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<th>Actual computation for the complexified hyperbolic volume conjecture (Volume Conjecture and Its Related Topics)</th>
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<td>Author(s)</td>
<td>Murakami, Jun</td>
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Kyoto University
Actual computation for the complexified hyperbolic volume conjecture

Jun Murakami (Waseda University)

Abstract. In this report, some computations to check the complexified hyperbolic volume conjecture in [4] are explained. This conjecture concerns the first term of the asymptotics of \( \lim_{N \to \infty} J_N(K) \) of a hyperbolic knot \( K \). In the workshop at IIAS, Hikami explained his observation about the second term of the asymptotics of \( \lim_{N \to \infty} J_N(K) \), which is also positively checked in the computation of this report.

Acknowledgement. I would like to thank all participants of the workshop at IIAS for useful discussion, especially K. Hikami for introducing his technique to do actual computation by using Pari-Gp.

1. Introduction.

1.1. Complexified hyperbolic volume conjecture. For a hyperbolic knot \( K \) in \( S^3 \), let \( \text{Vol}(K) \) and \( \text{CS}(K) \) be the hyperbolic volume and the Chern-Simons invariant respectively of the complement of \( K \). Then the complexified hyperbolic volume conjecture [4] is the following formula to explain \( \text{Vol}(K) \) and \( \text{CS}(K) \) as a certain limit of the colored Jones invariants.

\[
\lim_{N \to \infty} J_N(K) = \exp \left( \frac{N}{2\pi} (\text{Vol}(K) + \sqrt{-1} \ \text{CS}(K)) \right)
\]

Here \( J_N(K) \) is the colored Jones polynomial corresponding to the \( N \) dimensional representation \( \mathcal{U}_q(sl_2) \) with the parameter \( q \) specialized to \( \exp(2\pi \sqrt{-1}/N) \), the primitive \( N \)-th root of unity. This invariant \( J_N(K) \) is proved in [3] to be equal to the Kashaev's invariant \( K_N(K) \). The exact meaning of the imaginary part is given in (3).
This conjecture is based on the following Kashaev conjecture. In [1], Kashaev conjectured that

\[
\lim_{N \to \infty} |K_N(K)| = \exp \frac{N \text{Vol}(K)}{2\pi}.
\]

He checked this relation exactly for the figure-eight knot $4_1$ and numerically for the knots $5_2$ and $6_1$.

The arguments of $K_N(K) (= J_N(K))$ are investigated in [4] and the complexified hyperbolic volume conjecture (1) is proposed. To give the exact meaning of the imaginary part of (1), it may be better to consider the following relation:

\[
\lim_{N \to \infty} \frac{J_{N+1}(K)}{J_N(K)} = \exp \left( \frac{1}{2\pi} (\text{Vol}(K) + \sqrt{-1} \text{CS}(K)) \right),
\]

which is checked numerically for some examples in the rest of this report.

Kashaev conjectured (2) for hyperbolic knots, and it is generalized in [3] for any knot $K$ as follows:

\[
|J_N(K)| \sim \exp \left( N \frac{v_3 |S^3 \setminus K|}{2\pi} \right),
\]

where $|S^3 \setminus K|$ denotes Gromov's simplicial volume of $S^3 \setminus K$, and $v_3$ is the volume of the ideal regular tetrahedron of the hyperbolic 3-space $H^3$, i.e.

\[
v_3 = 1.014941606409653625021202554
\]

It may be natural to consider about a complexification of the volume conjecture, which might be the following form.

\[
J_N(K) \sim \exp \left( N \left( \frac{v_3 |S^3 \setminus K|}{2\pi} + \sqrt{-1} \text{CS}(K) \right) \right),
\]

or, its quotient version

\[
\frac{J_{N+1}(K)}{J_N(K)} \sim \exp \left( N \left( \frac{v_3 |S^3 \setminus K|}{2\pi} + \sqrt{-1} \text{CS}(K) \right) \right).
\]
2. HIKAMI'S OBSERVATION.

Hikami observed that

\[ 2 \pi \log |J_N(K)| \sim \text{Vol}(K) N + 3 \pi \log N + O \left( \frac{1}{N} \right) \]

for several hyperbolic prime knots. The volume \( \text{Vol}(K) \) in the first term corresponds to the Kashaev's conjecture (2). The second term explains a mysterious behavior of \( |J_N(K)| \), since the coefficient \( 3\pi \) appears for every prime knot he checked. Moreover, Kashaev and Tirkkonen [2] proved that

\[ |J_N(K)| \sim N^{3/2} \]

for any torus knot \( K \). This implies that

\[ 2 \pi \log |J_N(K)| \sim 3 \pi \log N. \]

Now reformulate (7) for \( J_{N+1}(K)/J_N(K) \) to compare the complexified hyperbolic volume conjecture. Since

\[ \log(N + 1) - \log N \sim \frac{1}{N} \]

Hikami's observation (7) is reformulated as follows.

\[ 2 \pi \log \left| \frac{J_{N+1}(K)}{J_N(K)} \right| \sim \text{Vol}(K) + \frac{3\pi}{N} + O \left( \frac{1}{N^2} \right). \]

This relation seems to be true for all the examples given in this report.
3. Actual Computations for Several Knots

3.1. Preliminaries. Let $N$ be a positive integer and

$$q = \exp 2\pi \sqrt{-1}/N.$$  

Let

$$(x)_k = \prod_{i=1}^{k}(1-x^i).$$

It is known that

$$(q)_{N-1} = (\bar{q})_{N-1} = N,$$

and so

$$(11) \quad \frac{1}{(q)_i} = \frac{(\bar{q})_{N-1-i}}{N}, \quad \frac{1}{(\bar{q})_i} = \frac{(q)_{N-1-i}}{N}.$$  

3.2. Figure-eight knot $4_1$. For the figure-eight knot $K$,

$$(12) \quad J_N(K) = \sum_{i=0}^{N-1}(q)_i(\bar{q})_i$$

Since $(q)_i(\bar{q})_i$ is a positive real number, numerical computation of the summension may have good accuracy. By using “pari-Gp 2.0.14” [5], which uses 28 digits for real numbers, $J_{N+1}/J_N(K)$ is computed by the following program.

Program. The feature of this program is to compute the formula consisting of a sum of the terms of $(q)_i$ and $(\bar{q})_i$ as a polynomial modulo $x^N-1$. We replace $q$ by an indeterminate $x$ and compute everything as a polynomial in $x$ modulo $x^N-1$. Here we use the function of Pari to handle polynomials modulo a polynomial. At the end of the computation of $J_N(K)$, $x$ is replaced by $q = \exp 2\pi \sqrt{-1}/N$.

The following program is for $N = 40$ case. The parameter 1 represents a list for

$$(x)_i \mod x^N - 1.$$  

The $i$-th component of 1 is $(x)_{i-1}$. Similarly, the parameter 1m is a list for

$$(x^{-1})_i \mod x^N - 1.$$
and \texttt{lms} for

\[(x)_{i-1}(x^{-1})_i \mod x^N - 1.\]

The parameters \texttt{ansn1} and \texttt{ansn2} contain the value of $J_N(K)$ and $J_{N+1}$ respectively. In Pari, a polynomial $P(x)$ modulo a polynomial $Q(x)$ is represented by

\[
\text{Mod}(P(x), Q(x))
\]

and the part $P(x)$ is obtained by

\[
\text{component} \left( \text{Mod}(P(x), Q(x)), 2 \right)
\]

By using the above conventions, the program to compute $J_{N+1}(K)/J_N(K)$ is the following.

\begin{verbatim}
N = 40
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(i=0, N-1, labs[i+1])
ansn1 = subst(component(ans, 2), x, \exp(2*Pi*sqrt(-1)/N))
N = N+1
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(i=0, N-1, labs[i+1])
ansn2 = subst(component(ans, 2), x, \exp(2*Pi*sqrt(-1)/N))
2*Pi*log(ansn2/ansn1)
\end{verbatim}

\textbf{Results.} The results of $\frac{J_{N+1}(K)}{J_N(K)}$ are obtained as in Table 1.
<table>
<thead>
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<th>( N )</th>
<th>( 2\pi \log \left( \frac{J_{N+1}(K)}{J_N(K)} \right) )</th>
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\[
2\pi \log \frac{J_{1001}(K)}{J_{1000}} - \frac{3\pi}{1000} \]

Vol(K) \[=2.029875015579984388038536163 \]

Table 1
Graph. The points \( \left( \frac{1}{N}, \frac{J_{N+1}(K)}{J_N(K)} \right) \) of the above data is plotted as follows.

Fitting. From the above result, we can predict the actual limit by estimating the asymptotics of \( J_N(K) \) by fitting with certain function, which is determined by the least square method. Here, we try to use the function of the form

\[
(13) \quad a_0 + \frac{a_1}{N} + \frac{a_2}{N^2}.
\]

\( a_0, a_1 \) and \( a_2 \) are obtained by the following function of Mathematica

\[
\text{Fit}[1/. \{x_-\}, y-] \rightarrow \{1/x, y\}, \{1, x, x^{-2}\}, x]
\]

where 1 is a list of the pairs

\[
\{N, J_{N+1}(K)/J_N(K)\}
\]

in the above table with \( N \geq 40 \). The result of the fitting is

\[
2.02988 + 9.42629 \frac{1}{N} - 8.34016 \frac{1}{N^2}.
\]

Note that the constant term is equal to the volume of \( S^3 \setminus K \) up to 6 digits, and the coefficient of \( x \) is almost equal to \( 3\pi = 9.42478 \ldots \).
3.3. **Knot $5_2$.** Let $K$ be the knot $5_2$. Then

$$J_N(K) = \sum_{i=1}^{N-1} \sum_{j=1}^{i} \frac{(q)_i^2}{(\bar{q})_j}.$$ 

This knot is achiral and $J_N(K)$ has a non-trivial imaginary part. The following results suggest that

$$\lim_{N \to \infty} \log \frac{J_{N+1}(K)}{J_N(K)} = \text{Vol}(K) + \sqrt{-1} \text{CS}(K),$$

where

$$\text{Vol}(K) = 2.82812208833, \quad \text{CS}(K) = -3.02412837657.$$ 

The program to compute $J_{N+1}(K)/J_N(K)$ for $N = 40$ is

```plaintext
N = 40
1 = listcreate(N)
listinsert(1, 1, 1)
for(i=2, N, listinsert(1, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
12 = listcreate(N)
for(i=1, N, listinsert(12, 1[i]*l[i], i))
ans = sum(i=0, N-1, 12[i+1]*sum(j=0,i, 
    1[N-1-j+1]*Mod(x^component(Mod(-j*(i+1),N), 2), x^N-1)))
ans1 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
N = N+1
1 = listcreate(N)
listinsert(1, 1, 1)
for(i=2, N, listinsert(1, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
12 = listcreate(N)
for(i=1, N, listinsert(12, 1[i]*l[i], i))
ans = sum(i=1, N-2, 12[i+1]*sum(j=0,i, 
    1[N-1-j+1]*Mod(x^component(Mod(-j*(i+1),N), 2), x^N-1)))
ans2 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
2*Pi*log(ansn2*(N-1)/ansn1/N)
```

At the last line, $N$ and $N-1$ are added since, in the computation of ans, we use the relation (11).

The results are given in Table 2.
$\mathrm{CS}(K) = 2\pi^2 \mathrm{cs}(K)$ where $\mathrm{cs}(K)$ is the Chern-Simons invariant obtained by SnapPea.

**Graphs.** The real and imaginary parts of the points \( \left( \frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)} \right) \) are plotted as follows.

![Graph](image)

**Figure 2.** Plotting of the points \( \left( \frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)} \right) \) of the knot 5_2.

**Fitting.** The result of the fitting is

\[
2.82812 - 3.02413\sqrt{-1} + (9.42398 + 0.00306762\sqrt{-1}) \frac{1}{N} - (8.80117 - 1.82018\sqrt{-1}) \frac{1}{N^2}.
\]
3.4. **Knot $6_1$.** Let $K$ be the knot $6_1$. Then

$$J_N(K) = \sum_{0 \leq m \leq N-1} \frac{|(q)_m|^2}{(q)_k(q)_l} q^{(m-k-l)(m-k+1)}.$$ 

The program to compute $J_{N+1}(K)/J_N(K)$ for $N = 40$ is

```
N = 40
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(m=0, N-1, labs[m+1]*\n  sum(k=0, m, lm[N-k-1]+1)*\n  sum(ll=0, m-k, l[N-l-1]+1)*\n  Mod(x^component(Mod((m-k-11)*(m-k+1), N), 2), x^N-1)
))
ans1 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
N = N+1
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(m=0, N-1, labs[m+1]*\n  sum(k=0, m, lm[N-k-1]+1)*\n  sum(ll=0, m-k, l[N-l-1]+1)*\n  Mod(x^component(Mod((m-k-11)*(m-k+1), N), 2), x^N-1)
))
ans2 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
2*Pi*log(ans2*(N-1)^2/ans1/N^2)
```

The results are given in Table 3.
3. \[ CS(K) = -2\pi^2 \mathrm{cs}(K) + \pi^2 \]

Graphs. The real and imaginary parts of the points \( \left( \frac{1}{N}, \frac{J_{N+1}(K)}{J_{N}(K)} \right) \) are plotted as follows.

![real part](image1.png) ![imaginary part](image2.png)

**Figure 3.** Plotting of the points \( \left( \frac{1}{N}, \frac{J_{N+1}(K)}{J_{N}(K)} \right) \) of the knot 6_1.

Fitting. The result of the fitting is

\[
(3.16404 - 6.79075 \sqrt{-1}) + (9.40652 + 0.00370946 \sqrt{-1}) \frac{1}{N} - (7.70212 - 5.27915 \sqrt{-1})
\]
3.5. Knot 63. Let \( K \) be the knot 63. Then

\[
J_N(K) = \sum_{k,l,m \geq 0 \atop k+l+m \leq N-1} \left| \frac{(q)^{k+l+m}}{(\bar{q})_l (q)_m} \right|^2 (q)_{k+l} (\bar{q})_{m+k} q^{(m-l)(k+1)}.
\]

The program to compute \( J_{N+1}(K)/J_N(K) \) for \( N = 40 \) is

\[
\begin{align*}
N &= 40 \\
l &= \text{listcreate}(N) \\
\text{listinsert}(l, 1, 1) \\
\text{for}(i=2, N, \text{listinsert}(l, \text{Mod}(1-x^{-(i-1)}, x^{-N-1}) \cdot l[i-1], i)) \\
lm &= \text{listcreate}(N) \\
\text{listinsert}(lm, 1, 1) \\
\text{for}(i=2, N, \text{listinsert}(lm, \text{Mod}(1-x^{-(N-i+1)}, x^{-N-1}) \cdot lm[i-1], i)) \\
labs &= \text{listcreate}(N) \\
\text{for}(i=1, N, \text{listinsert}(labs, l[i] \cdot lm[i], i)) \\
\text{ans} &= \text{sum}(m=0, N-1, \text{labs}[N-1-m+1] \cdot \text{listcreate}(N)) \\
\text{sum}(p=0, N-1-m, \text{labs}[p+m+1] \cdot \text{lm}[p+1] \cdot \text{listcreate}(N)) \\
\text{sum}(k=0, p, \text{labs}[N-1-p+k+1] \cdot \text{lm}[k+1] \cdot \text{listcreate}(N)) \\
\text{Mod}(x \cdot \text{component}(\text{Mod}(-(m-p+k) \cdot (k+1), N), 2), x^{-N-1})) \\
\text{ans1} &= \text{subst}(\text{component}(\text{ans}, 2), x, \exp(2\pi i \sqrt{-1}/N)) \\
N &= N+1 \\
l &= \text{listcreate}(N) \\
\text{listinsert}(l, 1, 1) \\
\text{for}(i=2, N, \text{listinsert}(l, \text{Mod}(1-x^{-(i-1)}, x^{-N-1}) \cdot l[i-1], i)) \\
lm &= \text{listcreate}(N) \\
\text{listinsert}(lm, 1, 1) \\
\text{for}(i=2, N, \text{listinsert}(lm, \text{Mod}(1-x^{-(N-i+1)}, x^{-N-1}) \cdot lm[i-1], i)) \\
labs &= \text{listcreate}(N) \\
\text{for}(i=1, N, \text{listinsert}(labs, l[i] \cdot lm[i], i)) \\
\text{ans} &= \text{sum}(m=0, N-1, \text{labs}[N-1-m+1] \cdot \text{listcreate}(N)) \\
\text{sum}(p=0, N-1-m, \text{labs}[p+m+1] \cdot \text{lm}[p+1] \cdot \text{listcreate}(N)) \\
\text{sum}(k=0, p, \text{labs}[N-1-p+k+1] \cdot \text{lm}[k+1] \cdot \text{listcreate}(N)) \\
\text{Mod}(x \cdot \text{component}(\text{Mod}(-(m-p+k) \cdot (k+1), N), 2), x^{-N-1})) \\
\text{ans2} &= \text{subst}(\text{component}(\text{ans}, 2), x, \exp(2\pi i \sqrt{-1}/N)) \\
2\pi i \log(\text{ans2} \cdot (N-1)^{4}/\text{ans1}/N^{4})
\end{align*}
\]

The results are given in Table 4.
### Table 4

<table>
<thead>
<tr>
<th>$N$</th>
<th>$2\pi \log \frac{J_{N+1}(K)}{J_N(K)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>5.991757632388930862295686837</td>
</tr>
<tr>
<td>40</td>
<td>5.920010510909063767712690688</td>
</tr>
<tr>
<td>47</td>
<td>5.887310870362322038241138727</td>
</tr>
<tr>
<td>50</td>
<td>5.8760918004707597393608402</td>
</tr>
<tr>
<td>60</td>
<td>5.846282921844738453303387249</td>
</tr>
<tr>
<td>70</td>
<td>5.824859282414985211663083205</td>
</tr>
<tr>
<td>80</td>
<td>5.808687819822659085249294793</td>
</tr>
<tr>
<td>94</td>
<td>5.791733883431946311125566885</td>
</tr>
<tr>
<td>100</td>
<td>5.78589893155213353224223121</td>
</tr>
<tr>
<td>120</td>
<td>5.770610335748061213979602476</td>
</tr>
<tr>
<td>150</td>
<td>5.755245033266310556638346366</td>
</tr>
<tr>
<td>$2\pi \log \frac{J_{251}(K)}{J_{250}(K)} - \frac{3\pi}{150}$</td>
<td>5.69241318019451469186903498</td>
</tr>
<tr>
<td>Vol($K$)</td>
<td>5.69302109128</td>
</tr>
</tbody>
</table>

**Graph.** The real and imaginary parts of the points \(\left(\frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)}\right)\) are plotted as follows.

![Graph](image)

**Figure 4.** Plotting of the points \(\left(\frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)}\right)\) of the knot 63.

**Fitting.** The result of the fitting is

\[
5.69297 + 9.43385 \frac{1}{N} - 14.1061 \frac{1}{N^2}.
\]
3.6. Knot 89. Let $K$ be the knot 89. Then

$$J_N(K) = \sum_{0 \leq l, m_1, m_2, n_1, n_2 \leq N-1} \frac{(q)_{l-m_1}(q)_{l}(q)_{l-m_2}}{(q)_{m_1}(q)_{m_2}(q)_{n_1}(q)_{n_2}} \times \frac{(q)_{l-n_1}(q)_{l-n_2}}{(q)_{l-m_1-n_1}(q)_{l-m_2-n_2}}$$

$$q^{(m_2-m_1)(l-m_1-m_2)+(n_2-n_1)(l-n_1-n_2)+m_2-m_1+n_2-n_1}.$$

The program to compute $J_{N+1}(K)/J_N(K)$ is almost equal to those for the previous examples. The only different lines are the following.

$$\ldots$$

ans = sum(11=0, N-1, labs[11+1]*
    sum(m1=0, 11, labs[11-m1+1]*labs[N-1-m1+1]*
        sum(n1=0, 11-m1, labs[N-1-n1+1]*lab[11-n1+1]*lab[11-11+m1+n1+1]*
            sum(m2=0, 11-m1, labs[11-m2+1]*labs[N-1-m2+1]*
                sum(n2=0, 11-m2, labs[N-1-n2+1]*lab[11-n2+1]*lab[11-11+m2+n2+1]*
                    Mod(x^(((m2-m1)*(11-m1-m2)+(n2-n1)*(11-n1-n2)+m2-m1+n2-n1)%N),
                        x^N-1))))))

$$\ldots$$

$$2*Pi*log(ans2*(N-1)^10/ans1/N^10)$$

The results are given in Table 5.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$2\pi \log \frac{J_{N+1}(K)}{J_N(K)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>8.036805097240829695180371009</td>
</tr>
<tr>
<td>10</td>
<td>8.373856508425248006124939747</td>
</tr>
<tr>
<td>15</td>
<td>8.152791235806956158626064554</td>
</tr>
<tr>
<td>20</td>
<td>8.021952312877980724820244796</td>
</tr>
<tr>
<td>25</td>
<td>7.941218675423634478989298960</td>
</tr>
<tr>
<td>30</td>
<td>7.885684247868739884080382928</td>
</tr>
<tr>
<td>40</td>
<td>7.81441575286245769627810490</td>
</tr>
<tr>
<td>50</td>
<td>7.770664225432679874868903250</td>
</tr>
</tbody>
</table>

$$2\pi \log \frac{J_5(K)}{J_{50}(K)} - \frac{3\pi}{50} = 7.582168666217292280561144647$$

| Vol(K) | 7.5881802236416 |

**Table 5**
Graph. The points \( \left( \frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)} \right) \) are plotted as follows.

![Graph of points](image)

**Figure 5.** Plotting of the points \( \left( \frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)} \right) \) of the knot 89.

### 3.7. Knot 820

Let \( K \) be the knot 820. Then

\[
J_N(K) = \sum_{\substack{j, l \leq k \leq i+l \leq j+m \leq N \leq i \leq j \leq i}} \frac{\{(\bar{q})_i(q)_k(\bar{q})_m\}^2}{\{(\bar{q})_j(q)_l}\} q^{k+m+im+km-il}.
\]

**Program.**

```plaintext
... ans = sum(i=0, N-1, lm2[i+1]*\ 
        sum(j=0, i, 12[N-1-j+1]*lm[N-1-i+j+1]*\ 
            sum(ll=0, N-1, lm2[N-1-ll+1]*\ 
                sum(k=max(ll,j), min(N-1, i+ll), \ 
                    12[k+1]*lm[N-1-k+ll+1]*l[N-1-i+k-ll+1]*lm[N-1-k+j+1]*\ 
                        sum(m=i-j+ll, N-1, lm2[m+1]*l[N-1-j-m+i+ll+1]*\ 
                            Mod(x^((k+m+i*m+k*m-i*ll)%N), x^N-1)*\ 
                            Mod(x^((k+m+i*m+k*m-i*ll)%N), x^N-1)*\ 
                        )\ 
                )\ 
            )\ 
        )\ 
    )\ 
... 2*Pi*log(ans2*(N-1)^9/ans1/N^9)
```
results.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$2\pi \log \frac{J_{N+1}(K)}{J_N(K)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4.9931348302832317119922109736 - 8.534810138421228059039058370 $\sqrt{-1}$</td>
</tr>
<tr>
<td>7</td>
<td>6.058772085097703463174557594 - 13.01002462670787288866716257 $\sqrt{-1}$</td>
</tr>
<tr>
<td>10</td>
<td>4.83814631375578870051905369 - 11.9472940926942721637213050 $\sqrt{-1}$</td>
</tr>
<tr>
<td>14</td>
<td>4.733597316958210595817845225 - 11.870057520625791877540683 $\sqrt{-1}$</td>
</tr>
<tr>
<td>20</td>
<td>4.577298093617009639760204539 - 11.88396870671513344794156134 $\sqrt{-1}$</td>
</tr>
<tr>
<td>25</td>
<td>4.488646016707939440733681448 - 11.8936255292428507802734017 $\sqrt{-1}$</td>
</tr>
<tr>
<td>30</td>
<td>4.42968129481447562532108244 - 11.89859594888974090236089072 $\sqrt{-1}$</td>
</tr>
<tr>
<td>35</td>
<td>4.38740236780920648262851653 - 11.9015392777342954376145556 $\sqrt{-1}$</td>
</tr>
<tr>
<td>40</td>
<td>4.355311811237171523425351164 - 11.90347183192930081507773909 $\sqrt{-1}$</td>
</tr>
<tr>
<td>50</td>
<td>4.310046749251591944060510784 - 11.90576469622971586668761426 $\sqrt{-1}$</td>
</tr>
</tbody>
</table>

$\frac{2\pi \log J_{61}(K)}{J_{50}(N)} = \frac{3\pi}{56} \quad 4.12155119036204349752752181 - 11.90576469622971586668761426 \sqrt{-1}$

$\text{Vol}(K) + \sqrt{-1} \text{CS}(K)$

4.1249032518077 -11.9099170709 $\sqrt{-1}$

Table 6. $\text{CS}(K') = -2\pi^2 \text{cs}(K) - \pi^2$.

Graphs.

![real part](image1)

![imaginary part](image2)

**Figure 6.** Plotting of the points $\left(\frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot $8_{20}$.

3.8. **Whitehead link.** The Whitehead link is the most simplest hyperbolic 2-component link. It is not a one-component knot, but complexified hyperbolic volume conjecture seems to hold for this link as follows.

Let $K$ be the Whitehead link. Then
\[ J_N(K) = \sum_{0 \leq i,j,k \leq N-1} \frac{\{(\overline{q})_i (\overline{q})_j\}^2}{(q)_k^4 (\overline{q})_{i-k} (\overline{q})_{j-k}} q^{-(N-1)N/2} \]

Program.

\[ \text{ans} = \text{sum}(k=0, N-1, \text{lm}4[N-1-k+1]*\{$\text{sum}(i=k, N-1, \text{lm}2[i+1]*\{1[N-1-i+k+1]*\}{$\text{sum}(j=k, N-1, \text{lm}2[j+1]*\{1[N-1-j+k+1]))\}$}\{2*\pi*\log(\text{ans2}*(N-1)^6/\text{ans1}/N^6)\} \]

Here \( q^{-(N-1)N/2} \) is omitted since it is equal to \( \pm 1 \), which contributes to \( \text{CS}(K) \) by a multiple of \( 2\pi^2 \).

Results.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( 2\pi \log \frac{J_{N+1}(K)}{J_N(K)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>3.892920359101811097809525583 + 2.457483997330866045812504703 ( \sqrt{-1} )</td>
</tr>
<tr>
<td>50</td>
<td>3.848161466402914225154530180 + 2.461039474018016569869745301 ( \sqrt{-1} )</td>
</tr>
<tr>
<td>60</td>
<td>3.818029013349499312708236153 + 2.462976748675980254703390855 ( \sqrt{-1} )</td>
</tr>
<tr>
<td>70</td>
<td>3.796362501209537691078944556 + 2.464147191795881614582476451 ( \sqrt{-1} )</td>
</tr>
<tr>
<td>80</td>
<td>3.780034327560022195082015385 + 2.464907923404764622274395868 ( \sqrt{-1} )</td>
</tr>
<tr>
<td>100</td>
<td>3.757062258985477857247991239 + 2.465803785962819679236327339 ( \sqrt{-1} )</td>
</tr>
<tr>
<td>120</td>
<td>3.741674608179023673159144258 + 2.466291085896660260686606142 ( \sqrt{-1} )</td>
</tr>
<tr>
<td>150</td>
<td>3.726228649726558596057828429 + 2.466690204011030007962113880 ( \sqrt{-1} )</td>
</tr>
</tbody>
</table>

\[ 2\pi \log \frac{J_{151}(K)}{J_{150}(N)} - \frac{3\pi}{150} = 3.66339679665476275738575561 + 2.466690204011030007962113880 \sqrt{-1} \]

<table>
<thead>
<tr>
<th>( \sqrt{-1} \text{CS}(K) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.6638623767089 + 2.46740110027234 ( \sqrt{-1} )</td>
</tr>
</tbody>
</table>

**Table 7.** \( \text{CS}(K) = 2\pi^2 \text{cs}(K) \), where \( \text{cs}(K) = 1/8 \).
Fitting.

\[
3.66386 + 2.46742 \sqrt{-1} + (9.42575 - 0.00497353 \sqrt{-1}) \frac{1}{N} - \\
(10.5298 + 15.7048 \sqrt{-1}) \frac{1}{N^2}
\]

Remark. The volume conjecture (4) is not hold for all links, because \(J_N(L) = 0\) if \(L\) is a split link \(L = K_1 \sqcup K_2\). In this case,

\[
|S^3 \setminus L| = |S^3 \setminus K_1| + |S^3 \setminus K_2|
\]

and so

\[
\lim_{N \to \infty} \exp(N |S^3 \setminus L|)
\]

does not equal to 0 if \(K_1\) and \(K_2\) are both hyperbolic knots.
4. Conclusion

In the above computations, we see the behavior of \( \frac{J_{N+1}(K)}{J_N(K)} \) to check the formula (6). We also compare with Hikami’s observation (7). Both conjectures (6) and (7) seem to be true for the examples given here. Moreover, the imaginary part of the coefficient of \( \frac{1}{N} \) in the asymptotic expansion of \( \frac{J_{N+1}(K)}{J_N(K)} \) seems to be 0 for these examples.

References

[5] Pari-GP (a software package for computer-aided number theory),
http://www.parigp-home.de/
[6] SnapPea (a program for creating and studying hyperbolic 3-manifolds),
http://humber.northnet.org/weeks/index/SnapPea.html