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Logarithmic Correction of Kashaev's Invariant

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1 Introduction

In Refs. 1, 2 Kashaev defined an invariant $\langle\mathcal{K}\rangle_N$ for knot $\mathcal{K}$ using a quantum dilogarithm function at the N-th root of unity ($\omega \equiv e^{2\pi i/N}$), and proposed a stimulating conjecture that for a hyperbolic knot $\mathcal{K}$ the invariant $\langle\mathcal{K}\rangle_N$ gives a hyperbolic volume of a knot complement in a limit $N \to \infty$;

$$\lim_{N\to\infty} \frac{2\pi}{N} \log\left|\langle\mathcal{K}\rangle_N\right| = \mathrm{Vol}(S^3 \setminus \mathcal{K}), \tag{1.1}$$

where Vol denotes a hyperbolic volume (see, e.g., Ref. 3). Here the invariant $\langle\mathcal{K}\rangle_N$ is defined for a $(1,1)$-tangle of knot $\mathcal{K}$ with the enhanced Yang–Baxter operator defined by

$$R^{ij}_{k\ell} = \frac{N\,\omega^{1-(k-j+1)(\ell-i)}}{(\omega)_{[\ell-k-1]}\,(\omega)^*_{[j-\ell]}\,(\omega)_{[i-j]}\,(\omega)^*_{[k-i]}} \cdot \theta\begin{bmatrix} i & j \\ k & \ell \end{bmatrix}, \tag{1.2a}$$

$$(R^{-1})^{ij}_{k\ell} = \frac{N\,\omega^{-1+(\ell-i-1)(k-j)}}{(\omega)^*_{[\ell-k-1]}\,(\omega)_{[j-\ell]}\,(\omega)^*_{[i-j]}\,(\omega)_{[k-i]}} \cdot \theta\begin{bmatrix} i & j \\ k & \ell \end{bmatrix}, \tag{1.2b}$$

$$\mu^k_\ell = -\delta_{k,\ell+1}\,\omega^{\frac{1}{2}}, \tag{1.2c}$$

where $[x] \in \{0, 1, \dots, N-1\}$ modulo N, and

$$\theta\begin{bmatrix} i & j \\ k & \ell \end{bmatrix} = 1, \quad \text{if and only if} \begin{cases} i \leq k < \ell \leq j, \\ j \leq i \leq k < \ell, \\ \ell \leq j \leq i \leq k \ (\text{with } \ell < k), \\ k < \ell \leq j \leq i. \end{cases}$$

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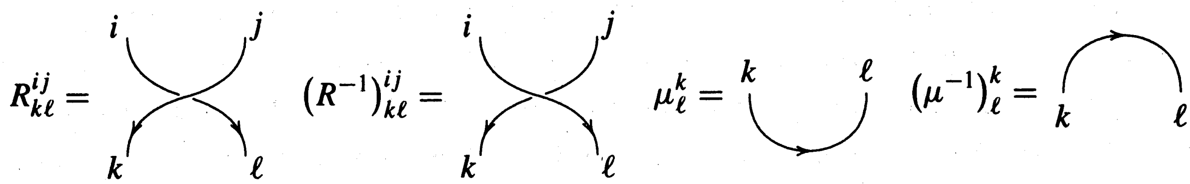
It was proved later that Kashaev's invariant $\langle\mathcal{K}\rangle_N$ is identified with the N-colored Jones polynomial at the N-th primitive root of unity [4], whose R-matrix is given by

$$R^{ij}_{k\ell} = \sum_{n=0}^{\min(N-1-i,j)} \delta_{\ell,i+n}\,\delta_{k,j-n}\,(-1)^{i+j+n}\,\frac{(\omega)^*_{i+n}\,(\omega)_j}{(\omega)_i^*\,(\omega)_{j-n}\,(\omega)_n^*}\,\omega^{ij+\frac{1}{2}(i+j-n)}, \tag{1.3a}$$

$$\left(R^{-1}\right)^{ij}_{k\ell} = \sum_{n=0}^{\min(N-1-j,i)} \delta_{\ell,i-n}\,\delta_{k,j+n}\,(-1)^{i+j+n}\,\frac{(\omega)_i^*\,(\omega)_{j+n}}{(\omega)^*_{i-n}\,(\omega)_j\,(\omega)_n}\,\omega^{-ij-\frac{1}{2}(i+j-n)}, \tag{1.3b}$$

$$\mu^k_\ell = -\delta_{k,\ell}\,\omega^{k+\frac{1}{2}}, \tag{1.3c}$$



and the conjecture (1.1) is rephrased as for arbitrary knot $\mathcal{K}$,

$$\lim_{N\to\infty}\frac{2\pi}{N}\log\left|\langle\mathcal{K}\rangle_N\right| = v_3\cdot\left\|S^3\setminus\mathcal{K}\right\|, \tag{1.1'}$$

where v_3 and $\|\cdot\|$ are a hyperbolic volume of regular ideal tetrahedron and the Gromov norm respectively. Further recalling a close relationship between the hyperbolic volume and the Chern–Simons invariant [5, 6], discussed is a complexification of above conjecture [7, 8],

$$\lim_{N\to\infty}\frac{2\pi}{N}\log\left(\langle\mathcal{K}\rangle_N\right) = v_3\cdot\left\|S^3\setminus\mathcal{K}\right\| + \mathrm{i}\ \mathrm{CS}(\mathcal{K}), \tag{1.4}$$

where CS denotes the Chern–Simons invariant defined by

$$\mathrm{CS}(\mathcal{M}) = 2\pi^2\ \mathrm{cs}(\mathcal{M})$$

$$\mathrm{cs}_{\mathcal{M}}(A) = \frac{1}{8\pi^2}\int_{\mathcal{M}}\mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

It is expected that this *volume conjecture* will be a key to solve a geometrical meaning of the quantum invariants which have been introduced since a discovery of the Jones polynomial.

Since the conjecture (1.1) was proposed, some attempts have been made, and especially the relationship between the R-matrix (1.2) and the hyperbolic ideal octahedron has been clarified [9–11]. Though, it is a hard analytic problem to prove a conjecture (1.1) rigorously from an explicit form of invariants (a limit to replace the q-series $(q)_n$ with the dilogarithm function works well generally [2]). Purpose of this article is to study analytically eq. (1.1) for the torus knot and numerically for several hyperbolic knots and links. These results support the volume conjecture (1.1′), and further indicate that there is a logarithmic correction to Kashaev's invariant,

$$\log\left|\langle\mathcal{K}\rangle_N\right| \xrightarrow[N\to\infty]{} v_3 \cdot \left\|S^3 \setminus \mathcal{K}\right\| \cdot \frac{N}{2\pi} + \frac{3}{2}\#(\mathcal{K}) \cdot \log N + O(N^0), \tag{1.5}$$

where $\#(\mathcal{K})$ is the number of prime factors (as connected-sum of prime knots) of a knot $\mathcal{K}$.

This paper is organized as follows. In Section 2 we study the invariants of the torus knot $\mathrm{Trs}(m, p)$ with m and p being coprime integers following Ref. 12. We first consider an asymptotic expansion, and see that the left hand side of eq. (1.1) gives zero, which agrees with a fact that the torus knot is not hyperbolic. Furthermore there is a logarithmic correction,

$$\log\left|\langle\mathrm{Trs}(m, p)\rangle_N\right| \simeq \frac{3}{2}\log N,$$

which supports eq. (1.5). On the other hand, explicit form of the invariants itself gives an interesting fact. From a view point of the q-series, such invariants can be regarded as a reduction of the q-series in a case of q being the N-th root of unity ω, and in a case of the trefoil knot the q-series defined from an invariant $\langle\text{trefoil}\rangle_N$ generates the number of "regularized linearized chord diagrams", i.e., an upper bound of the number of linearly independent Vassiliev invariants [13, 14]. Based on this observation, we propose asymptotic expansion of certain sets of q-series which arises from Kashaev's invariant of the $(2m + 1, 2)$-torus knot.

In Section 3 we consider numerically the volume conjecture (1.1) for hyperbolic knots up to 6-crossing (figure-eight knot 4_1, 5_2, 6_1, 6_2, 6_3), Whitehead link, and Borromean rings. We find that numerical calculation supports eq. (1.1), and that there is also a logarithmic correction to the hyperbolic volume,

$$\frac{2\pi}{N}\log\left|\langle\mathcal{K}\rangle_N\right| \sim \mathrm{Vol}(S^3 \setminus \mathcal{K}) + \frac{3}{2}\cdot\frac{2\pi}{N}\log N + O(N^{-1}),$$

for $\mathcal{K} \in$ {up to 6-crossing knots, Whitehead link, Borromean rings}. Based on these results, we may propose a conjecture (1.5), as the invariant is defined for a $(1, 1)$-tangle of an arbitrary knot $\mathcal{K}$. The last section is devoted to concluding remarks.

Throughout this paper we use a standard notation for a q-product,

$$(x)_n \equiv (x;q)_n = \prod_{i=0}^{n-1} \left(1 - x\,q^i\right), \qquad (x_1,\dots,x_k;q)_n = \prod_{i=1}^{k} (x_i;q)_n .$$

In a case of q being the N-th primitive root of unity, we set

$$\omega = \exp\left(\frac{2\pi\,\mathrm{i}}{N}\right), \tag{1.6}$$

and use a notation

$$(\omega)_n = \prod_{i=1}^{n} \left(1-\omega^i\right), \qquad (\omega)_n^* = \prod_{i=1}^{n} \left(1-\omega^{-i}\right).$$

2 Torus Knot

We consider the (m,p)-torus knot $\mathrm{Trs}(m,p)$, where m and p are coprime integers (Fig. 1). The torus knot is not hyperbolic, and an explicit form of the colored Jones polynomial is known. We review the asymptotic expansion of their knot invariants following Kashaev & Tirkkonen [12]. We compute explicitly Kashaev's invariant for the $(2m+1,2)$-torus knot by use of the R-matrix (1.2), and obtain a formula for an asymptotic expansion for certain sets of ω-series. In the case of the trefoil knot, this formula is nothing but one studied in Ref. 14. We further define the q-series which reduces to the knot invariant in a case of q being ω, and we study an asymptotic expansion of these q-series.

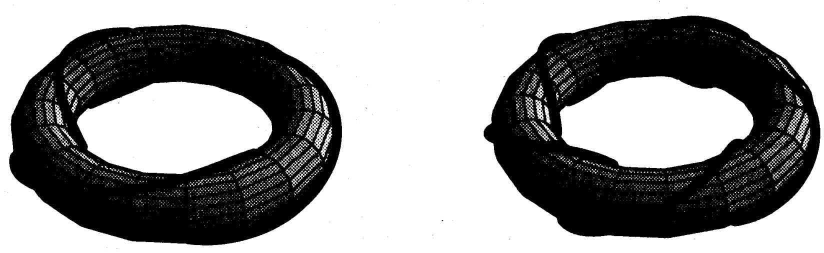


Figure 1: Trefoil ($\mathrm{Trs}(3,2)$) and Solomon's Seal knot ($\mathrm{Trs}(5,2)$).

2.1 Integral Formula

We start from the colored Jones polynomial for the (m, p)-torus knot $\mathcal{K} = \mathrm{Trs}(m, p)$. We set N as a "color". We know that the colored Jones invariant for the (m, p)-torus knot in a case that m and p are coprime integers is explicitly written by [15, 16]

$$2\,\mathrm{sh}\left(\frac{N\,h}{2}\right)\frac{J_{\mathcal{K}}(h;N)}{J_{\mathcal{O}}(h;N)} = \sum_{\epsilon=\pm 1}\sum_{r=-(N-1)/2}^{(N-1)/2} \epsilon\,\exp\left(h\,m\,p\,r^2 + h\,r(m+\epsilon\,p) + \frac{1}{2}\epsilon\,h\right), \tag{2.1}$$

where $\mathcal{O}$ denotes unknot, and we have

$$J_{\mathcal{O}}(h;N) = \frac{\mathrm{sh}(N\,h/2)}{\mathrm{sh}(h/2)}.$$

As was shown in Ref. 4 Kashaev's invariant is related with a specific value of the colored Jones polynomial, and we see that‡

$$\langle \mathrm{Trs}(m,p)\rangle_N = \mathrm{e}^{\pi \mathrm{i}(N+\frac{1}{N})} \lim_{h\to 2\pi\mathrm{i}/N} \frac{J_{\mathcal{K}}(h;N)}{J_{\mathcal{O}}(h;N)}. \tag{2.2}$$

To rewrite the invariant in the integral form, we follow Ref. 12. We use the Gauss integral formula;

$$\sqrt{\pi\,h}\,\mathrm{e}^{h w^2} = \int_{\mathcal{C}} \mathrm{d}z\,\exp\left(-\frac{z^2}{h} + 2\,w\,z\right),$$

where a path $\mathcal{C}$ is to be chosen by the convergence condition, $\mathbb{R} \ni x \mapsto x\,\mathrm{e}^{\mathrm{i}\phi} \in \mathcal{C}$ satisfying $\Re(h\,\mathrm{e}^{-2\mathrm{i}\phi}) > 0$ [12]. We apply the Gaussian integral formula to eq. (2.1), and we get

$$\begin{aligned}
2\,\mathrm{e}^{\frac{h}{4}(\frac{m}{p}+\frac{p}{m})}\,\mathrm{sh}\left(\frac{N\,h}{2}\right)\frac{J_{\mathcal{K}}(h;N)}{J_{\mathcal{O}}(h;N)} &= \sum_{\epsilon=\pm 1}\sum_{r=-(N-1)/2}^{(N-1)/2} \epsilon\,\mathrm{e}^{h\,m\,p\left(r+\frac{m+\epsilon p}{2mp}\right)^2} \\
&= \sum_{\epsilon=\pm 1}\epsilon \sum_{r=-(N-1)/2}^{(N-1)/2} \frac{1}{\sqrt{\pi\,h\,m\,p}}\int_{\mathcal{C}} \mathrm{d}z\,\mathrm{e}^{-\frac{z^2}{hmp}+z(2r+\frac{1}{p}+\frac{\epsilon}{m})} \\
&= \frac{2}{\sqrt{\pi\,h\,m\,p}}\int_{\mathcal{C}} \mathrm{d}z\,\mathrm{e}^{-\frac{z^2}{hmp}+\frac{z}{p}}\,\frac{\mathrm{sh}(N\,z)\,\mathrm{sh}(\frac{z}{m})}{\mathrm{sh}(z)}.
\end{aligned}$$

Summing integrand with one replacing $z \to -z$, we have

$$= \frac{2}{\sqrt{\pi\,h\,m\,p}}\int_{\mathcal{C}} \mathrm{d}z\,\mathrm{e}^{-\frac{z^2}{hmp}}\,\frac{\mathrm{sh}(N\,z)\,\mathrm{sh}(\frac{z}{m})\,\mathrm{sh}(\frac{z}{p})}{\mathrm{sh}\,z}.$$

‡A prefactor in the right hand is dropped in Ref. 12.

Decomposing $\mathrm{sh}(N\,z)$ and using an invariance under $z \to -z$, we see that

$$= \frac{2}{\sqrt{\pi\, h\, m\, p}} \int_{\mathscr{C}} \mathrm{d}z\, \mathrm{e}^{-\frac{z^2}{h m p} + N z}\, \frac{\mathrm{sh}(\frac{z}{m})\,\mathrm{sh}(\frac{z}{p})}{\mathrm{sh}\, z}$$

$$= \sqrt{\frac{m\, p}{\pi\, h}} \int_{\mathscr{C}} \mathrm{d}z\, \mathrm{e}^{m p (N z - \frac{z^2}{h})}\, \frac{2\,\mathrm{sh}(m\, z)\,\mathrm{sh}(p\, z)}{\mathrm{sh}(m\, p\, z)}.$$

To obtain Kashaev's invariant $\langle \mathrm{Trs}(m, p)\rangle_N$ defined in eq. (2.2), we differentiate above integral with respect to h, and we obtain

$$\langle \mathrm{Trs}(m, p)\rangle_N = \left(\frac{m\, p\, N}{2}\right)^{3/2} \mathrm{e}^{\pi \mathrm{i} (N + \frac{1}{N}) - \frac{\pi \mathrm{i}}{2N}(\frac{m}{p} + \frac{p}{m}) - \frac{\pi \mathrm{i}}{4}} \int_{\mathscr{C}} \mathrm{d}z\, \mathrm{e}^{m p N \pi (z + \frac{\mathrm{i}}{2} z^2)}\, z^2\, \frac{\mathrm{sh}(m\, \pi\, z)\,\mathrm{sh}(p\, \pi\, z)}{\mathrm{sh}(m\, p\, \pi\, z)}. \tag{2.3}$$

When we shift the path $\mathscr{C}$ to $\mathscr{C} + \mathrm{i}$, we get

$$\langle \mathrm{Trs}(m, p)\rangle_N = \left(\frac{m\, p\, N}{2}\right)^{3/2} \mathrm{e}^{\pi \mathrm{i} (N + \frac{1}{N}) - \frac{\pi \mathrm{i}}{2N}(\frac{m}{p} + \frac{p}{m}) - \frac{\pi \mathrm{i}}{4}} \times \left(\mathrm{Res}(m, p) + \int_{\mathscr{C} + \mathrm{i}} \mathrm{d}z\, \mathrm{e}^{m p N \pi (z + \frac{\mathrm{i}}{2} z^2)}\, z^2\, \frac{\mathrm{sh}(m\, \pi\, z)\,\mathrm{sh}(p\, \pi\, z)}{\mathrm{sh}(m\, p\, \pi\, z)} \right).$$

Here the first term, $\mathrm{Res}(m, p)$, comes from residues of the integral at $z = \frac{n}{m\, p}\, \pi\, \mathrm{i}$ for $n = 1, 2, \dots, mp - 1$, and it is written as

$$\mathrm{Res}(m, p) = \frac{2\, \mathrm{i}}{(m\, p)^3} \sum_{n=1}^{mp-1} (-1)^{n+1}\, n^2\, \mathrm{sh}\left(\frac{n\, \pi}{p}\, \mathrm{i}\right) \mathrm{sh}\left(\frac{n\, \pi}{m}\, \mathrm{i}\right) \mathrm{e}^{N \pi \mathrm{i} (n - \frac{n^2}{2 m p})}. \tag{2.4}$$

In the second term, we introduce $z = w + \mathrm{i}$, and using a fact that even function only survive in the integrand, we get

$$\langle \mathrm{Trs}(m, p)\rangle_N = \left(\frac{m\, p\, N}{2}\right)^{3/2} \mathrm{e}^{\pi \mathrm{i} (N + \frac{1}{N}) - \frac{\pi \mathrm{i}}{2N}(\frac{m}{p} + \frac{p}{m}) - \frac{\pi \mathrm{i}}{4}} \times \left(\mathrm{Res}(m, p) + 2\, \mathrm{i}\, (-1)^{mp + m + p}\, \mathrm{e}^{\frac{1}{2} m p N \pi \mathrm{i}} \int_{\mathscr{C}} \mathrm{d}w\, \mathrm{e}^{\frac{1}{2} \mathrm{i} m p N \pi w^2}\, w\, \frac{\mathrm{sh}(m\, \pi\, w)\,\mathrm{sh}(p\, \pi\, w)}{\mathrm{sh}(m\, p\, \pi\, w)} \right). \tag{2.5}$$

We intend to replace the integral with an infinite series. We define T-numbers $T_n^{(m,p)}$ by a series expansion of a generating function as

$$\frac{\mathrm{sh}(m\,w)\,\mathrm{sh}(p\,w)}{\mathrm{sh}(m\,p\,w)} = \sum_{n=0}^{\infty} \frac{T_n^{(m,p)}}{(2n+1)!} (-1)^n\, w^{2n+1}. \tag{2.6}$$

It is noted that the left hand side is also expanded as

$$\frac{\mathrm{sh}(m\,w)\,\mathrm{sh}(p\,w)}{\mathrm{sh}(m\,p\,w)} = \frac{1}{2}\sum_{n=0}^{\infty} \chi_{2mp}(n)\, \mathrm{e}^{-nw}, \tag{2.7}$$

where $\chi_{2mp}(n)$ is a periodic function modulo $2\,m\,p$;

$n \mod 2\,m\,p$	$m\,p-m-p$	$m\,p-m+p$	$m\,p+m-p$	$m\,p+m+p$	other
$\chi_{2mp}(n)$	1	-1	-1	1	0

(2.8)

By use of the Mellin transformation to eqs. (2.6) and (2.7), we can define the T-numbers $T_n^{(m,p)}$ in terms of the associated L-series as

$$\begin{aligned} T_n^{(m,p)} &= \frac{1}{2}(-1)^{n+1}\, L(-2n-1,\, \chi_{2mp}) \\ &= \frac{1}{2}(-1)^n\, \frac{(2\,m\,p)^{2n+1}}{2n+2} \sum_{a=1}^{2mp} \chi_{2mp}(a)\, B_{2n+2}\left(\frac{a}{2\,m\,p}\right), \end{aligned} \tag{2.9}$$

where $B_n(x)$ is the Bernoulli polynomial.

Substituting the expansion (2.6) into an integrand in eq. (2.5), we have

$$\langle \mathrm{Trs}(m,\,p)\rangle_N \simeq \left(\frac{m\,p\,N}{2}\right)^{3/2} \mathrm{e}^{\pi i N + \frac{\pi i}{N}\left(1-\frac{1}{2}(\frac{p}{m}+\frac{m}{p})\right)-\frac{\pi i}{4}} \times$$

$$\times \left(\mathrm{Res}(m,\,p) + 2\,\mathrm{i}\,(-1)^{mp+m+p}\, \mathrm{e}^{\frac{1}{2}\pi \mathrm{i} N m p} \sum_{n=0}^{\infty} \frac{T_n^{(m,p)}}{(2n+1)!} (-1)^n\, \pi^{2n+1} \int_{\mathscr{C}} \mathrm{d}w\, \mathrm{e}^{\frac{1}{2}\mathrm{i} m p N \pi w^2}\, w^{2n+2} \right)$$

After using an identity,

$$\int_{\mathscr{C}} \mathrm{d}w\, \mathrm{e}^{p\pi \mathrm{i} N w^2}\, w^{2n+2} = \frac{\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}}{\sqrt{p}} \cdot \frac{(-1)^{n+1}}{(p\,\pi\,\mathrm{i})^{n+1}} \cdot \frac{(2n+1)!}{n!\,2^{2n+1}} \cdot N^{-(n+\frac{3}{2})},$$

we finally get an asymptotic expansion of the invariant of the torus knot as

$$\begin{aligned} \langle \mathrm{Trs}(m,\,p)\rangle_N \simeq{}& \left(\frac{m\,p\,N}{2}\right)^{3/2} \mathrm{e}^{\pi i N + \frac{\pi i}{N}\left(1-\frac{1}{2}(\frac{p}{m}+\frac{m}{p})\right)-\frac{\pi i}{4}}\, \mathrm{Res}(m,\,p) \\ &+ (-1)^{(m+1)(p+1)}\, \mathrm{e}^{\pi i N(1+\frac{1}{2}mp)+\frac{\pi i}{N}\left(1-\frac{1}{2}(\frac{p}{m}+\frac{m}{p})\right)} \sum_{n=0}^{\infty} \frac{T_n^{(m,p)}}{n!} \left(\frac{\pi}{2\,m\,p\,N\,\mathrm{i}}\right)^n. \end{aligned} \tag{2.10}$$

As a result, we see in a limit $N \to \infty$ that

$$\log \left|\langle \mathrm{Trs}(m, p)\rangle_N\right| \sim \frac{3}{2} \log N. \tag{2.11}$$

which supports volume conjecture (1.1) and that there exits a logarithmic correction (1.5).

As seen from eq. (2.4), residue $\mathrm{Res}(m, p)$ is generally given as a summation of several terms whose asymptotics is controlled by $e^{-N\pi i \frac{n^2}{2mp}}$. In view from a complexification of the volume conjecture (1.4), this summation seems to represent a decomposition of the contribution by irreducible flat connections. In fact, when we consider the fundamental group of $S^3 \setminus \mathrm{Trs}(m, p)$ which has a presentation

$$\pi_1\big(S^3 \setminus \mathrm{Trs}(m, p)\big) = \big\langle x, y \mid x^m = y^p \big\rangle, \tag{2.12}$$

there are $(m-1)(p-1)/2$ irreducible representations, $\pi_1(S^3 \setminus \mathrm{Trs}(m, p)) \to SU(2)$, up to conjugacy as was discussed in Ref. 17. We hope to report a relationship with our result (2.10) and Ref. 18 in a future issue.

We shall give this asymptotic behavior explicitly for the $(2m+1, 2)$-torus knot below. These expressions are for our purpose to introduce an asymptotic expansion of q-series, and to propose a "strange identity" in a sense of Zagier.

Trefoil 3_1: $(m, p) = (3, 2)$ By explicit computation of the knot invariant using the R-matrix (1.2), we know that the invariant of the trefoil is given simply by [2]

$$\langle \mathrm{Trs}(3, 2)\rangle_N = \sum_{a=0}^{N-1} (\omega)_a. \tag{2.13}$$

Thus the formula (2.10) determines the asymptotic behavior of eq. (2.13) in a limit $N \to \infty$. By setting $(m, p) \to (3, 2)$ in eq. (2.4) we see that the residue term is given as

$$\mathrm{Res}(3, 2) = \frac{i}{3\sqrt{3}} e^{\pi i N (1 - \frac{1}{12})},$$

and that our T-numbers $T_n \equiv T_n^{(3,2)}$ are nothing but the Glaisher T-numbers T_n defined by

$$\frac{\sin(2x)}{2\cos(3x)} = \sum_{n=0}^{\infty} \frac{T_n}{(2n+1)!} x^{2n+1}. \tag{2.14}$$

n	0	1	2	3	4	5	6
T_n	1	23	1681	257543	67637281	27138236663	15442193173681

With these T-numbers, we obtain the asymptotic formula of eq. (2.13) as

$$\begin{aligned}\langle \mathrm{Trs}(3,2)\rangle_N &= \sum_{a=0}^{N-1} (\omega)_a \\ &\simeq N^{\frac{3}{2}} \exp\left(\frac{\pi\,\mathrm{i}}{4} - \frac{\pi\,\mathrm{i}\,N}{12} - \frac{\pi\,\mathrm{i}}{12\,N}\right) + \mathrm{e}^{-\frac{\pi \mathrm{i}}{12N}} \sum_{n=0}^{\infty} \frac{T_n}{n!}\left(\frac{\pi}{12\,\mathrm{i}\,N}\right)^n .\end{aligned} \tag{2.15}$$

This is nothing but a formula which was conjectured in Ref. 14.

As a result we obtain in a limit $N \to \infty$

$$\frac{2\pi}{N}\log\big(\langle \mathrm{Trs}(3,2)\rangle_N\big) \sim -\frac{\pi^2}{6}\mathrm{i} + \frac{3}{2}\cdot 2\pi\,\frac{\log N}{N} + O(N^{-1}), \tag{2.16}$$

where, according to a conjecture (1.4), the first term is expected to give the Chern–Simons invariant of the trefoil,

$$\mathrm{CS}\big(\mathrm{Trs}(3,2)\big) = -\frac{\pi^2}{6}$$

Solomon's Seal Knot 5_1: $(m,p) = (5,2)$ The invariant of the knot 5_1 is computed explicitly as[¶]

$$\langle \mathrm{Trs}(5,2)\rangle_N = \sum_{\substack{a,b=0 \\ 0\le a+b\le N-1}}^{N-1} \omega^{-ab}(\omega)_{a+b}. \tag{2.17}$$

Then the asymptotic expansion of this invariant follows from eq. (2.10) by setting $(m,p) \to (5,2)$. The residue term (2.4) is calculated as

$$\mathrm{Res}(5,2) = \frac{2\,\mathrm{i}}{25}\,\mathrm{e}^{N\pi\mathrm{i}}\left(2\,a\,\mathrm{e}^{-\frac{N\pi\mathrm{i}}{20}} - b\,\mathrm{e}^{-\frac{9N\pi\mathrm{i}}{20}}\right),$$

where we have set

$$a = \sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{5}}{2}\sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{5}}\right)}, \qquad b = \sin\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}}{2}\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{5}}\right)}.$$

The infinite series can be written in terms of $T_n^{(5,2)}$ defined by

$$\frac{\sin(2\,x)}{2\cos(5\,x)} = \sum_{n=0}^{\infty} \frac{T_n^{(5,2)}}{(2n+1)!}\,x^{2n+1}. \tag{2.18}$$

n	0	1	2	3	4	5	6
$T_n^{(5,2)}$	1	71	14641	6242711	455513328	5076970085351	8024733763147921

[¶]Y. Yokota showed this expression to the author during workshop.

As a result, we obtain

$$
\begin{aligned}
\langle \mathrm{Trs}(5,2)\rangle_N &= \sum_{0\leq a+b\leq N-1} \omega^{-ab}\,(\omega)_{a+b} \\
&\simeq \frac{2}{\sqrt{5}}\, N^{\frac{3}{2}} \mathrm{e}^{\frac{\pi}{4}\mathrm{i}-\frac{9\mathrm{i}\pi}{20N}} \left(2\,a\,\mathrm{e}^{-\frac{N\pi\mathrm{i}}{20}} - b\,\mathrm{e}^{-\frac{9N\pi\mathrm{i}}{20}}\right) + \mathrm{e}^{-\frac{9\mathrm{i}\pi}{20N}} \sum_{n=0}^{\infty} \frac{T_n^{(5,2)}}{n!} \left(\frac{\pi}{20\,\mathrm{i}\,N}\right)^n . \qquad (2.19)
\end{aligned}
$$

Recalling a complexification of volume conjecture (1.4), the first term which follows from residues of the integral may give the Chern–Simons invariant as

$$
\mathrm{CS}\big(\mathrm{Trs}(5,2)\big) = \left\{-\frac{1}{10}\pi^2, -\frac{9}{10}\pi^2\right\}.
$$

(2 m + 1, 2)-Torus Knot: ($m > 2$) We compute Kashaev's invariant explicitly using the R-matrix (1.2), and we find that it is explicitly written as

$$
\begin{aligned}
&\langle \mathrm{Trs}(2m+1,2)\rangle_N \\
&= N \sum_{1\leq a_{2m-2}\leq\cdots\leq a_1\leq N-1} (-1)^{\sum_{j=1}^{2m-2} a_j} \cdot \frac{\omega^{\frac{1}{2}\sum_{j=1}^{2m-2} a_j(a_j-1)}}{\prod_{j=1}^{2m-3} (\omega)_{a_j-a_{j+1}}} \\
&= \sum_{\substack{a_1,a_2,\dots,a_{2m-2}=0 \\ 0\leq a_1+a_2+\cdots+a_{2m-2}\leq N-1}}^{N-1} \frac{(\omega)_{a_1+a_2+\cdots+a_{2m-2}}}{\prod_{j=2}^{2m-3}(\omega)_{a_j}} (-1)^{\sum_{j=3}^{2m-2} j\,a_j} \\
&\quad \times \omega^{-a_1a_2+\sum_{j=3}^{2m-2}\left(\frac{j}{2}-1-a_1\right)a_j+\frac{1}{2}\sum_{j=3}^{2m-2}(a_j+a_{j+1}+\cdots+a_{2m-2})^2} \qquad (2.20)
\end{aligned}
$$

The asymptotic form of this series is given from eq. (2.10) by substituting $(m, p) \to (2m+1, 2)$. The residue term (2.4) is computed as

$$
\mathrm{Res}(2m+1,2) = \frac{2\,\mathrm{i}}{(2m+1)^2} \sum_{j=0}^{m-1} (-1)^j\,(m-j)\,\sin\left(\frac{2j+1}{2m+1}\pi\right)\,\mathrm{e}^{N\pi\mathrm{i}\left(1-\frac{(2j+1)^2}{4(2m+1)}\right)},
$$

and the infinite series is written in terms of $T_n^{(2m+1,2)}$ defined by

$$\frac{\sin(2x)}{2\cos((2m+1)x)} = \sum_{n=0}^{\infty} \frac{T_n^{(2m+1,2)}}{(2n+1)!} x^{2n+1}.$$

n	0	1	2	3	4	5
$T_n^{(7,2)}$	1	143	58081	48571823	69471000001	151763444497103
$T_n^{(9,2)}$	1	239	160801	222359759	525750911041	1898604115708079
$T_n^{(11,2)}$	1	359	361201	746248439	2635820840161	14219082731542919
$T_n^{(13,2)}$	1	503	707281	2041111463	10069440665761	75868751534107223
$T_n^{(15,2)}$	1	671	1256641	4828434911	31713479172481	318124890738776351

To conclude, we have obtained an asymptotic expansion for a set of the ω-series as

$$\begin{aligned}
&\langle \mathrm{Trs}(2m+1,2)\rangle_N \\
&= \sum_{\substack{a_1,a_2,\dots,a_{2m-2}=0 \\ 0\le a_1+a_2+\cdots+a_{2m-2}\le N-1}}^{N-1} \frac{(\omega)_{a_1+a_2+\cdots+a_{2m-2}}}{\prod_{j=2}^{2m-3}(\omega)_{a_j}} (-1)^{\sum_{j=3}^{2m-2} j\, a_j} \\
&\quad \times \omega^{-a_1 a_2 + \sum_{j=3}^{2m-2}\left(\frac{j}{2}-1-a_1\right)a_j + \frac{1}{2}\sum_{j=3}^{2m-2}(a_j+a_{j+1}+\cdots+a_{2m-2})^2} \\
&\simeq \frac{2}{\sqrt{2m+1}} N^{\frac{3}{2}} \mathrm{e}^{\frac{\pi \mathrm{i}}{4} - \frac{\pi \mathrm{i}}{N}\frac{(2m-1)^2}{4(2m+1)}} \sum_{j=0}^{m-1} (-1)^j\,(m-j)\,\sin\left(\frac{2j+1}{2m+1}\pi\right) \mathrm{e}^{-N\pi \mathrm{i}\frac{(2j+1)^2}{4(2m+1)}} \\
&\qquad + \mathrm{e}^{-\frac{\pi \mathrm{i}}{N}\frac{(2m-1)^2}{4(2m+1)}} \sum_{n=0}^{\infty} \frac{T_n^{(2m+1,2)}}{n!} \left(\frac{\pi}{4(2m+1)N\mathrm{i}}\right)^n .
\end{aligned} \tag{2.21}$$

Based on the first sum the Chern–Simons invariant may be identified as

$$\mathrm{CS}\big(\mathrm{Trs}(2,2m+1)\big) = \left\{ -\frac{(2j+1)^2}{2(2m+1)}\pi^2 \,\middle|\, j=0,1,\dots,m-1 \right\}.$$

2.2 q-Series Identity and $(2m+1,2)$-Torus Knot

We shall propose some observations based on the asymptotic behavior studied in the previous section. Strategy is merely to regard the invariants as q-series with q being the N-th root of unity ($\omega = \mathrm{e}^{2\pi \mathrm{i}/N}$) as was formulated by Zagier [14]. In general, by setting q to be ω an infinite q-series terminates and gives finite number even though the q-series itself diverges. We explicitly study q-series which arise from Kashaev's invariants of the $(2m+1,2)$-torus knots.

Trefoil 3_1: $(m, p) = (3, 2)$ We define

$$F(q) = \sum_{n=0}^{\infty} (q)_n, \tag{2.22}$$

following Ref. 14. This series does not converge in any open set, but in a case of q being root of unity $\omega = \mathrm{e}^{2\pi \mathrm{i}/N}$ it reduces to Kashaev's invariant (2.13);

$$F(\omega) = \langle \mathrm{Trs}(3, 2) \rangle_N. \tag{2.23}$$

We note that

$$F(1 - x) = \sum_{n=0}^{\infty} a_n \, x^n, \tag{2.24}$$

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	5	15	53	217	1014	5335	31240	201608

and that a_n coincides with the upper bound of the number of linearly independent Vassiliev invariants of degree n [13].

Furthermore, we have in a limit $t \to 0$

$$\begin{aligned} F(\mathrm{e}^{-t}) &= \sum_{n=0}^{\infty} (1 - \mathrm{e}^{-t})(1 - \mathrm{e}^{-2t}) \cdots (1 - \mathrm{e}^{-nt}) \\ &= \mathrm{e}^{t/24} \sum_{n=0}^{\infty} \frac{T_n}{n!} \left(\frac{t}{24} \right)^n, \end{aligned} \tag{2.25}$$

where T_n is the Glaisher number (2.14). This identity can be proved based on Zagier's q-series identity as follows [14] (see also Ref. 19 for a generalization of eq. (2.25)). When we define a function $S(x)$ by

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty} (x)_{n+1} \, x^n \\ &= (x\, q)_\infty + (1 - x) \sum_{n=0}^{\infty} \big((x\, q)_n - (x\, q)_\infty \big) \, x^n, \end{aligned} \tag{2.26}$$

we can check that it solves the q-difference equation,

$$S(x) = 1 - q\, x^2 - q^2 \, x^3 \, S(q\, x). \tag{2.27}$$

On the other hand, we can easily see that a function

$$S(x) = \sum_{n=1}^{\infty} \chi_{12}(n)\, x^{\frac{1}{2}(n-1)}\, q^{\frac{1}{24}(n^2-1)}, \tag{2.28}$$

also solves the same q-difference equation, and that both describe the same function. Here $\chi_{12}(n)$ is from eq. (2.8) with $(m, p) = (3, 2)$, and it becomes the Dirichlet character;

$n \mod 12$	1	5	7	11	other
$\chi_{12}(n)$	1	-1	-1	1	0

It is remarked that $S(x = 1)$ coincides with the Dedekind η-function,

$$(q)_\infty = \sum_{n=1}^{\infty} \chi_{12}(n)\, q^{\frac{1}{24}(n^2-1)}. \tag{2.29}$$

From two expressions (2.26) and (2.28), we find that

$$(x\,q)_\infty + (1-x) \sum_{n=0}^{\infty} \big((x\,q)_n - (x\,q)_\infty\big)\, x^n = \sum_{n=0}^{\infty} \chi_{12}(n)\, x^{\frac{1}{2}(n-1)}\, q^{\frac{1}{24}(n^2-1)}. \tag{2.30}$$

By differentiating with respect to x and setting $x \to 1$, we get

$$(q)_\infty \cdot \left(\frac{1}{2} - \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \right) - \sum_{n=0}^{\infty} \big((q)_n - (q)_\infty\big) = \frac{1}{2} \sum_{n=0}^{\infty} n\, \chi_{12}(n)\, q^{\frac{1}{24}(n^2-1)}. \tag{2.31}$$

Thus in a limit $t \to 0$ we obtain

$$-2\, e^{-t/24}\, F(e^{-t}) \sim \sum_{n=0}^{\infty} n\, \chi_{12}(n)\, e^{-\frac{1}{24}n^2 t}, \tag{2.32}$$

because $(q)_\infty$ induces an infinite order of t in a limit $t \to 0$. Applying the Mellin transformation to an equality $\sum_{n=0}^{\infty} n\, \chi_{12}(n)\, e^{-\frac{1}{24}n^2 t} \sim \sum_{n=0}^{\infty} \gamma_n\, t^n$, we get

$$\gamma_n = \frac{(-1)^n}{24^n\, n!} L(-2n-1, \chi_{12}).$$

By use of a relationship (2.9) between the L-series and the T-numbers, we find

$$\gamma_n = -2 \frac{T_n}{24^n\, n!},$$

which proves eq. (2.25). Note that the right hand side of eq. (2.31) is regarded as a function given by "differentiating the Dedekind η-function (2.29) half a time".

Comparing eq. (2.25) with eq. (2.15), we notice that there seems to be a naïve analytic continuation as

$$N \longleftrightarrow \frac{2\pi}{\mathrm{i}t}, \tag{2.33}$$

in a limit $t \to 0$. In fact applying this relation to the integral in eq. (2.5), we may have

$$F(\mathrm{e}^{-t}) \simeq -\left(\frac{6\pi}{\mathrm{i}t}\right)^{3/2} \mathrm{e}^{\frac{t}{24}+\frac{\pi}{4}\mathrm{i}} \int_{\mathscr{C}} \mathrm{d}w\, \mathrm{e}^{\frac{6\pi^2}{t}w^2}\, w\, \frac{\mathrm{sh}(2\pi w)}{\mathrm{ch}(3\pi w)}. \tag{2.34}$$

Using a generating function

$$\frac{\mathrm{sh}(2x)}{\mathrm{ch}(3x)} = \sum_{n=0}^{\infty} \chi_{12}(n)\, \mathrm{e}^{-nx} = 2\sum_{n=0}^{\infty} \frac{T_n}{(2n+1)!} (-1)^n x^{2n+1},$$

we see that above integral reproduces eqs. (2.25) and (2.32).

In conclusion, we have seen that a naïve substitution (2.33) into the integral gives correct asymptotic expansion (2.25) for the q-series (2.22).

Solomon's Seal Knot 5_1: $(m, p) = (5, 2)$ We define a q-series by

$$F^{(5,2)}(q) = \sum_{a,b=0}^{\infty} q^{-ab}\, (q)_{a+b}, \tag{2.35}$$

which gives the invariant (2.17) of the (5, 2)-torus knot in a case of q being the N-th root of unity;

$$F^{(5,2)}(\omega) = \langle \mathrm{Trs}(5,2) \rangle_N. \tag{2.36}$$

It should be remarked that using `Mathematica` we have a positive integral series by

$$F^{(5,2)}(1-x) = \sum_{n=0}^{\infty} a_n^{(5)}\, x^n.$$

n	0	1	2	3	4	5	6	7	8	9	10
$a_n^{(5)}$	1	2	6	23	109	621	4149	31851	276408	2676388	28608866

In a case of the trefoil knot (2.24), these coefficient series give the upper bound of the number of linearly independent Vassiliev invariants, but we do not know the meaning of this series.

We assume that the correspondence (2.33) is also applicable in Solomon's seal knot. With this assumption, the function $F^{(5,2)}(\mathrm{e}^{-t})$ in a limit $t \to 0$ may be formally written in the integral form,

$$F^{(5,2)}(\mathrm{e}^{-t}) \simeq \mathrm{i}\left(\frac{10\pi}{t}\right)^{3/2} \mathrm{e}^{\frac{9}{40}t} \int_{\mathscr{C}} \mathrm{d}w \, \mathrm{e}^{\frac{10\pi^2}{t}w^2} \, w \, \frac{\mathrm{sh}(2\pi w)}{\mathrm{ch}(5\pi w)}. \tag{2.37}$$

When we substitute an expansion (2.7) with $(m, p) = (5, 2)$ into above integrand,

$$\frac{\mathrm{sh}(2x)}{\mathrm{ch}(5x)} = \sum_{m=0}^{\infty} \chi_{20}(m)\,\mathrm{e}^{-mx} = 2\sum_{n=0} \frac{T_n^{(5,2)}}{(2n+1)!}(-1)^n x^{2n+1}, \tag{2.38}$$

$n \mod 20$	3	7	13	17	other
$\chi_{20}(n)$	1	−1	−1	1	0

we find that the asymptotic expansion is given by

$$\begin{aligned} F^{(5,2)}(\mathrm{e}^{-t}) &= \sum_{a,b=0}^{\infty} \mathrm{e}^{tab}\,(1-\mathrm{e}^{-t})\,(1-\mathrm{e}^{-2t})\cdots(1-\mathrm{e}^{-(a+b)t}) \\ &= \mathrm{e}^{\frac{9}{40}t} \sum_{n=0}^{\infty} \frac{T_n^{(5,2)}}{n!}\left(\frac{t}{40}\right)^n \qquad (2.39) \\ &\sim -\frac{1}{2}\sum_{n=0}^{\infty} n\,\chi_{20}(n)\,\mathrm{e}^{-\frac{t}{40}(n^2-9)}, \qquad (2.40) \end{aligned}$$

where $T_n^{(5,2)}$ is defined in eq. (2.18). The last equality also follows from the Mellin transformation and eq. (2.9). Note that, owing to the Jacobi triple identity, eq. (2.40) denotes a half-differential of the infinite q-product which can also be written in an infinite sum due to the Rogers–Ramanujan identity;

$$\sum_{n=1}^{\infty} \chi_{20}(n)\, q^{\frac{1}{40}(n^2-9)} = (q, q^4, q^5; q^5)_\infty = (q)_\infty \cdot \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n}. \tag{2.41}$$

Numerical computation using `Mathematica` supports a validity of eq. (2.39), which was a consequence of an ansatz (2.33). This result indicates that there may be a q-series identity like eq. (2.31) for $F^{(5,2)}(q)$ (2.35), which we hope to discuss in a future issue.

(2m + 1, 2)-Torus Knot: ($m > 2$) We define a formal q-series

$$F^{(2m+1,2)}(q) = \sum_{a_1,\dots,a_{2m-2}=0}^{\infty} \frac{(q)_{a_1+a_2+\dots+a_{2m-2}}}{\prod_{j=2}^{2m-3}(q)_{a_j}} (-1)^{\sum_{j=3}^{2m-2} j\, a_j}$$
$$\times q^{-a_1 a_2 + \sum_{j=3}^{2m-2} (\frac{j}{2} - 1 - a_1) a_j + \frac{1}{2} \sum_{j=3}^{2m-2} (a_j + a_{j+1} + \dots + a_{2m-2})^2} \quad . \qquad (2.42)$$

This gives the knot invariant (2.20) of the $(2\,m + 1, 2)$-torus knot in a limit $q \to \omega = e^{2\pi i/N}$,

$$F^{(2m+1,2)}(\omega) = \langle \mathrm{Trs}(2\,m + 1, 2)\rangle_N. \qquad (2.43)$$

We note that the function $F^{(2m+1,2)}(q)$ gives the positive integral coefficients $a_n^{(2m+1)}$ by

$$F^{(2m+1,2)}(1 - x) = \sum_{n=0}^{\infty} a_n^{(2m+1)}\, x^n,$$

n	0	1	2	3	4	5	6	7
$a_n^{(7)}$	1	3	12	62	402	3162	29308	312975
$a_n^{(9)}$	1	4	20	130	1070	10738	127316	1741705
$a_n^{(11)}$	1	5	30	235	2345	28623	413441	6896695
$a_n^{(13)}$	1	6	42	385	4515	64911	1105573	21759966
$a_n^{(15)}$	1	7	56	588	7924	131124	2572640	58354762

which might be related with the Vassiliev invariants. It indicates that $a_n^{(2m+1)}$ is given by the n-th order polynomial of m, *e.g.*,

$$a_0^{(2m+1)} = 1, \qquad a_1^{(2m+1)} = m, \qquad a_2^{(2m+1)} = m\,(m+1),$$

$$a_3^{(2m+1)} = \frac{1}{6}\, m\,(m+1)\,(8\,m+7),$$

$$a_4^{(2m+1)} = \frac{1}{6}\, m\,(m+1)\,(14\,m^2 + 22\,m + 9),$$

$$a_5^{(2m+1)} = \frac{1}{30}\, m\,(m+1)\,(8\,m+7)\,(19\,m^2 + 25\,m + 9),$$

$$a_6^{(2m+1)} = \frac{1}{180}\, m\,(m+1)\left(2360\,m^4 + 6544\,m^3 + 6841\,m^2 + 3209\,m + 576\right).$$

We also assume that a naïve analytic continuation (2.33) is correct in a limit $t \to 0$ in this case. The formal integral expression is thus given as

$$F^{(2m+1,2)}(\mathrm{e}^{-t})$$

$$\simeq \mathrm{i}\left(\frac{2(2m+1)\pi}{t}\right)^{3/2} \mathrm{e}^{\frac{(2m-1)^2}{8(2m+1)}t} \int_{\mathscr{C}} \mathrm{d}w\, \mathrm{e}^{\frac{2(2m+1)\pi^2}{t} w^2}\, w\, \frac{\mathrm{sh}(2\pi w)}{\mathrm{ch}((2m+1)\pi w)}. \quad (2.44)$$

Substituting an expansion (2.7) with $(m, p) \to (2m+1, 2)$,

$$\frac{\mathrm{sh}(2x)}{\mathrm{ch}((2m+1)x)} = \sum_{n=0}^{\infty} \chi_{8m+4}(n)\, \mathrm{e}^{-nx} = 2\sum_{n=0}^{\infty} \frac{T_n^{(2m+1,2)}}{(2n+1)!} (-1)^n x^{2n+1}, \quad (2.45)$$

$n \mod (8m+4)$	$2m-1$	$2m+3$	$6m+1$	$6m+5$	other
$\chi_{8m+4}(n)$	1	-1	-1	1	0

we obtain

$$F^{(2m+1,2)}(\mathrm{e}^{-t}) = \mathrm{e}^{\frac{(2m-1)^2}{8(2m+1)}t} \sum_{n=0}^{\infty} \frac{T_n^{(2m+1,2)}}{n!} \left(\frac{t}{2^3(2m+1)}\right)^n \quad (2.46)$$

$$\sim -\frac{1}{2}\sum_{n=0}^{\infty} n\, \chi_{8m+4}(n)\, \mathrm{e}^{-\frac{t}{8(2m+1)}(n^2-(2m-1)^2)}. \quad (2.47)$$

We note that the right hand side is now a "half-differential" of the infinite q-product defined by

$$\sum_{n=1}^{\infty} \chi_{8m+4}(n)\, q^{\frac{1}{8(2m+1)}(n^2-(2m-1)^2)} = (q, q^{2m}, q^{2m+1}; q^{2m+1})_\infty \quad (2.48)$$

$$= (q)_\infty \cdot \sum_{n_{m-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2+\cdots+n_{m-1}^2+n_1+\cdots+n_{m-1}}}{(q)_{n_{m-1}-n_{m-2}} \cdots (q)_{n_2-n_1}\, (q)_{n_1}}.$$

The validity of an asymptotic expansion (2.46) is checked numerically with a help of `Mathematica` for several m and n's, and this also supports that we may have a generalization of Zagier's q-series identity (2.31).

Summary: To conclude we have proposed conjectures, eqs. (2.39) and (2.46), concerning an asymptotic expansion for the q-series which arise from Kashaev's invariants of the $(2m+1, 2)$-torus knot. A case of the trefoil ($m = 1$) was studied in detail in Ref. 14, and this fact indicates strongly that there should be a generalization of Zagier's identity (2.31) for $m > 1$. It suggests that the q-series which is constructed based on the invariant for (m, p)-torus knot may generate a new asymptotic formula of the q-series in this manner.

3 Hyperbolic Knot

In a previous section, we have studied an asymptotic expansion of the invariant of the torus knot. We have seen that eq. (1.5) is checked analytically and that there is a logarithmic correction to the volume conjecture (1.1).

In this section, we shall check numerically with a help of PARI * on Alpha the logarithmic correction for the hyperbolic knots up to 6-crossing (the figure-eight knot 4_1, 5_2, 6_1, 6_2, and 6_3), Whitehead link, and Borromean rings. All these results support our conjecture (1.5).

Figure-Eight Knot: We study the figure-eight knot, whose invariant is given by [2]

$$\langle 4_1 \rangle_N = \sum_{a=0}^{N-1} \left| (\omega)_a \right|^2. \tag{3.1}$$

The asymptotic form in $N \to \infty$ is known exactly (see, e.g., Ref. 20), and we have

$$\lim_{N\to\infty} \frac{2\pi}{N} \log\bigl(\langle 4_1 \rangle_N\bigr) = 2\, D(\mathrm{e}^{\pi \mathrm{i}/3}) = 2.029883212819307... \tag{3.2}$$

which coincides with the hyperbolic volume of the complement of the figure-eight knot. Here we have used the Bloch–Wigner function $D(z)$;

$$\begin{aligned} D(z) &= \mathrm{Li}_2(z) + \arg(1-z) \cdot \log|z|, \\ \mathrm{Li}_2(z) &= -\int_0^z \frac{\log(1-t)}{t}\,\mathrm{d}t. \end{aligned}$$

Using PARI program, we perform numerical computation up to $N = 400000$ (see Fig. 2). Dots • in figure denote numerical data. We apply a method of least-squares to above data with a trial function

$$\begin{aligned} v_{\mathcal{K}}(N) &= \frac{2\pi}{N} \log\langle \mathcal{K} \rangle_N \\ &= c_1 + c_2 \cdot \frac{2\pi}{N} \log N + c_3\, N^{-1} + c_4\, N^{-2}, \end{aligned} \tag{3.3}$$

*GP/PARI calculator (`http://www.parigp-home.de/`)

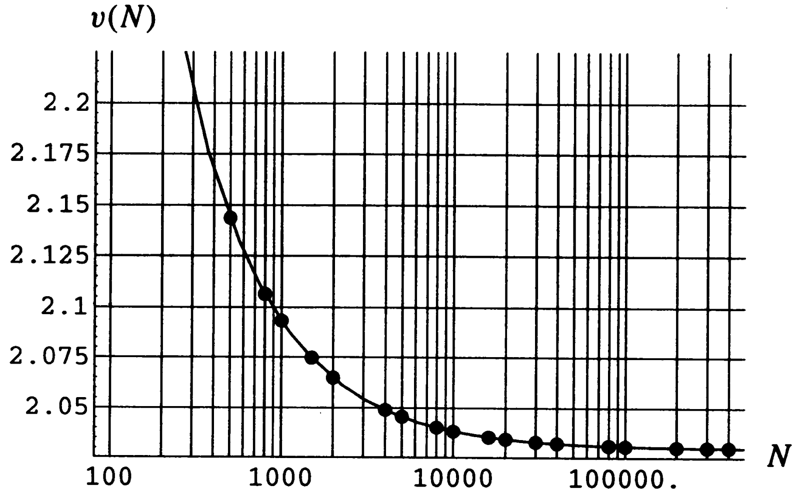


Figure 2: Figure-eight knot

which is motivated from analytic results of the torus knot. As a result, we obtain

$$c_1 = 2.029883193056962 \pm 7.771162 \times 10^{-9}$$
$$c_2 = 1.500026853413564 \pm 2.421627 \times 10^{-6}$$
$$c_3 = -1.726932111824925 \pm 0.00009535$$
$$c_4 = 3.575981132004609 \pm 0.00270356$$

A solid line in Fig. 2 denotes a function $v(N)$ with these parameters. We see that the first term c_1 matches with the exact result (3.2) and that the logarithmic correction c_2 is written as $c_2 \simeq 3/2$.

5_2 Knot: Using the R-matrix, we have [2]

$$\langle 5_2 \rangle_N = \sum_{0 \leq a \leq b \leq N-1} \frac{\left((\omega)_b\right)^2}{(\omega)_a^*} \omega^{-(b+1)a}. \tag{3.4}$$

We only consider a real part of $\frac{2\pi}{N} \log(\langle 5_2 \rangle_N)$, and numerical data by PARI is given in Fig. 3 (• in figure).

In this case, we also apply the method of least-squares with a trial function (3.3), and

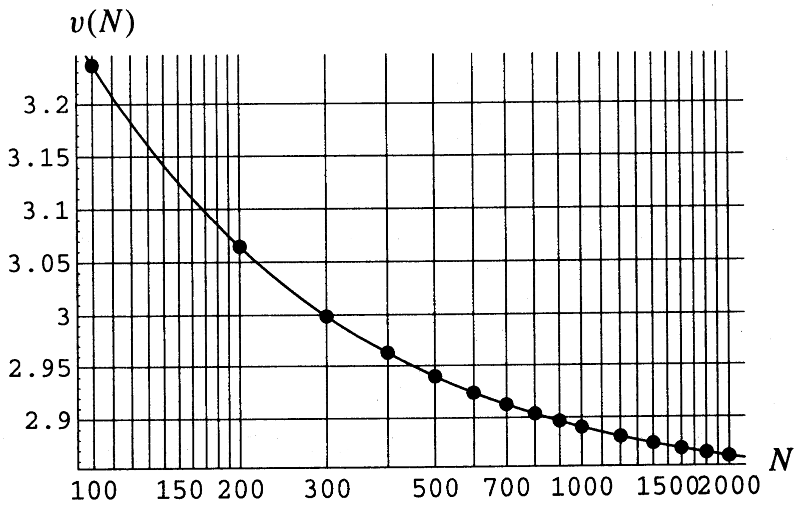


Figure 3: Knot 5_2

we get following results;

$$c_1 = 2.8281219744 \pm 1.5571 \times 10^{-8}$$
$$c_2 = 1.5000269858 \pm 2.01017 \times 10^{-6}$$
$$c_3 = -2.648116951 \pm 0.0000732$$
$$c_4 = 4.2278829125 \pm 0.0016885$$

It seems that c_1 coincides with the hyperbolic volume of the complement of knot 5_2,

$$\mathrm{Vol}(S^3 \setminus 5_2) = 2.828122088330783...$$

See that the logarithmic correction term seems to be $c_2 \simeq 3/2$.

6_1 Knot: We have [2]

$$\langle 6_1 \rangle_N = \sum_{\substack{a,b,c=0 \\ a+b \leq c}}^{N-1} \frac{\left|(\omega)_c\right|^2}{(\omega)_a \, (\omega)_b^*} \omega^{(c-a-b)(c-a+1)}. \tag{3.5}$$

We have used PARI and plotted a real part of $\frac{2\pi}{N} \log(\langle 6_1 \rangle_N)$ in Fig. 4.

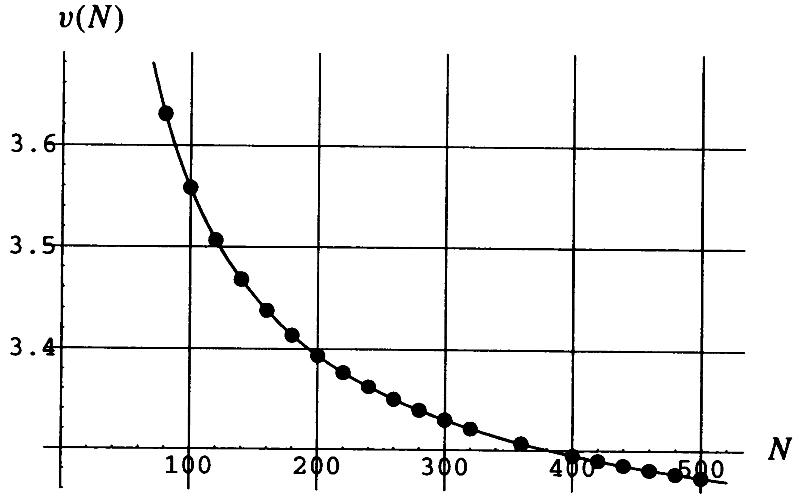


Figure 4: Knot 6_1

Due to the least-squares method, we obtain the following result for the function (3.3);

$$c_1 = 3.1639628602 \pm 3.0400 \times 10^{-8}$$
$$c_2 = 1.5000355979 \pm 1.8773 \times 10^{-6}$$
$$c_3 = -4.034362734 \pm 0.0000611$$
$$c_4 = 3.9717769748 \pm 0.0009704$$

A solid line in Fig. 4 denotes this function. We see that the first term c_1 is in good agreement with the hyperbolic volume of the complement of 6_1,

$$\mathrm{Vol}(S^3 \setminus 6_1) = 3.16396322888...$$

and that $c_2 \simeq 3/2$.

6_2 Knot: Kashaev's invariant is explicitly computed as

$$\langle 6_2 \rangle_N = \sum_{\substack{a,b,c=0 \\ a \leq b \\ 0 \leq a+c \leq N-1}}^{N-1} \omega^{-a(b+c+1)} \left(\frac{(\omega)_b}{|(\omega)_a|} \right)^2 \frac{(\omega)_{a+c}}{(\omega)_{b-a}}. \tag{3.6}$$

With PARI, we have computed numerically a real part of $\frac{2\pi}{N} \log\langle 6_1 \rangle_N$ for several N (• in

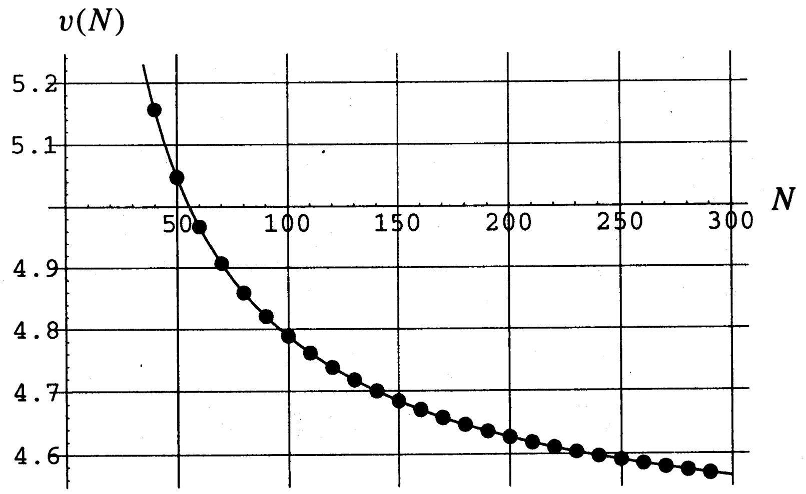


Figure 5: Knot 6_2

The least-square method with eq. (3.3) gives

$$c_1 = 4.400828513 \pm 2.9716 \times 10^{-7}$$
$$c_2 = 1.500213389 \pm 9.8267 \times 10^{-6}$$
$$c_3 = -4.6850950 \pm 0.00028$$
$$c_4 = 6.02178 \pm 0.00266$$

which indicates that c_1 agrees with a hyperbolic volume of the complement of 6_2,

$$\mathrm{Vol}(S^3 \setminus 6_2) = 4.40083251...$$

and $c_2 \simeq 3/2$.

6_3 Knot: We have

$$\langle 6_3 \rangle_N = \sum_{\substack{a,b,c=0 \\ a+b+c \leq N-1}}^{N-1} \left| \frac{(\omega)_{a+b+c}}{(\omega)_b \, (\omega)_c} \right|^2 (\omega)^*_{a+b} \, (\omega)_{a+c} \, \omega^{(a+1)(b-c)}. \tag{3.7}$$

Plotted as • in Fig. 6 is numerical result of a real part of $\frac{2\pi}{N} \log \langle 6_2 \rangle_N$.

A solid line is from eq. (3.3) with following coefficients determined by the least-square

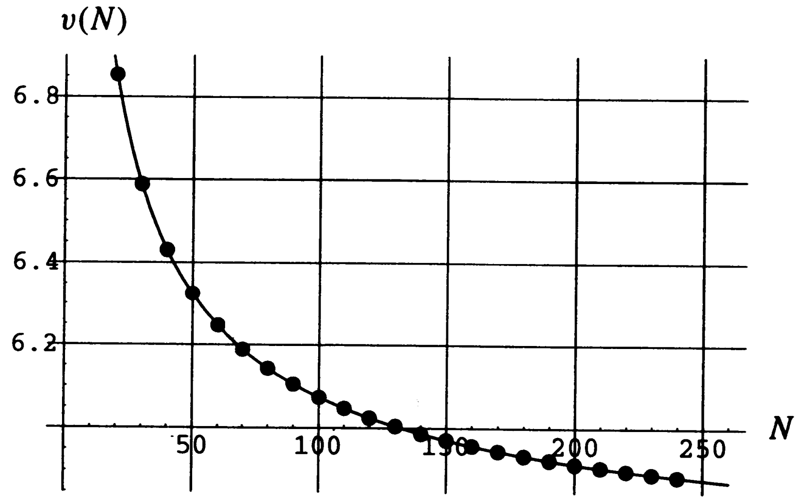


Figure 6: 6_3 Knot

method;

$$c_1 = 5.69289987 \pm 0.0000124$$
$$c_2 = 1.50410580 \pm 0.00025998$$
$$c_3 = -5.61617 \pm 0.00659$$
$$c_4 = 10.31505 \pm 0.03968$$

The first term c_1 is consistent with the exact hyperbolic volume of the complement of knot 6_3,

$$\mathrm{Vol}(S^3 \setminus 6_3) = 5.69302109...$$

and the logarithmic correction term indicates $c_2 \simeq 3/2$.

Whitehead Link: We have

$$\langle \mathrm{Whitehead} \rangle_N = \sum_{\substack{a,b,c=0 \\ b \leq a \\ a+c \leq N-1}}^{N-1} \frac{(\omega)^*_{a+c}\,(\omega)_a}{(\omega)_b\,(\omega)^*_c}\,\omega^{c(a-b)}. \tag{3.8}$$

Numerical results by PARI for a real part of $\frac{2\pi}{N} \log \langle \mathrm{Whitehead} \rangle_N$ are plotted as • in Fig. 7.

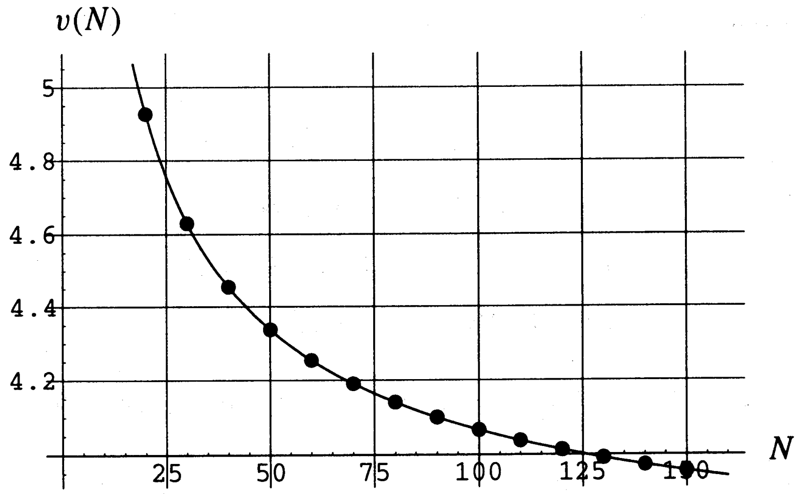


Figure 7: Whitehead link

We have applied the least square method with a function (3.3), and we obtain the following result;

$$c_1 = 3.663960 \pm 0.000113$$
$$c_2 = 1.499780 \pm 0.001902$$
$$c_3 = -3.272891 \pm 0.046130$$
$$c_4 = 6.184589 \pm 0.254868$$

In view of these data, it seems that c_1 coincides with the hyperbolic volume of the Whitehead link,

$$\mathrm{Vol}(S^3 \setminus \text{Whitehead}) = 3.66386237...$$

and that $c_2 \simeq 3/2$.

Borromean Rings: Using the R-matrix (1.3) of the colored Jones invariant, we have

$$\langle \text{Borromean} \rangle_N = \sum_{\substack{a,b,c,d=0 \\ a \leq b \leq a+c \leq N-1 \\ b+d \leq N-1}} \left| \frac{(\omega)_{a+c}\,(\omega)_{b+d}}{(\omega)_d\,(\omega)_{a+c-b}} \right|^2 \frac{1}{(\omega)_a\,(\omega)^*_{b-a}}\,\omega^{(b+1)(c-d+a-b)}. \tag{3.9}$$

See Fig. 8 for numerical results of a real part of $\frac{2\pi}{N}\log\langle\text{Borromean}\rangle_N$, and a solid line is a trial function (3.3) with

$$c_1 = 7.3276812 \pm 4.119463 \times 10^{-6}$$
$$c_2 = 1.5017634 \pm 0.0001082$$
$$c_3 = -8.764472 \pm 0.0029616$$
$$c_4 = 11.116386 \pm 0.0250$$

This is in agreement with the hyperbolic volume of the complement of the Borromean rings,

$$\mathrm{Vol}(S^3 \setminus \text{Borromean}) = 7.32772475...$$

and that $c_2 \simeq 3/2$.

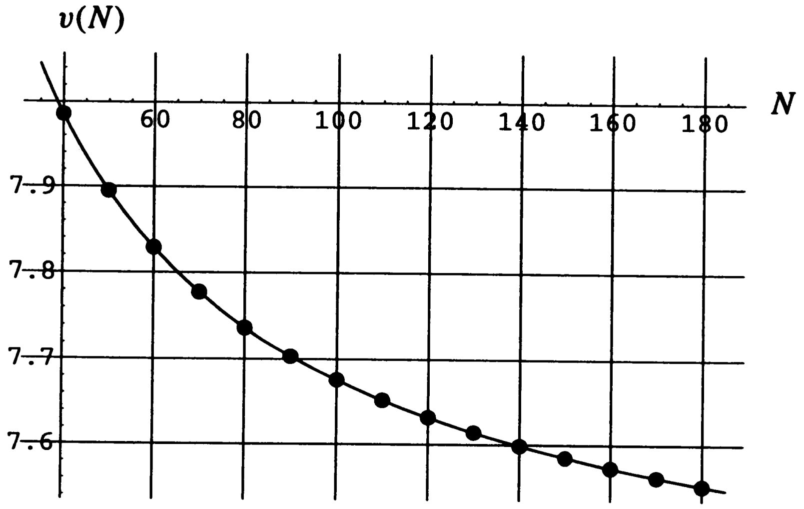


Figure 8: Borromean rings

4 Conclusion

We have studied an asymptotic expansion of Kashaev's knot invariant (or, a specific value of the colored Jones polynomial); analytical studies for the torus knot, and numerical computations for the hyperbolic knot (up to 6-crossing), Whitehead link, and Borromean rings. Collecting these results and recalling that Kashaev's invariant is given from the colored Jones polynomial for (1,1)-tangle of knot, we have proposed that Kashaev's invariant $\langle\mathcal{K}\rangle_N$

of knot $\mathcal{K}$ have an asymptotic form in a limit $N \to \infty$,

$$\log\left|\langle \mathcal{K} \rangle_N\right| \sim v_3 \cdot \left\| S^3 \setminus \mathcal{K} \right\| \cdot \frac{N}{2\pi} + \frac{3}{2} \#(\mathcal{K}) \cdot \log N + O(N^0). \tag{1.5}$$

Here v_3 is a hyperbolic volume of the regular ideal tetrahedron, and $\|\cdot\|$ and $\#(\mathcal{K})$ respectively denote the Gromov norm and the number of prime factors of a knot as connected-sum of prime knots. According to Ref. 14, in a case of the trefoil knot a factor $3/2$ in a logarithmic term in eq. (1.5) is connected with a weight $1/2$ of a *nearly* modular function $F(\omega)$. It will be interesting to study analytic properties of Kashaev's invariant of knot $\mathcal{K}$, and to see a difference between the torus knot and the hyperbolic knot.

We have also proposed several q-series identities, eqs. (2.39) and (2.46). The q-series is introduced so that it reduces to an explicit form of Kashaev's invariants of $(2m+1, 2)$-torus knot in a case of q being the N-th root of unity. In a case of trefoil ($m = 1$) the q-series identity was proved by Zagier. It remains for future studies whether we have such q-series identities for invariant of the $(2m + 1, 2)$-torus knot.

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