subset I_N$. Let $\mathfrak{S}_A$ be the full permutation group acting on A, and consider the commutant

$$C_A(g) := \{\, s \in \mathfrak{S}_A\ ;\ sgs^{-1} = g \,\}.$$

Let $n_l(g), \ell \ge 2$, be the number of cycles g_j such that $\ell_j = |\mathrm{supp}(g_j)| = \ell$. Then, the order $|C_A(g)|$ is equal to $\prod_{\ell\ge2} n_\ell(g)! \cdot \ell^{n_\ell(g)}$. However, since we consider independently for each j the cycle $\sigma g_j \sigma^{-1}$, the first factor $\prod_{\ell\ge2} n_\ell(g)!$ does not appear in the next discussion.

Let $g_j = (i_{j1}, i_{j2}, \ldots, i_{j\ell_j})$, then $h_j = \sigma g_j \sigma^{-1}$ is given by (9). This means that the cycle h_j determines the integers $\sigma(i_{j1}), \sigma(i_{j2}), \ldots, \sigma(i_{j\ell_j})$ modulo cyclic permutations. On the other hand, for integers $p \in I_N \setminus \mathrm{supp}(g)$, $\sigma(p)$'s can be given arbitrariry from $I_N \setminus \sigma\cdot\mathrm{supp}(g)$. Thus, taking into account the evaluation (16) and $\prod_{\ell\ge2} \ell^{n_\ell(g)} = \prod_{1\le j\le m} \ell_j$, we get

$$s(\,(g_j, B_j)_{1\le j\le m}) \le \prod_{1\le j\le m} \ell_j! \cdot \binom{|B_j| - 2}{\ell_j - 2} \times (N - |\mathrm{supp}(g)|)!.$$

This evaluation is necessarily from above because the evaluation (16) is given not counting any restriction coming from other $\sigma g_{j'}\sigma^{-1}$ for $j' \neq j$.

Fix the width $k_j = |B_j| \geq \ell_j$. Then, the number of such intervals in I_N is $(N - k_j + 1) < N$. Therefore the left hand side of (18) is evaluated from above by

$$\begin{aligned} & C \cdot \frac{N^m \cdot (N - |\mathrm{supp}(g)|)!}{N!} \cdot \prod_{1 \leq j \leq m} \sum_{\ell_j \leq k_j \leq N} \binom{k_j - 2}{\ell_j - 2} \rho^{k_j} \\ = \;& C \cdot \frac{N^m \cdot (N - |\mathrm{supp}(g)|)!}{N!} \cdot \frac{\rho^{|\mathrm{supp}(g)|}}{(1-\rho)^{|\mathrm{supp}(g)|-m}} \qquad \text{(by Lemma 5)}, \end{aligned}$$

where C denotes a constant independent of N and k_j's.

The above last term tends to 0 as $N \to \infty$. This proves that, for the positive definite function f in the theorem, its centralization f_N tends to the delta function δ_e pointwise on $\mathfrak{S}_\infty$. This proves our assertion. □

4 Comments to Proof of Theorem 2

To prove Theorem 2, we need an evaluation of the block length $||h||$ from below for $h \in \mathfrak{S}_N$, similar to (12) for the length $|h|$ but a little more finer.

Let $h = h_1 h_2 \cdots h_m$ be as in §2 a cycle decomposition of $h \in \mathfrak{S}_N$. Consider intervals $[h_j], 1 \leq j \leq m$, as before. If $[h_j]$ and $[h_{j'}]$ have a non-empty intersection, we join them to get a bigger interval. In this way, we devide the union $\cup_{1 \leq j \leq m}[h_j]$ into *connected components*. Let M be the number of such connected components. Then we have a partition of the index set $I_m = \{1, 2, \ldots, m\}$ into M subsets $J_p, 1 \leq p \leq M$, such that $C_p := \cup_{j \in J_p}[h_j]$ are these connected components.

Lemma 6. *For an element $h \in \mathfrak{S}_N$, let the notations be as above. Let the connnected components $C_p = \cup_{j \in J_p}[h_j]$ be $[c_p^-, c_p^+]$ for $1 \leq p \leq M$. Then the block length of h is given as*

$$||h|| = \sum_{1 \leq p \leq M} (|C_p| - 1) = \sum_{1 \leq p \leq M} (c_p^+ - c_p^-) - M. \tag{19}$$

We omit the proof of the lemma.

Using Lemma 6, we can prove Theorem 2 similarly as Theorem 1. Here we omit the details.

5 Closures in $\mathrm{Rep}(\mathfrak{S}_\infty)$ of unitary representations

In this section, we state a rather astonishing property of unitary representations of the infinite symmetric group $\mathfrak{S}_\infty$.

For a locally compact group G, a topology is introduced in the set $\mathrm{Rep}(G)$ of its unitary representations by means of 'weak containment', for which we refer [Di, §18]. In consequence, a topology is introduced in the dual $\widehat{G}$ of G.

For the infinite symmetric group $G = \mathfrak{S}_\infty$, any irreducible unitary representation (= IUR) known until now can be realized as an induced representation $\mathrm{Ind}_H^G \pi$ from a wreath product type subgroup H and its irreducible unitary representation π, as is proved in [Hi2].

Theorem 7. *For any irreducible unitary representation of the infinite symmetric group $G = \mathfrak{S}_\infty$ given in* [Hi2]*, its closure in* $\mathrm{Rep}(G)$*, with respect to the topology of weak containment, contains at least one of the trivial representation $\mathbf{1}_G$, the sign representation* sgn_G *and the regular representation λ_G.*

Method of Proof. Take an IUR ρ given as an induced representation $\mathrm{Ind}_H^G \pi$. Take a positive definite function f_π associated to π which is given as its matrix element. Then, a positive definite function F associated to ρ is given as an induced up of f_π: $F = \mathrm{Ind}_H^G f_\pi$, which is defined as an extension of f_π to G by putting 0 outside of H (see the next section).

Using explicit form of a wreath product subgroup H, we can work as in the previous sections. In more detail, chosing an appropriate increasing sequence of subgroups $G_N \nearrow \mathfrak{S}_\infty$ $(N \to \infty)$, $G_N = \mathfrak{S}_{J_N}$ with $J_N \nearrow \mathbf{N}$, we calculate the centralization

$$F^{G_N}(g) := \frac{1}{|G_N|} \sum_{\sigma \in G_N} F(\sigma g \sigma^{-1}) \qquad (g \in G = \mathfrak{S}_\infty) \tag{20}$$

on G_N of F, and prove that $F^{G_N}(g)$ converges respectively to the constant function 1, the sign $\mathrm{sgn}(g)$ or the delta function $\delta(g)$ pointwise, as $N \to \infty$.

The key points are

(i) a kind of reduction from F to f_π, and

(ii) an estimation of the order of $\{\sigma \in G_N\,;\ \sigma h \sigma^{-1} \in H\}$ for an element $h \in H, \neq e$.

According to the result in Theorem 7, we can propose certain conjectures.

Conjecture 1 (a weaker form): *For the infinite symmetric group $G = \mathfrak{S}_\infty$, every infinite-dimensional IUR is not closed in the dual space $\widehat{G}$ as a one point set, with respect to the weak containment topology.*

Recall that this topology can be defined in two different ways. The one is by means of the so-called hull-kernel topology according to the containment relation among kernels of representations, and the other is by means of the convergence of positive definite functions associated with representations, cf. for instance, [Di, §3, §18].

Recall further the following fact [Di, §4, §9, §18]. Let G' be a locally compact, unimodular and separable group. Assume that G' is of type I. Then, for an IUR π of G', the one point set $\{[\pi]\}$ in $\widehat{G'}$ is closed if and only if the representation π is CCR, or equivalently, $\pi(L^1(G')) \subset \mathcal{C}(\mathcal{H}_\pi)$ (cf. [Di, §13]). Here, $\mathcal{C}(\mathcal{H}_\pi)$ denotes the algebra of all compact operators on the representation space $\mathcal{H}_\pi$ of π.

In our present case, the group $G = \mathfrak{S}_\infty$ is not of type I. Here again, if an IUR π is CCR, then the one point set $\{[\pi]\}$ is closed. However the converse is not known to be true. Furthermore, since G is discrete, an IUR π of G is CCR if and only if $\pi(g)$ is compact for any $g \in G$, and so $\dim \pi$ is finite.

Thus the above Conjecture 1 makes sense, and we propose further the following more exact one.

Conjecture 2: *For the infinite symmetric group $G = \mathfrak{S}_\infty$, every infinite-dimensional IUR contains in its closure in* $\mathrm{Rep}(G)$ *at least one of the trivial representation $\mathbf{1}_G$, the sign representation* sgn_G *and the regular representation λ_G.*

6 Inducing up of positive definite functions

In a general setting, let G be a discrete group, and H its subgroup. Take a unitary representation π of H on a Hilbert space $\mathcal{V}_\pi$, and consider an induced representation $\rho = \mathrm{Ind}_H^G \pi$.

The representation space $\mathcal{H}_\rho$ of ρ is given as follows. For a vector $v \in \mathcal{V}_\pi$, and a representative g_0 of a right coset $Hg_0 \in H\backslash G$, put

$$E_{v,g_0}(g) = \begin{cases} \pi(h)v & (g = hg_0, h \in H), \\ 0 & (g \notin Hg_0). \end{cases} \tag{21}$$

Let $\mathcal{H}$ be a linear span of these $\mathcal{V}_\pi$-valued functions on G, and define an inner product on it as

$$\langle E_{v,g_0}, E_{v',g_0'} \rangle = \begin{cases} \langle \pi(h)v, v' \rangle & \text{if } hg_0 = g_0' \ (\exists h \in H), \\ 0 & \text{if } Hg_0 \neq Hg_0'. \end{cases} \tag{22}$$

The space $\mathcal{H}_\rho$ is nothing but the completion of $\mathcal{H}$.

The representation ρ is given as

$$\rho(g_1)E(g) = E(gg_1) \quad (g_1, g \in G, E \in \mathcal{H}_\rho). \tag{23}$$

Now take a non-zero vector $v \in \mathcal{V}_\pi$ and put $E = E_{v,e} \in \mathcal{H}_\rho$. Consider a positive definite function on H associated to π as

$$f_\pi(h) = \langle \pi(h)v, v \rangle \qquad (h \in H), \tag{24}$$

and also such a one on G associated to ρ as

$$F(g) = \langle \rho(g)E, E \rangle \qquad (g \in G). \tag{25}$$

Then, we can easily prove the following lemma.

Lemma 8. *The positive definite function F on G associatet to $\rho = \mathrm{Ind}_H^G \pi$ is equal to the inducing up of the positive definite function f_π on H associated to π: $F = \mathrm{Ind}_H^G f_\pi$, which is, by definition, equal to f_π on H and to zero outside of H.*

7 Case of characters $\mathbf{1}_G$ and sgn_G

Firstly we treat the case where the closure of an induced representation $\rho = \mathrm{Ind}_H^G \pi$ contains characters $\mathbf{1}_G$ or sgn_G.

Let H be a subgroup of $G = \mathfrak{S}_\infty$ of the product form $H = H_1 H_2$, where $H_1 = \mathfrak{S}_I$ and $H_2 \subset \mathfrak{S}_J$ with an infinite subset $I \subset \mathbf{N}$ and $J = \mathbf{N} \setminus I$. Denote by χ_1 a character $\mathbf{1}_{\mathfrak{S}_I}$ or $\mathrm{sgn}_{\mathfrak{S}_I}$ of the group $\mathfrak{S}_I \cong \mathfrak{S}_\infty$, and by π_2 a unitary representation (= UR) of H_2. Take a UR $\pi = \chi_1 \otimes \pi_2$ of $H_1 H_2$ and induce it up to G to get $\rho = \mathrm{Ind}_H^G \pi$.

Theorem 9. *Let a unitary representation $\pi = \chi_1 \otimes \pi_2$ of $H = H_1 H_2$ be as above. Then the closure of its induced representation $\rho = \mathrm{Ind}_H^G \pi$ of $G = \mathfrak{S}_\infty$ contains the character $\chi_G = \mathbf{1}_G$ or sgn_G corresponding to $\chi_1 = \mathbf{1}_{\mathfrak{S}_I}$ or $\mathrm{sgn}_{\mathfrak{S}_I}$.*

Proof. Let $J_N \subset \mathbf{N}$ be a series of increasing subsets such that $|J_N| = N$, $J_N \nearrow \mathbf{N}$, and that the ratio $|I \cap J_N|/|J_N| \to 1$ as $N \to \infty$, so that $|J \cap J_N|/N \to$

0. Then, $G_N := \mathfrak{S}_{J_N} \nearrow G = \mathfrak{S}_\infty$ and we consider the centralizations of a positive definite function F associated to ρ along the series of increasing subgroups G_N: for $g \in G$,

$$F^{G_N}(g) := \frac{1}{|G_N|} \sum_{\sigma \in G_N} F(\sigma g \sigma^{-1}) = \frac{1}{N!} \sum_{\sigma \in G_N} F(\sigma g \sigma^{-1}). \tag{26}$$

Take a unit vector v from the representation space $\mathcal{H}_{\pi_2}$ and put a positive definite function f_π associated to π as

$$f_\pi(h_1 h_2) = \chi_1(h_1) \cdot \langle \pi_2(h_2)v, v\rangle \qquad (h_1 \in H_1, h_2 \in H_2). \tag{27}$$

Then $F = \mathrm{Ind}_H^G f_\pi$ is such a one associated to $\rho = \mathrm{Ind}_H^G \pi$, by Lemma 8.

Now take an arbitrary $g \in G$. Since $J_N \nearrow \mathbf{N}$, if N is sufficiently large, there exists a $\sigma_0 \in G_N$ such that $g' = \sigma_0 g \sigma_0^{-1} \in H_1 \cap G_N = \mathfrak{S}_{I \cap J_N}$ or

$$S' := \mathrm{supp}(g') \subset I \cap J_N. \tag{28}$$

Then we have $F^{G_N}(g) = F^{G_N}(g')$.

Fix $g' \in \mathfrak{S}_I$, and consider the asymptotic behavior of the value $F^{G_N}(g')$ as $N \to \infty$. In the formula (26) for g', instead of g, we devide the sum over $\sigma \in G_N = \mathfrak{S}_{J_N}$ into three parts as follows.

Case 1: σ such that $\sigma g' \sigma^{-1} \in \mathfrak{S}_I \cap G_N$ or equivalently $\sigma S' \subset I \cap J_N$;

Case 2: σ such that $\sigma g' \sigma^{-1} \in H = H_1 H_2$, but not in Case 1;

Case 3: σ such that $\sigma g' \sigma^{-1} \notin H$.

In Case 1, $F(\sigma g' \sigma^{-1}) = f_\pi(\sigma g' \sigma^{-1}) = \chi_G(g') = \chi_G(g)$. The number of such $\sigma \in G_N = \mathfrak{S}_{J_N}$ is equal to

$$\frac{|I \cap J_N|!}{(|I \cap J_N| - |S'|)!} \times |J_N \setminus S'|! = \frac{|I \cap J_N|!}{(|I \cap J_N| - k)!} \times (N-k)! \tag{29}$$

with $k = |S'| = |\mathrm{supp}(g)|$. Therefore, since $|I \cap J_N|/N \to 1$, the partial sum for Case 1 in (26) is evaluated as follows when N tends to ∞:

$$\frac{1}{|G_N|} \sum_{\sigma \in G_N:\,\text{Case 1}} F(\sigma g' \sigma^{-1}) \;=\; C_N \cdot \chi_G(g), \tag{30}$$

$$C_N = \frac{1}{N!} \cdot \frac{|I \cap J_N|!}{(|I \cap J_N| - k)!} \cdot (N-k)! \;=\; \prod_{p=0}^{k-1} \frac{|I \cap J_N| - p}{N - p} \longrightarrow 1. \tag{31}$$

In Case 2, we have $|F(\sigma g' \sigma^{-1})| \le 1$ and the evaluation in Case 1 shows us that the partial sum for this case tends to zero as $N \to \infty$. (This follows directly from $\lim_{N\to\infty} C_N = 1$.) In Case 3, we have $F(\sigma g' \sigma^{-1}) = 0$ and there is no contribution to the sum in (26).

Altogether we get finally $F^{G_N}(g) \to \chi_G(g) \quad (g \in G)$. This proves our assertion. □

8 A reduction to a subgroup $\mathfrak{S}_I \cong \mathfrak{S}_\infty$, $I \subset \mathbf{N}$

To treat the case where the closure of $\rho = \mathrm{Ind}_H^G \pi$ contains the regular representation λ_G, it is better to prepare a preliminary step.

We take a subgroup $H \subset G = \mathfrak{S}_\infty$ of the product form $H = H_1 H_2$, where $H_1 \subset \mathfrak{S}_I \cong \mathfrak{S}_\infty$ and $H_2 \subset \mathfrak{S}_J$ with an infinite subset $I \subset \mathbf{N}$ and $J = \mathbf{N} \setminus I$. Take also an infinite-dimensional UR π_1 of H_1 and a UR π_2 of H_2. Then we take a UR $\pi = \pi_1 \otimes \pi_2$ of $H = H_1 H_2$ and its induced one $\rho = \mathrm{Ind}_H^G \pi$ of G.

For $j = 1, 2$, take a unit vector v_j from the representation space $\mathcal{H}_{\pi_j}$ and put a positive definite function f_π associated to π as

$$f_\pi(h_1 h_2) = f_{\pi_1}(h_1) \cdot f_{\pi_2}(h_2), \quad f_{\pi_j}(h_j) = \langle \pi_j(h_j) v_j, v_j \rangle \quad (h_j \in H_j).$$

Then $F = \mathrm{Ind}_H^G f_\pi$ is a positive definite function associated to $\rho = \mathrm{Ind}_H^G \pi$.

Let $J_N \subset \mathbf{N}$ be a series of increasing subsets with the same property as in the proof of Theorem 9, so that putting $J'_N = I \cap J_N$, we have

$$J'_N \nearrow I \quad \text{and} \quad |J'_N|/|J_N| = |J'_N|/N \to 1 \ (N \to \infty).$$

For our later use, we put $G' := \mathfrak{S}_I \supset H_1$, which is naturally isomorphic to $\mathfrak{S}_\infty$, and put $F' := \mathrm{Ind}_{H_1}^{G'} f_{\pi_1}$. Then, F' is a positive definite function on G' associated to $\mathrm{Ind}_{H_1}^{G'} \pi_1$.

We have $G_N := \mathfrak{S}_{J_N} \nearrow G = \mathfrak{S}_\infty$ and $G'_N := \mathfrak{S}_{J'_N} = G' \cap G_N \nearrow G'$. We compair centralizations F^{G_N} in (26) of a positive definite function $F = \mathrm{Ind}_H^G f_\pi$ with those $(F')^{G'_N}$ of $F' = \mathrm{Ind}_{H_1}^{G'} f_{\pi_1}$, concerning their limits as $N \to \infty$..

Take an arbitrary $g \in G$. Then, if N is sufficiently large, there exists a $\sigma_0 \in G_N$ such that $g' = \sigma_0 g \sigma_0^{-1} \in \mathfrak{S}_I \cap G_N = \mathfrak{S}_{J'_N}$ with $J'_N = I \cap J_N$ (in another notation, $g' \in G'_N \subset G'$), or equivalently $S' := \mathrm{supp}(g') \subset J'_N$. Then, $F^{G_N}(g) = F^{G_N}(g')$.

Fix $g' \in \mathfrak{S}_I = G'$, and devide the sum over $\sigma \in G_N = \mathfrak{S}_{J_N}$ in (26) for $F^{G_N}(g')$ into three parts according to Cases 1, 2 and 3 for σ as in the proof of Theorem 9.

Case 1: In Case 1, since $g' \in G'_N \subset G'$, and $\sigma g' \sigma^{-1} \in G'_N$, there exists a $\sigma' \in G'_N$ such that $\sigma g' \sigma^{-1} = \sigma' g' \sigma'^{-1}$. Since $G' \cap H = H_1$, we have $F(\sigma g' \sigma^{-1}) = F(\sigma' g' \sigma'^{-1}) = F'(\sigma' g' \sigma'^{-1})$.

Note that $(\sigma g' \sigma^{-1})(i) = i$ for $i \notin \sigma(S') := \{\sigma(j)\,;\, j \in S'\}$, then we see that the restriction $\sigma|S'$ of σ determines the element $\sigma g' \sigma^{-1}$ completely. So we count the number of $\sigma \in G_N = \mathfrak{S}_{J_N}$ (resp. $G_N \cap \mathfrak{S}_I = \mathfrak{S}_{J'_N} = G'_N$) in Case 1 that have the same restriction $\sigma|S'$ on $S' \subset I$. They are equal to $|J_N \setminus S'|! = (N-k)!$ and $(|J'_N| - k)!$ respectively, with $k = |S'| = |\mathrm{supp}(g)|$.

$$\frac{1}{N!}\sum_{\sigma\in G_N:\,\text{Case 1}} F(\sigma g'\sigma^{-1}) = C_N \times \frac{1}{|J'_N|!}\sum_{\sigma\in G_N\cap\mathfrak{S}_I=\mathfrak{S}_{J'_N}} F'(\sigma g'\sigma^{-1})$$

with $$C_N = \frac{|J'_N|!}{N!}\cdot\frac{(N-k)!}{(|J'_N|-k)!} \longrightarrow 1 \quad (N\to\infty).$$

Since $G_N \cap \mathfrak{S}_I = \mathfrak{S}_{J'_N} = G'_N$, the right hand side of the above equality, except the constant factor C_N, is nothing but the centralization, with respect to G'_N of positive definite function F' on G':

$$(F')^{G'_N}(g') := \frac{1}{|G'_N|}\sum_{\sigma\in G'_N} F'(\sigma g'\sigma^{-1}). \tag{32}$$

Cases 2 and 3: In Case 2, the partial sum over $\sigma\in G_N$ in this case tends to zero as $N\to\infty$ similarly as in the proof of Theorem 9. In Case 3, we have no contribution to the sum in (26).

Altogether we get the following lemma.

Lemma 10. *Let the notations be as above, in particular,* $H = H_1H_2, H_1 \subset \mathfrak{S}_I, H_2 \subset \mathfrak{S}_J$ *with* $|I| = \infty, J = \mathbf{N}\setminus I$, *and* $\pi = \pi_1\otimes\pi_2$ *with a UR* π_j *of* H_j, *and take* $f_\pi(h_1h_2) = f_{\pi_1}(h_1)f_{\pi_2}(h_2)$ $(h_j\in H_j)$. *Put* $F = \mathrm{Ind}_H^G f_\pi$ *for* $G = \mathfrak{S}_\infty$, *and* $F' = \mathrm{Ind}_{H_1}^{G'} f_{\pi_1}$ *for* $G' = \mathfrak{S}_I \cong \mathfrak{S}_\infty$.

For an increasing sequence of subsets $J_N \nearrow \mathbf{N}$, *put* $G_N = \mathfrak{S}_{J_N}, G'_N = G'\cap G_N = \mathfrak{S}_{J'_N}$ *with* $J'_N = I\cap J_N$. *For any* $g\in G = \mathfrak{S}_\infty$, *there exists a* $g'\in G'$ *conjugate to* g *in* G. *If the sequence* J_N *satisfies* $|J'_N|/|J_N| \to 1 (N\to\infty)$, *then,*

$$\lim_{N\to\infty} F^{G_N}(g) = \lim_{N\to\infty}(F')^{G'_N}(g'). \tag{33}$$

9 Case of the regular representation λ_G

We follow the notations in the previous section. For a subgroup $H = H_1H_2 \subset G = \mathfrak{S}_\infty$, we take as H_1 a so-called wreath product type subgroup imbedded into $G' = \mathfrak{S}_I \cong \mathfrak{S}_\infty$ in a saturated way, and $H_2 \subset \mathfrak{S}_J, J = \mathbf{N}\setminus I$. Let us explain for H_1 in more detail.

Take any finite group T and a countable infinite index set Y. Put $T_\eta = T$ for any $\eta\in Y$, and take a restricted direct product $D_Y(T) := \prod'_{\eta\in Y} T_\eta$. Denote by $\mathfrak{S}_Y$ the group of all finite permutations on Y, then it acts naturally on $D_Y(T)$ by permuting components of

$$d = (t_\eta)_{\eta\in Y} \in D_Y(T).$$

The semidirect product group $D_Y(T) \rtimes \mathfrak{S}_Y$ is called a wreath product of T with $\mathfrak{S}_Y$ and is denoted by $\mathfrak{S}_Y(T)$, where, for $\sigma \in \mathfrak{S}_Y$ and $d \in D_Y(T)$, $\sigma \cdot d \cdot \sigma^{-1} = (t'_\eta)$ with $t'_\eta = t_{\sigma^{-1}(\eta)}$ $(\eta \in Y)$.

We imbed $\mathfrak{S}_Y(T)$ into $\mathfrak{S}_I$ as follows. Take a faithful permutation representation of T into a finite symmetric group $\mathfrak{S}_n$, and identify T with its image in $\mathfrak{S}_n$. On the other hand, an ordered set $\mathcal{J} = (p_1, p_2, \ldots, p_n)$ of different n integers $p_j \in \mathbf{N}$ is called an *ordered n-set* and denote by $\overline{\mathcal{J}} := \{ p_1, p_2, \ldots, p_n \}$ its underlying subset of $\mathbf{N}$. We decompose I into infinite number of ordered n-sets $\mathcal{J}_\eta, \eta \in Y$: $I = \sqcup_{\eta \in Y} \overline{\mathcal{J}_\eta}$. For each η, denote by ι_η the order-preserving correspondence $p_j \mapsto j$ $(1 \le j \le n)$ from $\mathcal{J}_\eta = (p_1, p_2, \ldots, p_n)$ onto $I_n = \{ 1, 2, \ldots, n \}$. Then ι_η gives us an imbedding

$$\varphi_\eta : T_\eta = T \subset \mathfrak{S}_n \ni \sigma \mapsto \iota_\eta^{-1} \cdot \sigma \cdot \iota_\eta \in \mathfrak{S}_{\overline{\mathcal{J}_\eta}} \subset \mathfrak{S}_I. \tag{34}$$

This fixes imbeddings of $D_Y(T)$ and $\mathfrak{S}_Y$, and the one Φ of $\mathfrak{S}_Y(T)$ into $\mathfrak{S}_I$, which depends on a partition $\mathcal{I} = \{ \mathcal{J}_\eta \}_{\eta \in Y}$ of I into ordered n-sets.

We take $H_1 = \Phi(\mathfrak{S}_Y(T)) \subset \mathfrak{S}_I$, which is denoted also by $H(\mathcal{I}, T)$. In case T is trivial and imbedded into $\mathfrak{S}_1 = \{ e \}, n = 1$, we have $H(\mathcal{I}, T) = \mathfrak{S}_I$. Except this trivial case, we call such a subgroup as $H(\mathcal{I}, T)$ *properly of wreath product type.*

We take URs π_j of H_j for $j = 1, 2$, and then a tensor product representation $\pi = \pi_1 \otimes \pi_2$ of $H = H_1 H_2$, and induced it up to G: $\rho = \mathrm{Ind}_H^G \pi$. To get an irreducible UR of G by this method, we should choose as π_1 an IUR coming from an infinite tensor product (with respect to a reference vector) of a fixed irreducible finite-dimensional representation of T, and of course similar kinds of restrictions are necessary for H_2 and π_2. Further details are given in [Hi1] and [Hi2], and are summarized in §12 below. For our later use, we define for $\mathcal{I} = (\mathcal{J}_\eta)_{\eta \in Y}$ and $T \subset \mathfrak{S}_n$ the folowing

$$\mathrm{supp}(H(\mathcal{I}, T)) = \mathrm{supp}(\mathcal{I}) := \sqcup_{\eta \in Y} \overline{\mathcal{J}_\eta} \subset \mathbf{N}.$$

Theorem 11. *Let a subgroup $H \subset G = \mathfrak{S}_\infty$ be given as $H = H_1 H_2$, with a proper wreath product type subgroup $H_1 = H(\mathcal{I}, T)$ of $G' = \mathfrak{S}_I \cong \mathfrak{S}_\infty$, and $H_2 \subset \mathfrak{S}_J, J = \mathbf{N} \setminus I$. Let π_1 be an infinite-dimensional UR of H_1 and π_2 a UR of H_2. Take a tensor product representation $\pi = \pi_1 \otimes \pi_2$ of $H = H_1 H_2$. Then the closure of its induced representation $\rho = \mathrm{Ind}_H^G \pi$ of G contains the regular representation λ_G.*

Proof. By Lemma 10, we may and do assume that $H = H_1 = H(\mathcal{I}, T)$, that is, $I = \mathbf{N}$. The finite group T is contained in $\mathfrak{S}_n$ with $n \ge 2$. For $\pi = \pi_1$

and $f_\pi(h) = \langle \pi(h)v, v\rangle, v \in \mathcal{H}_\pi, \|v\| = 1$, we have $|F(h)| \le 1$ for $F = \mathrm{Ind}_H^G f_\pi$. Therefore, taking $G_N = \mathfrak{S}_{J_N}, J_N \nearrow \mathbf{N}$, we have the following evaluation for $g \in G$

$$|F^{G_N}(g)| \le \frac{1}{|G_N|} \sum_{\sigma \in G_N} |F(\sigma g \sigma^{-1})| \le \frac{D_N(g;H)}{|G_N|} = \frac{D_N(g;H)}{|J_N|!} \tag{35}$$

with $D_N(g;H) := |\{\sigma \in G_N\,;\, \sigma g \sigma^{-1} \in H\}|$.

We evaluate the number $D_N(g;H)$ from above. Replacing $T \subset \mathfrak{S}_n$ by $\mathfrak{S}_n$, we consider a bigger subgroup $\widetilde{H} \supset H = H(\mathcal{I}, T) = \Phi(\mathfrak{S}_Y(T))$, that is,

$$\widetilde{H} = H(\mathcal{I}, \mathfrak{S}_n) = \Phi(\mathfrak{S}_Y(\mathfrak{S}_n)).$$

Then, naturally $D_N(g;H) \le D_N(g;\widetilde{H})$, and thus we evaluate the latter.

Recall that these subgroups are defined by means of a partition of $I = \mathbf{N}$ into ordered n-sets as $I = \sqcup_{\eta \in Y} \overline{\mathcal{J}_\eta}$. We introduce a linear order $\eta_1, \eta_2, \dots$ in Y, and put $J_N := \sqcup_{1 \le i \le N} \overline{\mathcal{J}_{\eta_i}}$. Then, $|J_N| = nN$ and $J_N \nearrow \mathbf{N}$.

Take an arbitrary $g \in G, \ne e$, and decompose it into disjoint cycles as in (8):

$$g = g_1 g_2 \cdots g_m, \quad g_j = (i_{j1}\ \ i_{j2}\ \ \dots\ \ i_{j\ell_j}), \tag{36}$$

then, $\mathrm{supp}(g) = \sqcup_{1 \le j \le m} \mathrm{supp}(g_j)$, with $\mathrm{supp}(g_j) = \{i_{j1}, i_{j2}, \dots, i_{j\ell_j}\}$. For $\sigma \in G$, put $h = \sigma g \sigma^{-1}$ and $h_j = \sigma g_j \sigma^{-1}$, then,

$$h = \sigma g \sigma^{-1} = h_1 h_2 \cdots h_m, \quad h_j = (\sigma(i_{j1})\ \ \sigma(i_{j2})\ \ \dots\ \ \sigma(i_{j\ell_j})). \tag{37}$$

We treat the case where $D_N(g;H) > 0$ for sufficiently large N. Take a $\sigma \in G_N$ such that $h = \sigma g \sigma^{-1} \in \widetilde{H}$. Then, we have the following two cases:

Case I: For a certain $j, 1 \le j \le m$, $\mathrm{supp}(h_j) = \sigma\,\mathrm{supp}(g_j) \subset \overline{\mathcal{J}_{\eta_i}}$ for some $1 \le i \le N$.

Case II: For any $j, 1 \le j \le m$, $\mathrm{supp}(h_j) = \sigma\,\mathrm{supp}(g_j) \not\subset \overline{\mathcal{J}_{\eta_i}}$ for any $1 \le i \le N$.

Denote by $D_N^I(g;\widetilde{H})$ (resp. $D_N^{II}(g;\widetilde{H})$) the number of $\sigma \in G_N$ with $h = \sigma g \sigma^{-1} \in \widetilde{H}$ which is in Case I (resp. Case II). Then we have the following evaluations from above.

Lemma 12.

$$\begin{aligned} D_N^I(g;\widetilde{H}) &\le m \cdot N \cdot n(n-1) \cdot (N'-2)!, \qquad N' = nN, \\ D_N^{II}(g;\widetilde{H}) &\le \left(\sum_{j=1}^m \frac{\ell_j(\ell_j - 1)}{2} + |\mathrm{supp}(g)| \right) \cdot N \cdot n(n-1) \cdot (N'-2)!\,. \end{aligned}$$

Assume this lemma be granted, then

$$\frac{D_N(g;H)}{|G_N|} \leq \frac{D_N(g;\widetilde{H})}{|J_N|!} \leq \frac{D_N^I(g;\widetilde{H}) + D_N^{II}(g;\widetilde{H})}{N'!} \longrightarrow 0. \tag{38}$$

This has to be proved for Theorem 11. □

Here we omit the proof of the lemma.

10 Indecomposable positive definite class functions

For the infinite symmetric group $G = \mathfrak{S}_\infty$, all the indecomposable (or extremal) positive definite class-functions, which are also called characters or Thoma characters, are classified and are given explicitly in [Th2].

After Satz 3 in [Th2], they are written as follows. Let $\alpha = (\alpha_1, \alpha_2, \dots), \beta = (\beta_1, \beta_2, \dots)$ be decreasing sequences of non-negative real numbers such that

$$\textstyle\sum_{1 \leq k < \infty} \alpha_k + \sum_{1 \leq k < \infty} \beta_k \leq 1, \tag{39}$$

and put $\gamma_0 = 1 - (\|\alpha\| + \|\beta\|) \geq 0$, with $\|\alpha\| := \sum_{1 \leq k < \infty} \alpha_k$, $\|\beta\| := \sum_{1 \leq k < \infty} \beta_k$, so that $\|\alpha\| + \|\beta\| + \gamma_0 = 1$.

Take a $g \in G$ and let $g = g_1 g_2 \cdots g_m$ be a cycle decomposition in (36), where the length of cycle g_j is denoted by ℓ_j. For $\nu \geq 2$, let $n_\nu(g) = |\{ j \,;\, \ell_j = \nu \}|$ the number of g_j with length ν. Then the character $f_{\alpha,\beta}$ determined by the parameter (α, β) is given by

$$f_{\alpha,\beta}(g) = \left(\sum_{1 \leq k < \infty} \alpha_k^\nu + (-1)^{\nu+1} \sum_{1 \leq k < \infty} \beta_k^\nu \right)^{n_\nu(g)}. \tag{40}$$

The case where $\alpha_1 = 1$ (resp. $\beta_1 = 1$ and $\gamma_0 = 1$) corresponds to the identity representation $\mathbf{1}_G$ (resp. the sign representation sgn_G, and the regular representation λ_G). Except the cases of 1-dimensional representations $\mathbf{1}_G$ and sgn_G, such a character corresponds to the center of a II_1 type factor representation of G [Th1]. These factor representations can be decomposed into irreducible representations, but explicit decompositions are known only in the case where $\gamma_0 = 0$, in [Ob2].

Now let us rewrite the formula (40) in another form. Put

$$\chi_G^{(k)} = \mathbf{1}_G, \quad \chi_G^{(-k)} = \mathrm{sgn}_G, \qquad \alpha_{-k} = \beta_k$$

for $k = 1, 2, \ldots$. Then, when $\ell_j = \nu$, we have $(-1)^{\nu+1} = (-1)^{\ell_j+1} = \mathrm{sgn}_G(g_j) = \chi_G^{(-k)}(g_j)$. Therefore the formula (40) is written as

$$\begin{aligned} f_{\alpha,\beta}(g) &= \prod_{1\le j\le m} \left(\sum_{1\le k<\infty} \chi_G^{(k)}(g_j)\alpha_k^{\ell_j} + \sum_{1\le k<\infty} \chi_G^{(-k)}(g_j)(\alpha_{-k})^{\ell_j} \right) \\ &= \prod_{1\le j\le m} \left(\sum_{k\in \mathbf{Z}^*} \chi_G^{(k)}(g_j)\alpha_k^{\ell_j} \right) \quad \text{with} \quad \mathbf{Z}^* = \mathbf{Z}\setminus\{0\}. \end{aligned} \tag{41}$$

We expand this product into a sum of monomial products as follows. Let $K_+ = \max\{k \ ; \ \alpha_k > 0\}, K_- = \min\{k \ ; \ \alpha_k > 0\}$, and let $\mathbf{Z}_{\alpha,\beta}$ be the intersection of the interval $[K_-, K_+] \subset \mathbf{Z}$ with $\mathbf{Z}^*$. Then the sum over $k \in \mathbf{Z}^*$ in (41) is actually over $k \in \mathbf{Z}_{\alpha,\beta}$. Thus we get

$$f_{\alpha,\beta}(g) = \sum_{(k_1,k_2,\ldots,k_m)\in(\mathbf{Z}_{\alpha,\beta})^m} \ \prod_{1\le j\le m} \chi_G^{(k_j)}(g_j)(\alpha_{k_j})^{\ell_j}, \tag{42}$$

where $g = g_1g_2\cdots g_m$ is a cycle decomposition and ℓ_j is the length of cycle g_j.

As is shown later, this expression of $f_{\alpha,\beta}$ has its own intrinsic meaning in relation to the centralization of matrix elements of certain induced representations of G containing all irreducible unitary representations (= IURs) constructed in [Hi2].

11 IURs of $G = \mathfrak{S}_\infty$ as induced representations

Take a subgroup H of G of the form

$$H = H_0 H_P H_Q, \quad H_P = {\textstyle\prod'_{p\in P}} H_p, \quad H_Q = {\textstyle\prod'_{q\in Q}} H_q, \tag{43}$$

where $H_0 = \mathfrak{S}_B$ with a finite subset $B \subset \mathbf{N}$, $H_p = \mathfrak{S}_{I_p}$ with an infinite subset $I_p \subset \mathbf{N}$, and $H_q = H(\mathcal{I}_q, T_q)$ properly of wreath product type subgroup with $T_q \subset \mathfrak{S}_{n_q}, n_q > 1$, and an infinite partition $\mathcal{I}_q = (\mathcal{J}_{\eta_q})_{\eta_q\in Y_q}$ of $I_q :=$ $\mathrm{supp}(H(\mathcal{I}_q, T_q))$ into ordered n_q-sets $\mathcal{J}_{\eta_q}$. Thus H is determined by the data

$$\mathfrak{c} := (B, (I_p)_{p\in P}, (\mathcal{I}_q, T_q)_{q\in Q})$$

and is denoted also by $H^{\mathfrak{c}}$. We assume that H is "saturated" in G in the sense that

$$\mathbf{N} = B \sqcup (\sqcup_{p\in P} I_p) \sqcup (\sqcup_{q\in Q} I_q) \tag{44}$$

is a partition of $\mathbf{N}$. We admit the cases where some of B, P and Q are empty.

As an IUR of H, we take

$$\pi = \pi_0 \otimes (\otimes_{p\in P}\,\chi_p) \otimes \left(\otimes^b_{q\in Q}\,\pi_q\right), \tag{45}$$

where π_0 is an IUR of $H_0 = \mathfrak{S}_B$, χ_p is a character of $H_p = \mathfrak{S}_{I_p}$ (and so trivial one or sign), and π_q is an IUR of $H_q = H(\mathcal{I}_q, T_q)$, and the tensor product $\otimes^b_{q\in Q}\pi_q$ is taken with respect to a reference vector $b = (b_q)_{q\in Q}, b_q \in V(\pi_q), \|b_q\| = 1$, if $\dim\pi_q > 1$ for infinitely many $q \in Q$. Here $V(\pi_q)$ denotes the representation space of π_q.

As an IUR π_q of the group $H_q = H(\mathcal{I}_q, T_q) \cong \mathfrak{S}_{Y_q}(T_q) := D_{Y_q}(T_q) \rtimes \mathfrak{S}_{Y_q}$, we take the following one. Take an IUR ρ_{T_q} of the finite group T_q, and consider it as an IUR ρ_{α_q} of each component $T_{\eta_q} = T_q$ of $D_{Y_q}(T_q) = \prod'_{\eta_q\in Y_q} T_{\eta_q}$. Making their tensor product, we get an IUR π'_q of the restricted direct product $D_{Y_q}(T_q)$. Here, in case $\dim\rho_{T_q} > 1$, the tensor product is taken with respect to a reference vector

$$a_q = (a_{\eta_q})_{\eta_q\in Y_q} \quad \text{with} \quad a_{\eta_q} \in V(\rho_{\eta_q}), \quad \|a_{\eta_q}\| = 1.$$

For a $\sigma \in \mathfrak{S}_{Y_q}$, put for $\otimes_{\eta_q\in Y_q} w_{\eta_q} \in \otimes^{a_q}_{\eta_q\in Y_q} V(\rho_{\eta_q})$,

$$\pi'_q(\sigma)\left(\otimes_{\eta_q\in Y_q} w_{\eta_q}\right) := \chi_{Y_q}(\sigma)(\otimes_{\eta_q\in Y_q} w'_{\eta_q}), \quad w'_{\eta_q} = w_{\sigma^{-1}(\eta_q)},$$

where χ_{Y_q} is a character of $\mathfrak{S}_{Y_q}$. Then, $\pi'_q(d\cdot\sigma) := \pi'_q(d)\pi'_q(\sigma)$ gives an IUR of $\mathfrak{S}_{Y_q}(T_q)$. Pulling π'_q back to $H_q = H(\mathcal{I}_q, T_q)$ through an isomorphism similar to Φ in §9, we get an IUR π_q of H_q.

Thus the IUR π of $H = H^{\mathfrak{c}}$ is determined by the data $(\mathfrak{c}, \mathfrak{d})$ with

$$\mathfrak{d} := (\pi_0, (\chi_p)_{p\in P}, (b; (\rho_{T_q}, \chi_{Y_q}, a_q)_{q\in Q})),$$

and is denoted also by $\pi(\mathfrak{c}, \mathfrak{d})$.

We know in [Hi2] that, under the saturation condition (44), the induced representation

$$\rho(\mathfrak{c}, \mathfrak{d}) = \mathrm{Ind}^G_H \pi(\mathfrak{c}, \mathfrak{d})$$

is irreducible, and equivalence relations among these IURs are also clarified there. As far as I know, this big family of IURs of $G = \mathfrak{S}_\infty$ contains all IURs known until now.

12 Centralization of matrix elements of IURs

For an IUR given as $\rho(\mathfrak{c}, \mathfrak{d}) = \mathrm{Ind}^G_H \pi(\mathfrak{c}, \mathfrak{d})$, we take one of its matrix elements as a positive definite function on G and study limits of its centralizations. So, take a unit vector $v_0 \in V(\pi_0)$ and $v_Q \in \otimes^b_{q\in Q} V(\pi_q)$, and consider a matrix element f_π of $\pi = \pi(\mathfrak{c}, \mathfrak{d})$ given according to (45) as

$$f_\pi(h) = \langle \pi_0(h_0)v_0, v_0\rangle \cdot (\otimes_{p\in P}\chi_p)(h_P) \cdot \langle (\otimes^b_{q\in Q}\pi_q)(h_Q)v_Q, v_Q\rangle, \tag{46}$$

where $h = h_0 h_P h_Q \in H = H_0 H_P H_Q$ is a decomposition according to (43). Then $F = \mathrm{Ind}_H^G f_\pi$ is a matrix element of $\rho(\mathfrak{c}, \mathfrak{d}) = \mathrm{Ind}_H^G \pi$. Let us study the centralizations F^{G_N} of F for certain increasing sequences $G_N \nearrow G$ of subgroups.

Take $G_N = \mathfrak{S}_{J_N}, J_N \nearrow \mathbf{N}$, as follows. We demand an asymptotic condition

$$\frac{|I_p \cap J_N|}{|J_N|} \to \lambda_p \ \ (p \in P), \qquad \frac{|I_q \cap J_N|}{|J_N|} \to \mu_q \ \ (q \in Q), \tag{47}$$

then there holds

$$\textstyle\sum_{p\in P} \lambda_p + \sum_{q\in Q} \mu_q = 1. \tag{48}$$

Put for the family $\{ H_p = \mathfrak{S}_{I_p} \ ; \ p \in P \}$,

$$P_+ = \{ p \in P \ ; \ \chi_p = \mathbf{1}_{H_p} \}, \quad P_- = \{ p \in P \ ; \ \chi_p = \mathrm{sgn}_{H_p} \}, \tag{49}$$

then we have the following inequality similar as (39)

$$\textstyle\sum_{p\in P_+} \lambda_p + \sum_{p\in P_-} \lambda_p \le 1. \tag{50}$$

At this stage, first let us give our results in the following theorem and the succeeding corollaries, and then give the proof of the theorem in the next section.

From a technical reason for proving the convergence of sequences F^{G_N} as $N \to \infty$, we assume in the following an additional condition on the way of growing up of J_N's, in such a form that, for each $q \in Q$,

$$I_q \cap J_N \quad \text{is a union of subsets } \ \overline{\mathcal{J}_{\eta_q}}, \ \eta_q \in Y_q \ \ (N >> 0). \tag{51}$$

Theorem 13. *Let $H = H_0 H_P H_Q$ be a subgroup of $G = \mathfrak{S}_\infty$, and π be its irreducible unitary representation given above in (43)–(44) and in (45) respectively. For a positive definite function f_π given in (46) as a matrix element of π, put $F = \mathrm{Ind}_H^G f_\pi$. Then it is a positive definite function associated to the induced representation $\rho = \mathrm{Ind}_H^G \pi$.*

According to an increasing sequence $G_N = \mathfrak{S}_{J_N} \nearrow G$ of subgroups, the centralizations F^{G_N} of F converges pointwisely to a Thoma character $f_{\alpha,\beta}$ if $J_N \nearrow \mathbf{N}$ satisfies the asymptotic condition (47). Here the parameters

$\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ are determined from $(\lambda_p)_{p\in P_+}, (\lambda_p)_{p\in P_-}$, respectively by rearranging λ_p's as decreasing sequences.

The inequality (50) *corresponds exactly to* (39), *and $\gamma_0 = \sum_{q\in Q} \mu_q$.*

Put $p_+ = |P_+|, p_- = |P_-|$. Then the lengths of α and β are limitted by p_+ and p_- in such a sense that $\alpha_k = 0\ (k > p_+), \beta_k = 0\ (k > p_-)$.

Corollary 14. (i) *In the case of $Q = \emptyset$, as limits of centralizations of $F = \mathrm{Ind}_H^G f_\pi$, there appear all $f_{\alpha,\beta}$ with $\alpha = (\alpha_1, \alpha_2, \dots)$ limited by p_+ and $\beta = (\beta_1, \beta_2, \dots)$ limited by p_- satisfying the equality*

$$\|\alpha\| + \|\beta\| = \sum_{1\le k<\infty} \alpha_k + \sum_{1\le k<\infty} \beta_k = 1. \tag{52}$$

(ii) *In the case of $Q \neq \emptyset$, as limits of centralizations of $F = \mathrm{Ind}_H^G f_\pi$, there appear all $f_{\alpha,\beta}$ with $\alpha = (\alpha_1, \alpha_2, \dots)$ limited by p_+ and $\beta = (\beta_1, \beta_2, \dots)$ limited by p_- satisfying the inequality* (39): *$\|\alpha\| + \|\beta\| \le 1$, and in particular, $f_{0,0} = \delta_e$ with $\alpha = \beta = \mathbf{0} = (0, 0, \dots)$ and $\gamma_0 = 1$.*

The invariant positive definite function $f_{\alpha,\beta}$ is a matrix element of a II_1 factor representation of G, associated to its cyclic vector. Therefore, in terms of the weak containment topology in the space $\mathrm{Rep}(G)$ of representations [Di, §18], we can translate the above corollary as follows.

Corollary 15. (i) *In the case of $Q = \emptyset$, the closure in* $\mathrm{Rep}(G)$ *of one point set $\{\rho\}$ of irreducible unitary representation $\rho = \mathrm{Ind}_H^G \pi$ contains all* II_1 *factor representations corresponding to $f_{\alpha,\beta}$ with α limited by p_+ and β limited by p_- satisfying the equality* (52).

(ii) *In the case of $Q \neq \emptyset$, the closure in* $\mathrm{Rep}(G)$ *of one point set $\{\rho\}$ contains all* II_1 *factor representations corresponding to $f_{\alpha,\beta}$ with α limited by p_+ and β limited by p_- satisfying the inequality* (39), *and in particular, it contains the regular representation λ_G.*

Notation 12.1. For an IUR $\rho = \mathrm{Ind}_H^G \pi$, $\rho = \rho(\mathfrak{c}, \mathfrak{d}), \pi = \pi(\mathfrak{c}, \mathfrak{d})$, denote by $\mathcal{TC}(\rho)$ the set of all Thoma characters obtained here as limits of centralizations of the matrix element $F = \mathrm{Ind}_H^G f_\pi$. Then,

$$\mathcal{TC}(\rho) := \{\, f_{\alpha,\beta}\ ; \quad \alpha, \beta \text{ coming from } (\lambda_p)_{p\in P_+}, (\lambda_p)_{p\in P_-} \text{ satisfying Condition (TC) } \},$$

$$\text{CONDITION (TC):} \qquad \begin{cases} \sum_{p\in P} \lambda_p = 1 & \text{if } Q = \emptyset\ ; \\ \sum_{p\in P} \lambda_p \le 1 & \text{if } Q \neq \emptyset\ . \end{cases}$$

13 Proof of Theorem 13

13.1. Case of $Q = \emptyset$.

Let us first consider a case where $Q = \emptyset$. Take a $g \in \mathfrak{S}_\infty$ and let

$$g = g_1 g_2 \cdots g_m, \tag{53}$$

be its cycle decomposition. The centralization of $F = \mathrm{Ind}_H^G f_\pi$ over $G_N = \mathfrak{S}_{J_N}$ is

$$F^{G_N}(g) = \frac{1}{|G_N|} \sum_{\sigma \in G_N} F(\sigma g \sigma^{-1}) = \frac{1}{|J_N|!} \sum_{\substack{\sigma \in G_N \\ \sigma g \sigma^{-1} \in H}} f_\pi(\sigma g \sigma^{-1}). \tag{54}$$

Here, $H = H_0 H_P = H_0 \prod'_{p \in P} H_p$, and $f_\pi(h) = \langle \pi_0(h_0) v_0, v_0 \rangle \cdot \prod_{p \in P} \chi_p(h_p)$ for $h = h_0 \prod_{p \in P} h_p \in H_0 \prod'_{p \in P} H_p$.

Suppose N is sufficiently large so that $J_N \supset \sqcup_{1 \le j \le m} K_i$ with $K_i := \mathrm{supp}(g_j)$. Recall that $H_0 = \mathfrak{S}_B, H_p = \mathfrak{S}_{I_p}$ $(p \in P)$, and $\mathrm{supp}(\sigma g_j \sigma^{-1}) = \sigma K_j$, then we see that the condition $\sigma g \sigma^{-1} \in H$ is equivalent to that each $\sigma K_j, 1 \le j \le m$, is contained in some of B, I_p $(p \in P)$. Put

$$\begin{aligned} S(g) &:= \{ \sigma \in G_N = \mathfrak{S}_{J_N} \; ; \; \sigma g \sigma^{-1} \in H \}, \\ S_P(g) &:= \{ \sigma \in S(g) \; ; \; \sigma g \sigma^{-1} \in H_P \}, \\ S^B(g) &:= \{ \sigma \in S(g) \; ; \; \sigma g \sigma^{-1} \text{ has non-trivial} \\ &\qquad\qquad \text{component in } H_0 = \mathfrak{S}_B \}. \end{aligned} \tag{55}$$

Then, $S(g) = S_P(g) \sqcup S^B(g)$, and moreover $S_P(g)$ is decomposed into disjoint sum of its subsets as follows. Let $\delta = \{ J_p \; ; \; p \in P \}$ be a partition indexed by P of the set $I_m = \{ 1, 2, \ldots, m \}$ of indices of g_j's ($J_p = \emptyset$ except for finite number of p), and put

$$S_\delta(g) := \{ \sigma \in S(g) \; ; \; \sigma K_j \subset I_p \; \text{ or } \; \sigma g_j \sigma^{-1} \in \mathfrak{S}_{I_p} = H_p \; (j \in J_p, p \in P) \}.$$

Then $S_P(g) = \sqcup_{\delta \in \mathcal{P}_m} S_\delta(g)$, where $\mathcal{P}_m$ denotes the set of all partitions of I_m indexed by P. Thus we get

$$S(g) := S^B(g) \sqcup \big(\sqcup_{\delta \in \mathcal{P}_m} S_\delta(g) \big). \tag{56}$$

The right hand side of (57) below is a sum over $\sigma \in S(g)$, decomposed into partial sums according to the above decomposition of $S(g)$,

$$F^{G_N}(g) = \frac{1}{|J_N|!} \sum_{\sigma \in S^B(g)} f_\pi(\sigma g \sigma^{-1}) + \sum_{\delta \in \mathcal{P}_m} \frac{1}{|J_N|!} \sum_{\sigma \in S_\delta(g)} f_\pi(\sigma g \sigma^{-1}). \tag{57}$$

We study the second term. Put $h_j = \sigma g_j \sigma^{-1}$, then $\sigma g \sigma^{-1} = h_1 h_2 \cdots h_m$. For $\delta = \{ J_p \,;\, p \in P \} \in \mathcal{P}_m$, $h_j \in H_p$ and $\chi_p(h_p) = 1$ or $= \mathrm{sgn}(g_j) = (-1)^{\ell_j - 1}$ with $\ell_j = \ell(g_j)$. Denote this value by $\chi_p(g_j)$, then $f_\pi(\sigma g \sigma^{-1}) = \prod_{p\in P} \prod_{j \in J_p} \chi_p(g_j)$. Hence we have

$$\frac{1}{|J_N|!} \sum_{\sigma \in S_\delta(g)} f_\pi(\sigma g \sigma^{-1}) = \prod_{p\in P} \prod_{j\in J_p} \chi_p(g_j) \cdot \frac{|S_\delta(g)|}{|J_N|!}. \tag{58}$$

The number of elements $|S_\delta(g)|$ is given from the condition $\sigma K_j \subset I_p \cap J_N$ $(j \in J_p)$. Since $|K_j| = \ell_j$, we can choose for $\sqcup_{j\in J_p} \sigma K_j$ freely $\sum_{j \in J_p} \ell_j$ number of elements from $I_p \cap J_N$. Noting that $\sum_{p\in P} \sum_{j \in J_p} \ell_j = \sum_{i \in I_m} \ell_j$, we get

$$\begin{aligned} |S_\delta(g)| &= \prod_{p \in P} |I_p \cap J_N| (|I_p \cap J_N| - 1) \cdots \Big(|I_p \cap J_N| - \textstyle\sum_{j\in J_p} \ell_j + 1\Big) \\ &\quad \times \Big(|J_N| - \textstyle\sum_{j \in I_m} \ell_j\Big)!. \end{aligned} \tag{59}$$

When J_N grows up to N under the condition $|I_p \cap J_N|/|J_N| \to \lambda_p$ $(p \in P)$, we have

$$\textstyle\sum_{p\in P} \lambda_p = 1. \tag{60}$$

Furthermore, deviding the both sides of (59) by $|J_N|!$, and taking limits as $N \to \infty$, we obtain

$$\lim_{N\to\infty} \frac{|S_\delta(g)|}{|J_N|!} = \prod_{p \in P} \prod_{j \in J_p} \chi_p(g_j)\, \lambda_p^{\ell_j} \quad \text{with} \quad \ell_j = \ell(g_j).$$

Thus the limit of the second term of (57) gives us

$$\sum_{\delta \in \mathcal{P}_m} \prod_{p\in P} \prod_{j \in J_p} \chi_p(g_j)(\lambda_p)^{\ell(g_j)} = \prod_{j=1}^{m} \left(\sum_{p \in P} \chi_p(g_j) \lambda_p^{\ell(g_j)} \right). \tag{61}$$

On the other hand, for the first term of (57), an evaluation similar to that of $|S_\delta(g)|$ proves that its limit as $N \to \infty$ is equal to zero (see, **13.2** below). Or this fact follows also from (60) through the theory of positive definite functions.

Compairing the above formula (61) with the formula (41) or (42), we see that the proof of Theroem 13 in the case $Q = \emptyset$ is now complete.

13.2. Case of $Q \neq \emptyset$.

Here we study the general case of $Q \neq \emptyset$. Let $S(g) = \{ \sigma \in G_N = \mathfrak{S}_{J_N} \,;\, \sigma g \sigma^{-1} \in H \}$ and $S^B(g), S_P(g)$ be as in **13.1**, and in addition put

$$S^Q(g) := \{ \sigma \in S(g) \,;\, \sigma g \sigma^{-1} \text{ has non-trivial component in } H_Q \}. \tag{62}$$

Then, $S(g) = \big(S^B(g) \cup S^Q(g)\big) \sqcup S_P(g)$, and accordingly the formula (57) is rewritten as

$$F^{G_N}(g) = \frac{1}{|J_N|!} \sum_{\sigma \in S^B(g) \cup S^Q(g)} f_\pi(\sigma g \sigma^{-1}) + \sum_{\delta \in \mathcal{P}_m} \frac{1}{|J_N|!} \sum_{\sigma \in S_\delta(g)} f_\pi(\sigma g \sigma^{-1}). \quad (63)$$

Denote by $\Sigma_I(g;N)$ and $\Sigma_{II}(g;N)$ the first term and the second term in the right hand side. We want to prove that $\Sigma_I(g;N) \to 0$ as $N \to \infty$, under the condition

$$\frac{|I_p \cap J_N|}{|J_N|} \to \lambda_p \ \ (p \in P), \quad \frac{|I_q \cap J_N|}{|J_N|} \to \mu_q \ \ (q \in Q). \quad (64)$$

If this is done, the proof of Theorem 13 will be completed, because the limit of the second term $\Sigma_{II}(g;N)$ can be obtained just as in **13.1**.

Now let $\delta' = \{\, J_0, J_p \ (p \in P), J_q \ (q \in Q)\,\}$ be a partition of I_m for which at least one of J_0, J_q $(q \in Q)$ is non-empty. For $\sigma \in S(g)$, put $h = \sigma g \sigma^{-1}, h_j = \sigma g_j \sigma^{-1}$ $(j \in I_m)$, then $h = h_1 h_2 \cdots h_m$. Define

$$S_{\delta'}(g) := \{\sigma \in S(g) \ ; \ h_j = \sigma g_j \sigma^{-1} \ (j \in I_m) \ \text{ satisfy Condition (SQ)}\,\}$$

$$\text{Condition (SQ):} \qquad \begin{cases} h_j \in H_0 = \mathfrak{S}_B \ \text{ or } \sigma K_j \subset B \ \ (j \in J_0), \\ h_j \in H_p = \mathfrak{S}_{I_p} \ \text{ or } \sigma K_j \subset I_p \ \ (j \in J_p, p \in P), \\ h_j \in H_q = H(\mathcal{I}_q, T_q) \ \ (j \in J_q, q \in Q). \end{cases}$$

Denote by $\mathcal{P}'_m$ the set of all possible such partitions δ'. Noting that $|f_\pi(\sigma g \sigma^{-1})| \leq 1$, we get the evaluation

$$|\Sigma_I(g;N)| \leq \textstyle\sum_{\delta' \in \mathcal{P}'_m} \dfrac{|S_{\delta'}(g)|}{|J_N|!}. \quad (65)$$

So we should evaluate the number $|S_{\delta'}(g)|$.

For a subset $J \subset I_m$ and a subgroup H' of H, we denote by $DF(J, H')$ the number of possible ways for choosing integers $\sigma(k) \in J_N$ $(k \in \cup_{j \in J} K_j)$ under Condition (SQ) in such a way that $\sigma(\prod_{j \in J} g_j)\sigma^{-1} = \prod_{j \in J} h_j \in H'$. ($DF$ = degree of freedom). Similarly, for $K = J_N \setminus \mathrm{supp}(g) = J_N \setminus \cup_{j \in I_m} K_j$, denote by $DF'(K, H)$ the number of possible ways for choosing integers $\sigma(k) \in J_N$ $(k \in K)$ under Condition (SQ) in such a way that $\sigma g \sigma^{-1} = h \in H$ (after choosing all of $\sigma(k), k \in \mathrm{supp}(g)$). Then,

$$\begin{aligned} |S_{\delta'}(g)| &= DF(J_0, H_0) \cdot \textstyle\prod_{p \in P} DF(J_p, H_p) \\ &\times \textstyle\prod_{q \in Q} DF(J_q, H_q) \times DF'(J_N \setminus \cup_{j \in I_m} K_j, H), \end{aligned} \quad (66)$$

where $K_j = \mathrm{supp}(g_j), \cup_{j \in I_m} K_j = \mathrm{supp}(g)$.

In **13.1**, we calculated $DF(J_p, H_p = \mathfrak{S}_{I_p})$ as given below, noting that the condition (SQ) for this term is equivalent to $\sigma(K_j) \subset I_p\, (j \in J_p)$ and that $|\bigcup_{j\in J_p} K_j| = \sum_{j\in J_p} \ell_j$,

$$DF(J_p, H_p) = |I_p \cap J_N|(|I_p \cap J_N| - 1) \cdots \left(|I_p \cap J_N| - \sum_{j\in J_p} \ell_j + 1\right).$$

Similarly $DF(J_0, H_0 = \mathfrak{S}_B)$ is given as follows if N is sufficiently large so that $B \subset J_N$:

$$DF(J_0, H_0) = |B|(|B| - 1) \cdots \left(|B| - \sum_{j\in J_0} \ell_j + 1\right). \tag{67}$$

After taking all of $\sigma(k), k \in \mathrm{supp}(g) = \bigcup_{j\in I_m} K_j$, other elements $\sigma(i), i \in J_N \setminus \mathrm{supp}(g)$ can be chosen freely, and so

$$DF'(J_N \setminus \bigcup_{j\in I_m} K_j, H) = \left(|J_N| - \sum_{j\in I_m} \ell_j\right)!. \tag{68}$$

Note that, as $N \to \infty$, the factor $1/|J_N|!$ in (63) can be replaced by a simpler one if we note

$$\frac{1}{|J_N|!} \times \left(|J_N| - \sum_{j\in I_m} \ell_j\right)! \times \prod_{j\in I_m} |J_N|^{\ell_j} \longrightarrow 1 \quad (N \to \infty).$$

Then we see that the contribution to the limit from a partial sum for δ' is majorized by

$$\begin{aligned}
\lim_{N\to\infty} \frac{|S_{\delta'}(g)|}{|J_N|!} &= \lim_{N\to\infty} \frac{|B|}{|J_N|} \cdot \frac{|B|-1}{|J_N|} \cdot \cdots \cdot \frac{|B| - \sum_{j\in J_0} \ell_j + 1}{|J_N|} \\
&\times \prod_{p\in P} \lim_{N\to\infty} \frac{|I_p \cap J_N|}{|J_N|} \cdot \frac{|I_p \cap J_N| - 1}{|J_N|} \cdot \cdots \cdot \frac{|I_p \cap J_N| - \sum_{j\in J_p} \ell_j + 1}{|J_N|} \\
&\times \prod_{q\in Q} \lim_{N\to\infty} \frac{DF(J_q, H_q)}{\prod_{j\in J_q} |J_N|^{\ell_j}}.
\end{aligned} \tag{69}$$

Therefore, if $J_0 \neq \emptyset$ in δ', or if the first factor (containing $|B|$) actually exists in the right hand side of (69), then it is equal to zero and so the left hand side (contribution to the limit) is also zero.

13.3. Calculation for wreath product subgroup $H_q = H(\mathcal{I}_q, T_q)$.
Now assume $J_0 = \emptyset$ in δ'. Then it is enough for us to prove that the ratio

$$DF(J_q, H_q)/\prod_{j\in J_q} |J_N|^{\ell_j} \tag{70}$$

tends to zero as $N \to \infty$ for $J_q \neq \emptyset$. Recall that

$$H_q = H(\mathcal{I}_q, T_q) \cong \mathfrak{S}_{Y_q}(T_q) := D_{Y_q}(T_q) \rtimes \mathfrak{S}_{Y_q},$$

with a subgroup $T_q \subset \mathfrak{S}_{n_q}, n_q > 1$, and an infinite partition $\mathcal{I}_q = (\mathcal{J}_\eta)_{\eta \in Y_q}$ of $I_q := \mathrm{supp}(H(\mathcal{I}_q, T_q))$ into ordered n_q-sets $\mathcal{J}_\eta$. Replasing $T_q \subset \mathfrak{S}_{n_q}$ by $\mathfrak{S}_{n_q}$, we get a bigger group $\widetilde{H}_q = H(\mathcal{I}_q, \mathfrak{S}_{n_q})$, so that $H_q \subset \widetilde{H}_q \subset \mathfrak{S}_{I_q}$. Since $DF(J_q, H')$ is defined by the condition $\prod_{j \in J_q} h_j \in H'$ for $h_j = \sigma g_j \sigma^{-1}$, there holds

$$DF(J_q, H_q) \le DF(J_q, \widetilde{H}_q) \le DF(J_q, \mathfrak{S}_{I_q}).$$

Here the last term is given by a formula similar to that for $DF(J_p, H_p)$ by means of $\cup_{j \in J_q} K_j$ and I_q. Evaluating the middle term, we get the desired result. Here we omit the details.

By 13.2–13.3, the proof of Theorem 13 in the case of $Q \neq \emptyset$ is now complete.

14 Case of non-irreducible unitary representations

We keep to the notation in §11. Assume $Q \neq \emptyset$ in (43), and consider a subgroup $H' = H_0 H_P$ omitting H_Q (or replacing H_Q by $H'_Q = \{ e \}$), and also a subgroup $H'' = H_P$ in place of $H = H_0 H_P H_Q$. These subgroups are small and far from saturated in G. Take an IUR π' of H', and such a one π'' of H'' given as

$$\pi' = \pi_0 \otimes (\otimes_{p \in P} \chi_p), \qquad \pi'' = \otimes_{p \in P} \chi_p , \tag{71}$$

and consider induced representations of G as

$$\rho' = \mathrm{Ind}_{H'}^G \pi', \qquad \rho'' = \mathrm{Ind}_{H''}^G \pi'',$$

which are very far from to be irreducible. Let $f_{\pi'}$ and $f_{\pi''}$ be positive definite functions given as matrix elements of π' and π'' as

$$\begin{aligned} f_{\pi'}(h') &= \langle \pi_0(h'_0) v_0, v_0 \rangle \cdot \Big(\textstyle\prod_{p \in P} \chi_p\Big)(h'_P), \\ f_{\pi''}(h'') &= \Big(\textstyle\prod_{p \in P} \chi_p\Big)(h'_P), \end{aligned}$$

for $h' = h'_0 h'_P \in H' = H_0 H_P$, and a unit vector $v_0 \in V(\pi_0)$, and $h'' = h''_P \in H'' = H_P$ respectively. Put

$$F' = \mathrm{Ind}_{H'}^G f_{\pi'}, \qquad F'' = \mathrm{Ind}_{H''}^G f_{\pi''},$$

then F' and F'' are positive definite functions on G, or matrix elements associated to the induced representations $\rho' = \mathrm{Ind}_{H'}^G \pi'$ and $\rho'' = \mathrm{Ind}_{H''}^G \pi''$ respec-

Taking limits of centralizations of F' or F'', similarly as for $F = \mathrm{Ind}_H^G f_\pi$ in (46) with $H = H_0 H_P H_Q$ in (43) and π in (45), we get exactly the same family of Thoma characters $f_{\alpha,\beta}$, extremal invariant positive definite functions on G.

In more detail, repeating the discussions in §13 (essentially those in **13.1**), we obtain the following result, rather astonishing.

Theorem 16. *Let $G_N = \mathfrak{S}_{J_N}$ be an incresing sequence of subgroups going up to $G = \mathfrak{S}_\infty$. Assume that for every $p \in P$,*

$$|I_p \cap J_N|/|J_N| \longrightarrow \lambda_p \quad (N \to \infty).$$

Then, the centralizations of F' and F'' over G_N tend respectively to a Thoma character $f_{\alpha,\beta}$ pointwisely, where the decreasing sequences of non-negative integers $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ are reorderings of $\{\, \lambda_p \,;\, p \in P_+ \,\}$ and $\{\, \lambda_p \,;\, p \in P_- \,\}$ respectively.

These convergences are quite similar as for $F = \mathrm{Ind}_H^G f_\pi$, and are proved word for word as for the second term in (63) (cf. **13.1**).

15 Remarks and comments

15.1. Irreducible decompositions of factor representations.

Here we treat two extreme cases of Thoma character $f_{\alpha,\beta}$, where $\gamma_0 = 0$ or $\gamma_0 = 1$, with $\|\alpha\| + \|\beta\| + \gamma_0 = 1$.

Case of $\gamma_0 = 0$ or $\|\alpha\| + \|\beta\| = 1$.

An irreducible decomposition of a factor representation π_f (of type II_1) associated to $f = f_{\alpha,\beta}$ is given in [Ob2].

His result says the following. Let i_+ and i_- be natural numbers such that

$$\alpha_{i_+} > \alpha_{i_+ + 1} = 0 \quad \text{and} \quad \beta_{i_-} > \beta_{i_- + 1} = 0.$$

Then, the factor representation π_f is decomposed as an integral of IURs $\rho(\mathfrak{c}, \mathfrak{d}) = \mathrm{Ind}_H^G \pi(\mathfrak{c}, \mathfrak{d})$ with infinite multiplicity, where $\mathfrak{c} = (I_p)_{p \in P}, \mathfrak{d} = (\chi_p)_{p \in P}$, satisfying the condition

$$p_+ := |P_+| = i_+, \quad p_- := |P_-| = i_-. \tag{72}$$

Here $H = \prod'_{p \in P} \mathfrak{S}_{I_p}$ a restricted direct product of $\mathfrak{S}_{I_p} \cong \mathfrak{S}_\infty$, and $\pi(\mathfrak{c}, \mathfrak{d}) = \otimes_{p \in P} \chi_p$ a character of H, and P_+, P_- are defined in (49).

From this result, we can define $\mathrm{supp}(\pi_f) \subset \widehat{G}$ the dual space of G for $f = f_{\alpha,\beta}$ by

$$\begin{aligned}\mathrm{supp}(\pi_f) \;:=\; & \big\{\, \rho(\mathfrak{c},\mathfrak{d}) = \mathrm{Ind}_H^G\big(\otimes_{p\in P}\,\chi_p\big)\;;\;\mathfrak{c} = (I_p)_{p\in P},\\ & \mathfrak{d} = (\chi_p)_{p\in P}, H = \textstyle\prod'_{p\in P}\mathfrak{S}_{I_p}, |P_+| = i_+, |P_-| = i_-\,\big\}.\end{aligned}$$

We can characterize $\mathrm{supp}(\pi_f)$ from the view point of the topology in $\mathrm{Rep}(G)$, or more exactly from the set $\mathcal{TC}(\rho)$ in Notation 12.1 of Thoma characters obtained as limits of centralizations of matrix element $F = \mathrm{Ind}_H^G f_\pi$ of an IUR $\rho = \mathrm{Ind}_H^G \pi$.

Fix $f = f_{\alpha,\beta}$ and consider an IUR $\rho = \mathrm{Ind}_H^G\pi$ such that $\mathcal{TC}(\rho) \ni f_{\alpha,\beta}$. Let $\rho = \rho(\mathfrak{c},\mathfrak{d}), \pi = \pi(\mathfrak{c},\mathfrak{d})$, and $\mathfrak{c} = (B, (I_p)_{p\in P}, (\mathcal{I}_q, T_q)_{q\in Q})$. We say that ρ can attain $f_{\alpha,\beta}$ (or that $\mathcal{TC}(\rho)$ contains $f_{\alpha,\beta}$) *without redundancy* if $B = \emptyset$, $|P_+| = i_+, |P_-| = i_-$, and in addition $Q = \emptyset$ in case $\|\alpha\| + \|\beta\| = 1$ (or $\gamma_0 = 0$). The meaning of this terminology is that $B \neq \emptyset$ has no effect to the set $\mathcal{TC}(\rho)$, and that, if $|P_+| > i_+$ for exmple, we put $\lambda_p = 0$ for some $p \in P_+$ (in other words, kill the role of p) to get $f_{\alpha,\beta}$. Put

$$\mathcal{IUR}(f_{\alpha,\beta}) := \big\{\, \rho = \mathrm{Ind}_H^G\pi\;;\;\mathcal{TC}(\rho) \ni f_{\alpha,\beta},\;\; \text{without redundancy}\,\big\}$$

Proposition 17. *Assume* $\|\alpha\| + \|\beta\| = 1$. *For the indecomposable positive definite class function* $f = f_{\alpha,\beta}$, *the support* $\mathrm{supp}(\pi_f)$ *of* II_1 *factor representation* π_f *is characterized as follows:*

$$\mathrm{supp}(\pi_f) = \mathcal{IUR}(f), \qquad f = f_{\alpha,\beta}. \tag{73}$$

Remark 15.1. The expression given in (42) of $f_{\alpha,\beta}$ plays an important role for our calculation in §13. It has also an intimate relation to Obata's method in [Ob2] of giving irreducible decompositions of π_f, $f = f_{\alpha,\beta}$.

Case of $\gamma_0 = 1$ or $\alpha = \beta = 0$ (regular representation).

The regular representation λ_G is a factor representation associated to $f_{0,0} = \delta_e$. When we extend the above situation in the case of the factor representation π_f associated to $f = f_{\alpha,\beta}$ with $\gamma_0 = 0$ to the case of $f_{0,0}$ with $\gamma_0 = 1$, we can make a speculation about the support of λ_G (or support of Plancherel measure for G). Note that, for an IUR $\rho = \mathrm{Ind}_H^G\pi$, "$\mathcal{TC}(\rho)$ contains $f_{0,0}$ without redundancy" means that $B = P = \emptyset$ for ρ. Then, in this case, $\mathcal{TC}(\rho) = \{\, f_{0,0}\,\}$. We may take our speculation as

First working hypothesis:

$\mathrm{supp}(\lambda_G)$ *is equal to or is contained in the set* $\mathcal{IUR}(f_{0,0})$.

15.2. Classification of indecomposable positive definite functions. Aiming to apply our method of "*taking limits of centralizations*" of positive definite functions to other types of infinite discrete groups, we analyse relations of our present results to Thoma's results in [Th2].

Main important points in [Th2] can be considered as the following.

(1) Criterion for indecomposability of positive definite class functions;

(2) Sufficient condition for positive definiteness;

(3) Necessary condition for positive definiteness.

In that paper, after establishing a simple criterion for (1), the author studied (2) and (3) at the same time by applying a deep theory of analytic functions defined on discs.

Here in this paper, we established the second part (2) by the method of 'taking limits of centralizations', a proof quite different from that in [Th2] and much simpler one.

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