数理解析研究所講究録 1278 巻 2002 年 75-85

Infinitesimal generators of one-parameter unitary groups on a Boson Fock space

摂南大学 工学部 廣島文生*(Fumio Hiroshima)

Abstract

It is shown that a certain one-parameter symplectic group induces a one-parameter unitary group on a Boson Fock space through the so-called Bogoliubov transformation. An infinitesimal generator Δ of a one-parameter unitary group is given, and it is shown that Δ is quadratic.

1 Introduction

This is a joint work with K. R. Ito.[1] In the white noise analysis infinite dimensional rotation groups acting on (S') have been studied so far by many authors, e.g., see Hida [2]. Here (S') is a dual of a subspace (S) of a Boson Fock space $\mathcal{F}$. Such rotation groups are indueced from e.g., the conformal group (shifts, dilations, $SO(n)$, and special conformal transformations), the Lévy group, etc. Their infiniteisimal generators define infinite dimensional Laplacians, e.g., the Gross Laplacian, the Lévy Laplacian, etc. Formally these Laplacians are quadratic with respect to the annihilation and the creation operators in (S'). Then these play an important role of the infinite dimensional harmonic analysis in the white noise analysis.

The Bogoliubov transformation can be regarded as a map from a symplectic group to unitary operators acting on $\mathcal{F}$. The Bogoliubov transformation leaves the canonical commutation relations of the annihilation and the creation operators invariant. As is seen in this paper below, the Bogoliubov transformation associated with an element $\mathcal{A}$ of a symplectic group has the form

$$U(\mathcal{A})=\det(1-K_1^*K_1)^{1/4}\times :e^{-\frac{1}{2}(\Delta_{K_1}^*+2N_{K_2}+\Delta_{K_3})}: . \tag{1.1}$$

Here Δ_{K_j}, $j=1,3$, and N_{K_2} are quadratic operators defined by $\mathcal{A}$. The formal expression (1.1) has a rigorous mathematical meaning as an unitary operator. See

*This work is supported by Grant-in-Aid 13740106 for Encouragement of Young Scientists from the Ministry of Education, Science, Sports and Culture. e-mail: hiroshima@mpg.setsunan.ac.jp

[1] e-mail: ito@mpg.setsunan.ac.jp

Berezin [1] and Ruijsenaars [5]. It has been also known that a Bogolibov transformation induces a projective unitary representation on $\mathcal{F}$ of a subgroup of a symplectic group. See Shale [4].

$$\begin{tikzcd} \mathcal{F} \arrow[r, phantom, "\subset"] \arrow[d, "U(\mathcal{A})"'] & (S') \arrow[d, "g"] \\ \mathcal{F} \arrow[r, phantom, "\subset"] & (S') \end{tikzcd}$$

Figure 1: $U(\mathcal{A})$ and rotation group g

In this paper we give an example such that a certain one-parameter subgroup of a symplectic group yields a one-parameter unitary group on a Boson Fock space through the Bogoliubov transformation. Morever we show that the generator of a one-parameter unitary group, which is a self-adjoint operator, is also quadratic with respect to the annihilation and the creation operators.

2 Boson Fock space

We review fundamental facts on a Boson Fock space. Let $\mathcal{H}$ be a Hilbert space over the complex field $\mathbb{C}$ and $\mathcal{F} = \mathcal{F}(\mathcal{H})$ denote the Boson Fock space over $\mathcal{H}$ given by

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s^n},$$

where $\mathcal{H}^{\otimes_s^n}$ denotes the n-fold symmetric tensor product of $\mathcal{H}$ with $\mathcal{H}^{\otimes_s^0} := \mathbb{C}$. Vector Ψ of $\mathcal{F}$ is written as $\Psi = \left\{\Psi^{(0)}, \Psi^{(1)}, \Psi^{(2)}, \cdots\right\}$ with $\Psi^{(n)} \in \otimes_s^n \mathcal{H}$. The vacuum Ω is defined by

$$\Omega := \{1, 0, 0, \cdots\}.$$

The creation operator $a^\dagger(f) : \mathcal{F} \to \mathcal{F}$ smeared by $f \in \mathcal{H}$ is given by

$$\left(a^\dagger(f)\Psi\right)^{(n)} := S_n(f \otimes \Psi^{(n-1)}),$$

where S_n denotes the symetrizer of n-degree. Let

$$\mathcal{F}_0 := \text{the linear hull of } \{a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega | f_j \in \mathcal{H}, j = 1, ..., n, n \geq 0\}.$$

It is known that $\mathcal{F}_0$ is dense in $\mathcal{F}$. The annihilation operator $a(f)$ is defined by

$$a(f) := \left(a^\dagger(\bar{f})\lceil_{\mathcal{F}_0}\right)^*,$$

where $\bar{}$ denotes the complex conjugate. It holds that

$$(\Psi, a^\dagger(f)\Phi)_{\mathcal{F}} = (a(\bar{f})\Psi, \Phi)_{\mathcal{F}}, \quad \Psi, \Phi \in \mathcal{F}_0,$$

and

$$a(f)\Omega = 0. \tag{2.1}$$

Conversely if $a(f)\Psi = 0$ for all $f \in \mathcal{H}$, then Ψ is a multiple of Ω, i.e., $\Psi = \alpha\Omega$ with some $\alpha \in \mathbb{C}$. The creation operator and the annihilation operator satisfy the canonical commutation relations (CCR):

$$[a(f), a^\dagger(g)] = (\bar{f}, g)_{\mathcal{H}},$$

$$[a(f), a(g)] = 0,$$

$$[a^\dagger(f), a^\dagger(g)] = 0$$

on $\mathcal{F}_0$, where $(f, g)_{\mathcal{K}}$ denotes the scalar product on Hilbert space $\mathcal{K}$, which is linear in g and antilinear in f. In addition, we denote by $\|f\|_{\mathcal{K}}$ the associated norm. From (2.1) and CCR it follows that

$$\|a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega\|^2 = \|f_1\|^2 \cdots \|f_n\|^2.$$

Let $R(f) := 2^{-1/2}(a(f) + a^\dagger(\bar{f}))$. Suppose that a bounded operator A commutes with $e^{iR(f)}$ for all $f \in \mathcal{H}$. Then it is proven that A is a multiple of the identity. This is called that $R(f)$ is *irreducible*.

3 Projective unitary representations

3.1 Symplectic group

Let $B = B(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$ and $H_2 = H_2(\mathcal{H})$ Hilbert Schmidt operators. Let us define

$$\overline{K}f := \overline{K\bar{f}}.$$

Since $\overline{(K^*)} = (\overline{K})^*$, we write simply as $\overline{K}^*$. For $S, T \in$ B we define

$$\mathcal{A} := \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$$

$$\mathcal{A}\begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} S\phi + \overline{T}\psi \\ T\phi + \overline{S}\psi \end{pmatrix}.$$

Let

$$J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define the symplectic group Σ and a subgroup Σ_2 as follows.

Definition 3.1 *(1) We say that* $\mathcal{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma$, *if*

$$\mathcal{A}J\mathcal{A}^* = \mathcal{A}^*J\mathcal{A} = J. \tag{3.1}$$

(2) We say that $\mathcal{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma_2$, *if* $\mathcal{A} \in \Sigma$ *and* $T \in H_2$.

Note that the inverse $\mathcal{A}^{-1}$ of $\mathcal{A}$ is given by

$$\mathcal{A}^{-1} = J\mathcal{A}^*J = \begin{pmatrix} S^* & -T^* \\ -\overline{T}^* & \overline{S}^* \end{pmatrix}. \tag{3.2}$$

We equip Σ_2 with the topology as follows. We say $\mathcal{A}_n = \begin{pmatrix} S_n & \overline{T}_n \\ T_n & \overline{S}_n \end{pmatrix} \to \mathcal{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix}$ as $n \to \infty$ if $S_n \to S$ in $B(\mathcal{H})$ and $T_n \to T$ in H_2. Σ equipped with this topology becomes the topological group.

3.2 Bogoliubov transformation

Let $K \in H_2$. Then there exist complete orthonormal systems (CONS's) $\{\psi_n\}$, $\{\phi_m\}$, and a positive sequence $\{\lambda_n\}$ such that

$$Kf = \sum_{n=0}^{\infty} \lambda_n (\psi_n, f)\phi_n, \quad f \in \mathcal{H},$$

with $\sum_{n=0}^{\infty} \lambda_n^2 = \|K\|_{H_2}^2$. We define for $\Psi \in \mathcal{F}_0$

$$\langle a^\dagger | K | a^\dagger \rangle \Psi := s - \lim_{N\to\infty} \sum_{n=0}^{N} \lambda_n a^\dagger(\overline{\psi}_n) a^\dagger(\phi_n)\Psi,$$

$$\langle a | K | a \rangle \Psi := s - \lim_{N\to\infty} \sum_{n=0}^{N} \lambda_n a(\overline{\psi}_n) a(\phi_n)\Psi.$$

Moreover for $S \in \mathrm{B}(\mathcal{H})$ we dfine

$$\langle a^\dagger|S|a\rangle := s - \lim_{N\to\infty} \sum_{n=0}^{N} a^\dagger(e_n) a(\overline{S^* e_n}),$$

where $\{e_n\}$ is a CONS. Note that $\langle a^\dagger|S|a\rangle$ is independent of the choice of $\{e_n\}$. Let $\Psi = a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega$. Then

$$\langle a|K|a\rangle\Psi = \sum_{i\neq j} (\overline{f}_j, K f_i) a^\dagger(f_1)\cdots \widehat{a^\dagger}(f_i)\cdots \widehat{a^\dagger}(f_j)\cdots a^\dagger(f_n)\Omega,$$

and

$$\langle a^\dagger|K|a\rangle\Psi = \sum_{j=1}^{n} a^\dagger(f_1)\cdots a^\dagger(K f_j)\cdots a^\dagger(f_n)\Omega,$$

where $\widehat{\ }$ denotes omitting the term below. We simply write

$$\langle a^\dagger|K|a^\dagger\rangle = \Delta_K^*,$$

$$\langle a^\dagger|S|a\rangle = N_S,$$

$$\langle a|K|a\rangle = \Delta_K.$$

Let N be the number operator and define

$$\mathcal{D}_\infty := \cap_{k=1}^\infty D(N^k).$$

Proposition 3.2 *(1) Suppose that*

$$(i) K \in H_2, \quad (ii) \overline{K}^* = K, \quad (iii) \|K\|_{B(\mathcal{H})} < 1.$$

Then

$$U_1(K) := s - \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!}\left(-\frac{1}{2}\langle a^\dagger|K|a^\dagger\rangle\right)^n \Psi$$

exists for $\Psi \in \mathcal{F}_0$, and $U_1(K)\Psi \in \mathcal{D}_\infty$.
(2) Suppose that $S \in \mathrm{B}$ and $K \in H_2$. Then

$$U_2(S) := s - \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} :\left(-\frac{1}{2}\langle a^\dagger|S|a\rangle\right)^n : \Psi$$

and

$$U_3(K) := s - \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!}\left(-\frac{1}{2}\langle a|K|a\rangle\right)^n \Psi$$

exist for $\Psi \in \mathcal{F}_0$, and $U_2(K)\Psi, U_3(L)\Psi \in \mathcal{F}_0$, where $:X:$ denotes the Wick ordering.

Proof: See Ruijsenaars [5]. □

$\mathcal{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix}$ induces the following action:

$$\mathcal{A} : a(f) \longmapsto a(Sf) + a^\dagger(Tf) := b_{\mathcal{A}}(f) \tag{3.3}$$

and

$$\mathcal{A} : a^\dagger(f) \longmapsto a(\overline{T}f) + a^\dagger(\overline{S}f) := b^\dagger_{\mathcal{A}}(f). \tag{3.4}$$

Formally we may write

$$(b_{\mathcal{A}}(f), b^\dagger_{\mathcal{A}}(f)) = (a(f), a^\dagger(f)) \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix}.$$

Suppose $\mathcal{A} \in \Sigma$. Then the canonical commutation relations

$$[b_{\mathcal{A}}(f), b^\dagger_{\mathcal{A}}(g)] = (\overline{f}, g),$$

$$[b_{\mathcal{A}}(f), b_{\mathcal{A}}(g)] = 0,$$

$$[b^\dagger_{\mathcal{A}}(f), b^\dagger_{\mathcal{A}}(g)] = 0,$$

and

$$(\Psi, b^\dagger_{\mathcal{A}}(f)\Phi)_{\mathcal{F}} = (b_{\mathcal{A}}(\overline{f})\Psi, \Phi)_{\mathcal{F}}, \quad \Psi, \Phi \in \mathcal{F}_0,$$

follow. The map (3.3) and (3.4) are the so-called *homogeneous Bogoliubov transformation.* It is well known that $b^\sharp_{\mathcal{A}}(f)$ is unitarily equivalent with $a^\sharp(f)$ if and only if $\mathcal{A} \in \Sigma_2$. See Berezin [1].

3.3 Construction of Bogoliubov transformation

Now we want to construct a unitary operator implementing a unitary equivalence between $b^\sharp_{\mathcal{A}}(f)$ and $a^\sharp(f)$. We need some preparations.

(3.1) is equivalent with

$$S^*S - T^*T = 1, \tag{3.5}$$

$$\overline{S}^*T - \overline{T}^*S = 0, \tag{3.6}$$

$$SS^* - \overline{TT}^* = 1, \tag{3.7}$$

$$TS^* - \overline{ST}^* = 0. \tag{3.8}$$

Lemma 3.3 *Let* $\mathcal{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma$. *Then (1)* $S^{-1} \in B$, *(2)* $\|TS^{-1}\| < 1$, *(3)* $\overline{TS^{-1}}^* = TS^{-1}$, *(4)* $\overline{S^{-1}T}^* = S^{-1}T$.

Proof: From (3.5) it follows that

$$S^*S = 1 + T^*T \geq 1. \tag{3.9}$$

Thus (1) follows. In the case of $\|T\| = 0$, $\|TS^{-1}\| = 0 < 1$. We may assum that $\|T\| = \epsilon > 0$. By (3.9) we have

$$TS^{-1} = (1 + T^*T)^{-1}S^*,$$

which implies that

$$(TS^{-1})(TS^{-1})^* = (1 + T^*T)^{-1}S^*S(1 + T^*T)^{-1} = (1 + T^*T)^{-2}(1 + T^*T).$$

Thus

$$\|TS^{-1}\| \leq \|(1 + T^*T)^{-1}\| \leq \frac{1}{1 + \epsilon^2} < 1.$$

Thus (2) follows. By (3.6) we have $\overline{S}^*TS^{-1} = \overline{T}^*$. Then $S^*\overline{TS^{-1}} = T^*$ follows. Note that $(S^*)^{-1} = (S^{-1})^*$. It is obtained that

$$TS^{-1} = \overline{(S^*)^{-1}T^*} = \overline{(S^{-1})^*T^*} = \overline{TS^{-1}}^*.$$

Hence (3) follows. Similarly (4) is obtained from (3.8). □

Let $\mathcal{A} := \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma_2$. We set

- $K_1 := TS^{-1}$,
- $K_2 := 1 - \overline{S^{-1}}^*$,
- $K_3 := -S^{-1}\overline{T}$.

Since $K_1 \in H_2$, $\overline{K_1^*} = K_1$ and $\|K_1\| < 1$ by Lemma 3.3,

$$\mathcal{N}(\mathcal{A}) := \det(1 - K_1^*K_1)^{1/4}$$

and

$$U(\mathcal{A}) := \mathcal{N}(\mathcal{A})U_1(K_1)U_2(2K_2)U_3(K_3)$$

are well defined, moreover $U(\mathcal{A})$ maps $\mathcal{F}_0$ to $\mathcal{D}_\infty$. It may be formally written as

$$U(\mathcal{A}) = \det(1 - K_1^*K_1)^{1/4} :e^{-\frac{1}{2}(\Delta_{K_1} + 2N_{K_2} + \Delta_{K_3})}: .$$

Lemma 3.4 *Let $\mathcal{A} \in \Sigma_2$. Then $U(\mathcal{A})$ has the unique unitary operator extension.*

Proof: In Ruijsenaars [5] it has been established that

$$U(\mathcal{A})a^\sharp(f)U(\mathcal{A})^{-1}\Psi = b^\sharp_{\mathcal{A}}(f)\Psi$$

for $\Psi \in \mathcal{F}_0$ and

$$\|U_1(K_1)\Omega\|^2 = \det(1-K_1^*K_1)^{-1/2}.$$

From this it follows that

$$\begin{aligned}\|U(\mathcal{A})a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega\|^2 &= \|b^\dagger_{\mathcal{A}}(f_1)\cdots b^\dagger_{\mathcal{A}}(f_n)U(\mathcal{A})\Omega\|^2\\ &= \det(1-K_1^*K_1)^{1/2}\|b^\dagger_{\mathcal{A}}(f_1)\cdots b^\dagger_{\mathcal{A}}(f_n)U_1(K_1)\Omega\|^2\\ &= \|f_1\|^2\cdots\|f_n\|^2 = \|a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega\|^2.\end{aligned}$$

Then $U(\mathcal{A})$ maps $\mathcal{F}_0$ onto $\mathcal{E}$:= the linear hull of $\{b^\dagger_{\mathcal{A}}(f_1)\cdots b^\dagger_{\mathcal{A}}(f_n)U_1(\mathcal{A})\Omega\}$. From (3.2) it follows that

$$(a(f), a^\dagger(f)) = (b_{\mathcal{A}}(f), b^\dagger_{\mathcal{A}}(f))\begin{pmatrix} S^* & -T^* \\ -\overline{T}^* & \overline{S}^* \end{pmatrix}. \tag{3.10}$$

By this we see that $a^\sharp(f)\mathcal{E} \subset \mathcal{E}$. Thus $\mathcal{E}$ is dense in $\mathcal{F}$. Hence we conclude that $U(\mathcal{A})$ can be uniquely extended to a unitary operator on $\mathcal{F}$. The lemma follows. □

We denote its unitary extension by the same symbol $U(\mathcal{A})$.

3.4 Projective unitary representation

Lemma 3.5 *Let $\mathcal{A}_1, \mathcal{A}_2 \in \Sigma_2$. Then there exists a constant $\omega(\mathcal{A}_1, \mathcal{A}_2)$ such that*

$$U(\mathcal{A}_2)U(\mathcal{A}_1) = \omega(\mathcal{A}_2, \mathcal{A}_1)U(\mathcal{A}_2 \cdot \mathcal{A}_1).$$

Proof: A direct calculation shows that

$$a^\sharp(f)U(\mathcal{A}_2\cdot\mathcal{A}_1)^{-1}U(\mathcal{A}_2)U(\mathcal{A}_1) = U(\mathcal{A}_2\cdot\mathcal{A}_1)^{-1}U(\mathcal{A}_2)U(\mathcal{A}_1)a^\sharp(f).$$

Since $a^\sharp(f)$ is irreducible,

$$U(\mathcal{A}_2\mathcal{A}_1)^{-1}U(\mathcal{A}_2)U(\mathcal{A}_1) = \omega(\mathcal{A}_2, \mathcal{A}_1)1$$

with some constant $\omega(\mathcal{A}_1, \mathcal{A}_2)$. We conclude the lemma. □

Lemma 3.6 *$U(\mathcal{A})$ is strongly continuous in $\mathcal{A} \in \Sigma_2$.*

Proof: See [3]. □

The one-dimensional subspace defined by

$$\hat{\Psi} = \{\lambda\Psi | \lambda \in \mathbb{C}\}$$

is called the *ray*. We say that $\Psi \sim \Phi$, Set $\mathcal{F}/\sim := \hat{\mathcal{F}}$. For $\mathcal{A} \in \Sigma_2$ we define $\hat{U}(\mathcal{A})$ by

$$\hat{U}(\mathcal{A})\hat{\Psi} = (\widehat{U(\mathcal{A})}\Psi).$$

Corollary 3.7 *The map $\Sigma_2 \ni \mathcal{A} \mapsto \hat{U}(\mathcal{A})$ gives a continuous unitary representation of Σ_2 on $\hat{\mathcal{F}}$.*

Proof: It follows from Lemmas 3.5 and 3.6. □

4 One-parameter unitary group

In this section we construct a one-parameter unitary group on $\mathcal{F}$ derived from a homogeneous Bogoliubov transformation and see an explicit form of its infinitesimal generator.

4.1 Unitary representation of $\Sigma_2^{\mathrm{real,con}}$

In the previous section we show that by virtue of a Bogoliubov transformation a *projective* unitary representation of Σ_2 is given. In the present section we construct a unitary representation of a subgroup of Σ_2.

Definition 4.1 *(1) We say $\mathcal{A} = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma_2^{\mathrm{real}}$ if $\mathcal{A} \in \Sigma_2$ and $\overline{S} = S$ and $\overline{T} = T$. (2) $\Sigma_2^{\mathrm{real,con}}$ is defined by the connected component of Σ_2^{real}, which includes the identity* 1.

From the construction of $U(\mathcal{A})$ it follows that for $\mathcal{A} \in \Sigma_2^{\mathrm{real}}$

$$\overline{U(\mathcal{A})\Phi} = U(\mathcal{A})\overline{\Phi}. \tag{4.1}$$

Lemma 4.2 *The map $\Sigma_2^{\mathrm{real,con}} \ni \mathcal{A} \longmapsto U(\mathcal{A})$ defines a contiunous unitary representaion of $\Sigma_2^{\mathrm{real,con}}$.*

Proof: Since $U(\mathcal{A}_1)U(\mathcal{A}_2) = \omega(\mathcal{A}_1, \mathcal{A}_2)U(\mathcal{A}_1\mathcal{A}_2)$, we have

$$\omega(\mathcal{A}_1, \mathcal{A}_2) = (U(\mathcal{A}_1\mathcal{A}_2)\Omega, U(\mathcal{A}_1)U(\mathcal{A}_2)\Omega).$$

From (4.1), $\omega(\mathcal{A}_1, \mathcal{A}_2)$ is real. Then $\omega(\mathcal{A}_1, \mathcal{A}_2)$ is $+1$ or -1. Since $U(\mathcal{A})$ is strongly continuous in $\mathcal{A}$, $\omega(\mathcal{A}_1, \mathcal{A}_2)$ is continuous in both of $\mathcal{A}_1$ and $\mathcal{A}_2$. Moreover $\omega(1,1) = 1$. Hence $\omega(\mathcal{A}_1, \mathcal{A}_2) = 1$ for all $\mathcal{A}_1, \mathcal{A}_2 \in \Sigma_2^{\mathrm{real,con}}$. Thus the lemma follows. □

4.2 Examples

We suppose that $A \in H_2$, $A = A^*$, and $\overline{A} = A$. Let

$$\mathcal{A}_t := \exp\left(t\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}\right) = \begin{pmatrix} \cosh(tA) & \sinh(tA) \\ \sinh(tA) & \cosh(tA) \end{pmatrix}, \quad t \in \mathbb{R}.$$

Then $\{\mathcal{A}_t\}_{t\in\mathbf{R}}$ is a one-parameter group and

$$\{\mathcal{A}_t\}_{t\in\mathbf{R}} \subset \Sigma_2^{\mathrm{real,con}}.$$

Define the unitary operators $U(t)$ on $\mathcal{F}$ by

$$U(t) := U(\mathcal{A}_t), \quad t \in \mathbb{R}.$$

Lemma 4.3 *We have*

$$U(t)U(s) = U(t+s), \tag{4.2}$$

$$U(0) = 1, \tag{4.3}$$

$$s\text{-}\lim_{t\to 0} U(t) = 1. \tag{4.4}$$

Proof: (4.2) and (4.3) follow from Lemma 4.2. From Lemma 3.6, (4.4) follows. □

Hence by the Stone theorem there exists a self-adjoint operator Δ acting on $\mathcal{F}$ such that

$$U(t) = e^{it\Delta}, \quad t \in \mathbb{R}.$$

Theorem 4.4 *We have* $\Delta = -i/2(\Delta_A^* - \Delta_A)$.

Proof: See [3] for details. □

5 Concluding remarks

In the previous section it is shown that the generator of $U(t)$ is $1/2(\Delta_A^* - \Delta_A)$. Here we give a remark on $N = \int a^\dagger(k)a(k)dk$. Note

$$[N, a^\dagger(f)] = a^\dagger(f),$$

$$[N, a(f)] = -a(f).$$

Let $\phi(f) = 2^{-1/2}\left\{a^\dagger(f) + a(f)\right\}$ be a field opertor, and $\pi(f) = i2^{-1/2}\left\{a^\dagger(f) - a(f)\right\}$ its conjugate momentum. They satisfy

$$[\phi(f), \pi(g)] = i\int f(k)g(k)dk.$$

$$U(\pi/2) = e^{i(\pi/2)N}.$$

Then one can regard $U(\pi/2)$ as the Fourier transformation on $\mathcal{F}$. See Segal [6]. Actually since $U(\pi/2)a^\dagger(f)U^*(\pi/2) = ia^\dagger(f)$ and $U(\pi/2)a(f)U^*(\pi/2) = -ia(f)$, it is obtained that

$$U(\pi/2)\phi(f)U^*(\pi/2) = \pi(f).$$

References

[1] E. A. Berezin, *The method of second quantization*, Academic press, 1966.

[2] T. Hida, *Brownian Motion*, Springer-Verlag, 1980.

[3] K. R. Ito and F. Hiroshima, Infinitesimal generators of proper canonical transformations on a Boson Fock space, preprint, 2002.

[4] D. Shale, Linear symmetries of free boson fields, *Trans. Amer. Math. Soc.* **103** (1962), 149–167.

[5] S. N. M. Ruijsenaars, On Bogoliubov transforms. II. The general case. *Ann. Phys.* **116** (1978), 105–134.

[6] I. E. Segal, Tensor algebra over Hilbert spaces. I, *Trans. Amer. Math. Soc.* **81** (1956), 106–134.