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Classification of Universal Notions of Stochastic Independence

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0. This note is a survey of our recent preprints [Mu5,Mu6].

1. The Three Independences

One of the main features of quantum probability is the diversity of noncommutative notions of stochastic independence. Various notions of independence have been investigated by many authors from various points of views. The most famous one is the free independence of D. V. Voiculescu [Voi1, VDN, Spe1]. It is a very universal notion of noncommutative independence. For many notions from classical probability theory such as, central limit theorem, Lévy-Hinčin formula, Brownian motion, Lévy processes, stochastic calculus, entropy etc., their free analogues have been developed [VDN, HiP]. The free independence is deeply connected with the notion of free product of operator algebras. Another examples of notion of independence with a ceratin universal character are the tensor independence of R. L. Hudson [CuH, Hud, GvW, vWa2] and the boolean independence of W. von Waldenfels [vWa1, SpW, Boz]. In [Sch1], M. Schürmann initiated the study of universal notions of independence as "products" of algebraic probabilitry spaces. The three independences (tensor, free, boolean) corresponds to the three products (tensor product $\otimes$, free product $\star$, boolean product $\diamond$). He conjectured that these three universal independences are the only possible ones.

2. Universal Products (Speicher's Setting)

An answer to this conjecture was given by R. Speicher [Spe2]. He formulated the notion of *universal product* for algebraic probability spaces, through the requirements of "associativity" and of the existence of "universal calculation rule for mixed moments."

Let $\mathcal{K}$ be the class of all algebraic probability spaces $(\varphi, \mathcal{A})$. Here an *algebraic probability space* $(\varphi, \mathcal{A})$ means a pair of an associative $\mathbf{C}$-algebra $\mathcal{A}$ and a linear functional φ over $\mathcal{A}$. We do not assume the existence of units for algebras. For each pair of algebras $\mathcal{A}_1$, $\mathcal{A}_2$, we denote by $(\mathcal{A}_1 \sqcup \mathcal{A}_2, i_1, i_2)$ its coproduct in the category of algebras and algebra homomorphisms, where $i_1 : \mathcal{A}_1 \to \mathcal{A}_1 \sqcup \mathcal{A}_2$, $i_2 : \mathcal{A}_2 \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ are the injections of the coproduct. Indeed it is nothing but the free product of (non-unital) algebras $\mathcal{A}_1 \sqcup \mathcal{A}_2 := \mathcal{A}_1 \star \mathcal{A}_2$. It is defined by

$$\mathcal{A}_1 \star \mathcal{A}_2 = \bigoplus_{\varepsilon \in \mathbf{A}} \mathcal{A}_{\varepsilon_1} \otimes \mathcal{A}_{\varepsilon_2} \otimes \cdots \otimes \mathcal{A}_{\varepsilon_n},$$

where $\mathbf{A}$ is the set of all finite sequences $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ of length $n \geq 1$ from $\{1,2\}$ satisfying $\varepsilon_i \neq \varepsilon_{i+1}$ $(i = 1, 2, \cdots, n-1)$.

For the definition of universal product, let us prepare some notions and notations. A partition π of the linearly ordered set $\{1, 2, \cdots, n\}$ is a collection $\pi = \{V_1, V_2, \cdots, V_p\}$ of subsets (called blocks) of $\{1, 2, \cdots, n\}$ such that $\cup_{l=1}^{p} V_i = \{1, 2, \cdots, n\}$ and $V_i \cap V_j = \emptyset$ $(i \neq j)$. We denote by $\mathcal{P}(n)$ the set of all partitions π of $\{1, 2, \cdots, n\}$. For partitions $\pi, \sigma \in \mathcal{P}(n)$, we write $\sigma \leq \pi$ if for each block W in σ, there exists some block V in π such that $W \subset V$. It defines the partial order on $\mathcal{P}(n)$. We denote by $O_n := \{\{1\}, \{2\}, \cdots, \{n\}\}$ the minimal element of $\mathcal{P}(n)$ in this partial order. Let a partition $\pi = \{V_1, V_2, \cdots, V_p\} \in \mathcal{P}(n)$ and a p-tuple of algebraic probability spaces $(\varphi_l, \mathcal{A}_l)$, $l = 1, 2, \cdots, p$, be given. Then we write

$$a_1 a_2 \cdots a_n \in \mathcal{A}_\pi$$

if $a_i \in \mathcal{A}_l$ whenever $i \in V_l$. When a partition π is given by some $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \mathbf{A}$ as $\pi = \{V_1, V_2\}$ with $V_1 = \{i | \varepsilon_i = 1\}$ and $V_2 = \{i | \varepsilon_i = 2\}$, we also use the notation $a_1 a_2 \cdots a_n \in \mathcal{A}_\varepsilon$ to mean $a_1 a_2 \cdots a_n \in \mathcal{A}_\pi$. Given a situation $a_1 a_2 \cdots a_n \in \mathcal{A}_\pi$, then for each $\sigma \leq \pi$ we put

$$\varphi_\sigma(a_1 a_2 \cdots a_n) := \prod_{W \in \sigma} \varphi_l \Big(\overrightarrow{\prod_{i \in W}} a_i \Big)$$

where $\overrightarrow{\prod_{i \in W}} a_i$ denotes the product of a_i in the same order as they appear in $a_1 a_2 \cdots a_n$. The index l in the RHS of the above expression denotes a unique l satisfying $W \subset V_l$. From now on, we identify any element $a \in \mathcal{A}_l$ with its natural image $i_l(a) \in \sqcup_{l=1}^{p} \mathcal{A}_l$ where i_l is the lth injection of the coproduct. So we write a for short insted of $i_l(a)$. Under this prepartion, the notion of universal product is defined as follows.

Definition 2.1 ([Spe2]). *A* **universal product** *over $\mathcal{K}$ is a map $((\varphi_1, \mathcal{A}_1), (\varphi_2, \mathcal{A}_2)) \mapsto (\varphi_1 \varphi_2, \mathcal{A}_1 \sqcup \mathcal{A}_2)$ from $\mathcal{K} \times \mathcal{K}$ to $\mathcal{K}$ satisfying the following three conditions.*

(UP1) associativity: Under the natural identification $(\mathcal{A}_1 \sqcup \mathcal{A}_2) \sqcup \mathcal{A}_3 \cong \mathcal{A}_1 \sqcup (\mathcal{A}_2 \sqcup \mathcal{A}_3)$,

$$(\varphi_1 \varphi_2) \varphi_3 = \varphi_1 (\varphi_2 \varphi_3).$$

(UP2) universal calculation rule for mixed moments: For each $n = 1, 2, 3, \cdots$, each $\pi \in \mathcal{P}(n)$, and each $\sigma \leq \pi$, there exist constants $t(\pi; \sigma)$ such that, for any p-tuple $(\varphi_l, \mathcal{A}_l)$, $l = 1, 2, \cdots, p$, of algebraic probability spaces, and $\varphi := \varphi_1 \varphi_2 \cdots \varphi_p$, we have

$$\varphi[a_1 a_2 \cdots a_n] = \sum_{\sigma \leq \pi} t(\pi; \sigma)\, \varphi_\sigma(a_1 a_2 \cdots a_n)$$

whenever $a_1 a_2 \cdots a_n \in \mathcal{A}_\pi$ with $\# \pi = p$.

(UP3) normalization:

$$t(O_1; O_1) = t(O_2; O_2) = 1.$$

Speicher proved the following.

Theorem 2.2 ([Spe2]). *There exist only three universal products, namely tensor product $\otimes$, free product $\star$ and boolean product $\diamond$.*

Here the definitions of these three products are given as follows.

Definition 2.3. *The* **tensor product** $\otimes$, *the* **boolean product** $\diamond$ *and the* **free product** $\star$ *over $\mathcal{K}$ are associative products given by the following calculation rules for $a_1a_2\cdots a_n \in \mathcal{A}_\varepsilon$, respectively.*

$$
\begin{aligned}
(\varphi_1\otimes\varphi_2)[a_1a_2\cdots a_n] &= \varphi_1\Big[\overrightarrow{\prod_{k\in V_1}} a_k\Big]\,\varphi_2\Big[\overrightarrow{\prod_{l\in V_2}} a_l\Big],\\
(\varphi_1\diamond\varphi_2)[a_1a_2\cdots a_n] &= \Big(\prod_{k\in V_1}\varphi_1[a_k]\Big)\Big(\prod_{l\in V_2}\varphi_2[a_l]\Big),\\
(\varphi_1\star\varphi_2)[a_1a_2\cdots a_n] &= \sum_{\substack{I\subset\{1,2,\cdots,n\}\\ I\neq\{1,2,\cdots,n\}}} (-1)^{n-\#I+1}\Big((\varphi_1\star\varphi_2)\Big[\overrightarrow{\prod_{k\in I}} a_k\Big]\Big)\Big(\prod_{l\notin I}\varphi_{\varepsilon_l}[a_l]\Big).
\end{aligned}
$$

Here the calculation rule for free product $\star$ should be understood as a recurrence formula with the convention $(\varphi_1\star\varphi_2)\Big[\overrightarrow{\prod_{k\in\emptyset}} a_k\Big] := 1$.

3. Quasi-Universal Products

By the way, based on a previous work [Mu1,Mu2,Lu], we recently found in the setting of C^*-probability spaces an another example of independence and product with a certain universal character: the *monotonic independence* [Mu3] and the *monotone product* of C^*-probability spaces [Mu4]. So it is natural to revisit the classification problem of universal notions of independence. For the setting of algebraic probability spaces, the monotone product is defined as follows.

Definition 3.1. *The* **monotone product** $\rhd$ *and the* **anti-monotone product** $\lhd$ *over $\mathcal{K}$ are associative products given by the following calculation rules for $a_1a_2\cdots a_n \in \mathcal{A}_\varepsilon$, respectively.*

$$
\begin{aligned}
(\varphi_1\rhd\varphi_2)[a_1a_2\cdots a_n] &= \varphi_1\Big[\overrightarrow{\prod_{k\in V_1}} a_k\Big]\Big(\prod_{l\in V_2}\varphi_2[a_l]\Big),\\
(\varphi_1\lhd\varphi_2)[a_1a_2\cdots a_n] &= \Big(\prod_{k\in V_1}\varphi_1[a_k]\Big)\,\varphi_2\Big[\overrightarrow{\prod_{l\in V_2}} a_l\Big].
\end{aligned}
$$

Let us weaken the notion of universal product of Speicher. A linearly ordered partition (π,λ) of $\{1,2,\cdots,n\}$ is a pair of a (usual) partition π and a linear ordering λ among the blocks in π. We express such a linearly ordered partition as $(\pi,\lambda) = \{V_1 \prec V_2 \prec \cdots \prec V_p\}$. For example, $(\pi_1,\lambda_1) = \{\{1,3\}\prec\{2\}\}$ and $(\pi_2,\lambda_2) = \{\{2\}\prec\{1,3\}\}$ are different linearly ordered partitions of $\{1,2,3\}$. We denote by $\mathcal{LP}(n)$ the set of all linearly ordered partitions (π,λ) over $\{1,2,\cdots,n\}$. Let a linearly ordered partition $(\pi,\lambda) = \{V_1\prec V_2\prec\cdots\prec V_p\}\in\mathcal{LP}(n)$ and a p-tuple of algebraic probability spaces $(\varphi_l,\mathcal{A}_l)$, $l = 1,2,\cdots,p$, be given. We write

$$a_1a_2\cdots a_n \in \mathcal{A}_{(\pi,\lambda)}$$

if $a_i\in\mathcal{A}_l$ whenever $i\in V_l$. When a linearly ordered partition (π,λ) is given by some $\varepsilon = (\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\in\mathbf{A}$ as $\pi = \{V_1\prec V_2\}$ with $V_1 = \{i|\varepsilon_i = 1\}$ and $V_2 = \{i|\varepsilon_i = 2\}$, we identify (π, λ) with ε. For example, $(\pi, \lambda) = \{\{2,4\} \prec \{1,3\}\}$ is identified with $\varepsilon = (2121)$. Then we define a notion of quasi-universal product as follows.

Definition 3.2 ([Mu5]). *A* **quasi-universal product** *over $\mathcal{K}$ is a map $((\varphi_1, \mathcal{A}_1), (\varphi_2, \mathcal{A}_2)) \mapsto (\varphi_1\varphi_2, \mathcal{A}_1 \sqcup \mathcal{A}_2)$ from $\mathcal{K} \times \mathcal{K}$ to $\mathcal{K}$ satisfying the three conditions (UP1), (QUP2), (QUP3).*

(QUP2) quasi-universal calculation rule for mixed moments: For each $n = 1, 2, 3, \cdots$, each $(\pi, \lambda) \in \mathcal{LP}(n)$, and each $\sigma \leq \pi$, there exist constants $t(\pi, \lambda; \sigma)$ such that, for any p-tuple $(\varphi_l, \mathcal{A}_l)$, $l = 1, 2, \cdots, p$, of algebraic probability spaces, and $\varphi := \varphi_1\varphi_2 \cdots \varphi_p$, we have

$$\varphi[a_1 a_2 \cdots a_n] \;=\; \sum_{\sigma \leq \pi} t(\pi, \lambda; \sigma)\, \varphi_\sigma(a_1 a_2 \cdots a_n)$$

whenever $a_1 a_2 \cdots a_n \in \mathcal{A}_{(\pi,\lambda)}$ with $(\pi, \lambda) = \{V_1 \prec V_2 \prec \cdots \prec V_p\}$.

(QUP3) normalization:

$$t(1; O_1) \;=\; t(12; O_2) \;=\; t(21; O_2) \;=\; 1.$$

The monotone and anti-monotone products are examples of quasi-universal products over $\mathcal{K}$. The classification of quasi-universal products is given as follows.

Theorem 3.3 ([Mu5]). *There exist only five quasi-universal products over $\mathcal{K}$, namely tensor product $\otimes$, free product $\star$, boolean product $\diamond$, monotone product $\triangleright$ and anti-monotone product $\triangleleft$.*

4. Universal Products (Ben Ghorbal-Schürmann's Setting)

The notion of (quasi-)universal product is defined through the existence of "(quasi-)universal calculation rule for mixed moments." But there is an another formulation of universal product which was given by A. Ben Ghorbal and M. Schürmann [BGS]. They formulated the notion of universal product based on some commutative diagrams for arrows (= algebra homomorphisms). Their axioms are natural and sufficiently nice so that, for each universal product, the theory of Lévy processes can be developed on any dual group of Voiculescu [BGS, Sch2, Voi2].

Let us recall that, for each pair of algebra homomorphisms $j_1 : \mathcal{B}_1 \to \mathcal{A}_1$, $j_2 : \mathcal{B}_2 \to \mathcal{A}_2$, there exists a unique morphism $j_1 \amalg j_2 : \mathcal{B}_1 \sqcup \mathcal{B}_2 \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ such that the diagram

$$\begin{tikzcd}
& \mathcal{B}_1 \arrow[r, "j_1"] \arrow[dl, "\iota_1"'] & \mathcal{A}_1 \arrow[dr, "i_1"] & \\
\mathcal{B}_1 \sqcup \mathcal{B}_2 \arrow[rrr, "j_1 \amalg j_2"] & & & \mathcal{A}_1 \sqcup \mathcal{A}_2 \\
& \mathcal{B}_2 \arrow[ul, "\iota_2"] \arrow[r, "j_2"] & \mathcal{A}_2 \arrow[ur, "i_2"'] &
\end{tikzcd}$$

is commutative, where $(\mathcal{B}_1 \sqcup \mathcal{B}_2, \iota_1, \iota_2)$ is the coproduct of $\mathcal{B}_1$ and $\mathcal{B}_2$.

Definition 4.1 ([BGS]). *A* **universal product** *over $\mathcal{K}$ is a map $((\varphi_1, \mathcal{A}_1), (\varphi_2, \mathcal{A}_2)) \mapsto (\varphi_1\varphi_2, \mathcal{A}_1 \sqcup \mathcal{A}_2)$ from $\mathcal{K} \times \mathcal{K}$ to $\mathcal{K}$ satisfying the following four conditions.*

(U1) commutativity: Under the natural identification $\mathcal{A}_1 \sqcup \mathcal{A}_2 \cong \mathcal{A}_2 \sqcup \mathcal{A}_1$,

$$\varphi_1\varphi_2 = \varphi_2\varphi_1.$$

(U2) associativity: Under the natural identification $(\mathcal{A}_1 \sqcup \mathcal{A}_2) \sqcup \mathcal{A}_3 \cong \mathcal{A}_1 \sqcup (\mathcal{A}_2 \sqcup \mathcal{A}_3)$,

$$(\varphi_1\varphi_2)\varphi_3 = \varphi_1(\varphi_2\varphi_3).$$

(U3) universality: For any pair of algebra homomorphisms $j_1 : \mathcal{B}_1 \to \mathcal{A}_1$, $j_2 : \mathcal{B}_2 \to \mathcal{A}_2$,

$$(\varphi_1 \circ j_1)(\varphi_2 \circ j_2) = (\varphi_1\varphi_2) \circ (j_1 \amalg j_2).$$

(U4) normalization:

$$(\varphi_1\varphi_2) \circ i_1 = \varphi_1, \quad (\varphi_1\varphi_2) \circ i_2 = \varphi_2, \quad \textit{(extension)}$$
$$(\varphi_1\varphi_2)[i_1(a)i_2(b)] = (\varphi_1\varphi_2)[i_2(b)i_1(a)] = \varphi_1[a]\,\varphi_2[b] \quad (a \in \mathcal{A}_1,\ b \in \mathcal{A}_2). \quad \textit{(factorization)}$$

We remark here that the commutativity axiom (U1) is an algebraic interpretaion of the property of classical notion of independence that if two random variables X, Y are independent then Y, X are also independent ("independence" is not dependent on the order). Also the universality axiom (U3) is an algebraic interpretaion of the property of classical notion of independence that if X, Y are independent then $f(X)$, $g(Y)$ are also independent for any functions f and g.

It was proved by Ben Ghorbal and Schürmann that the two formulations of universal product (Definition 2.1 and Definition 4.1) are equivalent. So they obtained the following.

Theorem 4.2 ([BGS]). *There exist only three universal products satisfying the four conditions (U1), (U2), (U3) and (U4). Namely tensor product $\otimes$, free product $\star$ and boolean product $\diamond$.*

5. Natural Products

Since we already know another examples of product with a certain universal character (= monotone and anti-monotone products), it is natural to consider the classification problem for certain "weakened" universal products in the setting of Ben Ghorbal and Schürmann.

Definition 5.1. *A* **natural product** *over $\mathcal{K}$ is a map $((\varphi_1, \mathcal{A}_1), (\varphi_2, \mathcal{A}_2)) \mapsto (\varphi_1\varphi_2, \mathcal{A}_1 \sqcup \mathcal{A}_2)$ from $\mathcal{K} \times \mathcal{K}$ to $\mathcal{K}$ satisfying the three conditions (U2), (U3) and (U4).*

The classification theorem for natural products is as follows.

Theorem 5.2 ([Mu5]). *There exist only five natural products over $\mathcal{K}$, namely tensor product $\otimes$, free product $\star$, boolean product $\diamond$, monotone product $\triangleright$ and anti-monotone product $\triangleleft$.*

The proof of Theorem 5.2 is based on the reduction of natural products to quasi-universal products. For that purpose we used the theory of universal families developed in [BGS]. Also in the proof, we met with a rather complicated situation that we must determine all the possible values for the almost one handred unkowns in a certain sysytem of equations. But it could be solved without any problem [Mu6].

Finally we remark that also in the case of monotonic independence (resp. anti-monotonic independence), the theory of Lévy processes is possible on any dual groups of Voiculescu [Fra1,Fra2].

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