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PSL(2, Z) の有限型不変量について

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0. Preliminaries

PSL(2, Z) is the group of $2 \times 2$ matrices over $\mathbb{Z}$ with determinant 1 modulo $\pm E$. This group has the following generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfying the relations

$$S^2 = (TS)^3 = E.$$

Any element of $PSL(2, \mathbb{Z})$ can be presented as follows by using $S$ and $T$,

$$PSL(2, \mathbb{Z}) \ni T^{b_1}ST^{b_2}S \cdots T^{b_l}S.$$

From now on, we use the following sequence of integers to indicate the element.

$$[b_1, b_2, \cdots, b_l]$$

Then we get the following relations by using this symbol.

$$[b_1, b_2, \cdots, b_i, 0, b_{i+2}, \cdots, b_l] = [b_1, b_2, \cdots, b_i + b_{i+2}, \cdots, b_l]$$

$$[b_1, b_2, \cdots, b_i, 1, 1, 1, b_{i+4}, \cdots, b_l] = [b_1, b_2, \cdots, b_i, b_{i+4}, \cdots, b_l]$$

It is known that two symbols present the same element in $PSL(2, \mathbb{Z})$ if and only if they can be transformed to each other by finite sequence of the above relations.

1. The definition of the finite type invariant of $PSL(2, \mathbb{Z})$
Let $\overline{\Gamma}$ denote the free abelian group generated by all the elements in $\text{PSL}(2, \mathbb{Z})$ and $\overline{\Gamma}_n$ denote the group spanned by the following set

$$\left\{ \sum_{c_{i_{j}}=\pm 1} \text{the number of } (-1) \text{'s in } \{c_{i_{j}}\} \times [b_1, b_2, \cdots, b_l]_{c_{i_1}, c_{i_2}, \cdots, c_{i_l}} \right\},$$

where

$$[b_1, b_2, \cdots, b_{i_1}, \cdots, b_{i_l}]_{c_{i_1}, c_{i_2}, \cdots, c_{i_l}} = [b_1, b_2, \cdots, b_{i_1} - c_{i_1} + 1, \cdots, b_{i_l} - c_{i_l} + 1, \cdots, b_l].$$

Note that if $c_{i_j}$ is 1, then $b_{i_j}$ does not change and that if $c_{i_j}$ is -1, then $b_{i_j}$ is changed to $b_{i_j} + 2$.

Now we define the finite type invariant of $\text{PSL}(2, \mathbb{Z})$ as following.

**Definition.** An additive map from $\overline{\Gamma}/\overline{\Gamma}_{n+1}$ to $\mathbb{Q}$ is called an invariant of type $n$.

Let $\sim_n$ (we call this $n$-equivalence) denote the equivalence relation defined by $\overline{\Gamma}_{n+1}$ in $\overline{\Gamma}$.

**2. ON TYPE 0, 1 AND 2 INVARIANTS**

**Theorem 1.**

$$\overline{\Gamma}/\overline{\Gamma}_1 = \mathbb{Z}[[\ ],[0],[1],[0,1],[1,0],[1,1]].$$

Moreover, 0-equivalence class of \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) is determined by its congruence class modulo 2.

From now on, we restrict ourselves to the matrices 0-equivalent to the identity $E$ and consider finite type invariants. Let $\overline{\Gamma}(2)$ be the span over $\mathbb{Z}$ of matrices 0-equivalent to the identity. We know that any element of $\overline{\Gamma}(2)$ can be presented as a sequence of even integers with even length, subject to the following relation

$$[2a_1, 2a_2, \cdots, 2a_i, 0, 2a_{i+2}, \cdots, 2a_{2m}]$$

$$= [2a_1, 2a_2, \cdots, 2(a_i + a_{i+2}), \cdots, 2a_{2m}].$$

By similar calculation, we have the following
Theorem 2.

$$\bar{\Gamma}(2)/\bar{\Gamma}(2)_{2} = \mathbb{Z}\{[0, 0, 2], [2, 0]\}.$$ 

In fact, any element of $\bar{\Gamma}(2)$ is 1-equivalent to

$$(1 - A)[ ] + A_{0}[0, 2] + A_{1}[2, 0],$$

where

$$A = \sum_{i=1}^{2m} a_{i}, \quad A_{0} = \sum_{i=1}^{m} a_{2i}, \quad A_{1} = \sum_{i=1}^{m} a_{2i-1}.$$ 

Moreover, $1 - A$, $A_{0}$, $A_{1}$ are well-defined.

If $[2a_{1}, 2a_{2}, \cdots, 2a_{2m}] = \left(\begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array}\right)$, then

$$A_{0} = \sum_{i=1}^{\gamma/2} (-1)^{\left[(2i-1)\frac{\alpha}{\gamma}\right]}, \quad A_{1} = \sum_{i=1}^{\gamma/2} (-1)^{\left[(2i-1)\frac{\delta}{\gamma}\right]}.$$ 

Where $[\ ]$ denotes the greatest integer function.

To prove the formulas, we use Tuler's result of the linking number of a 2-bridge link ([2]).

Corollary 2.1. Any type 1 invariant is of the form

$$c_{1}(1 - A) + c_{2}A_{0} + c_{3}A_{1},$$

where $c_{i}$'s are constants.

Theorem 3.

$$\bar{\Gamma}(2)/\bar{\Gamma}(2)_{3} = \mathbb{Z}\{[0, 0, 2], [2, 0], [2, 2], [0, 2, 2, 0], [0, 4], [4, 0]\}.$$ 

In fact, any element of $\bar{\Gamma}(2)$ is 2-equivalent to

$$\frac{(A - 1)(A - 2)}{2}[ ] - A_{0}(A - 2)[0, 2] + A_{1}(A - 2)[2, 0]$$

$$+ \sum_{i=1}^{m} \sum_{j=i}^{m} a_{2i-1}a_{2j}[2, 2] + \sum_{i=1}^{m-1} \sum_{j=i}^{m-1} a_{2i}a_{2j+1}[0, 2, 2, 0].$$
\[
\frac{A_{0}(A_{0}-1)}{2} [0,4] + \frac{A_{1}(A_{1}-1)}{2} [4,0].
\]

If \([2a_1, 2a_2, \ldots, 2a_{2m}]=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\), then

\[
\sum_{i=1}^{m} \sum_{j=i}^{m} a_{2i-1}a_{2j} = \sum_{i=1}^{(\alpha-1)/2} \sum_{j=i}^{(\alpha-1)/2} (-1)^{[(2i-1)_{\alpha}]_{\delta}+[2j_{\alpha}]},
\]

\[
\sum_{i=1}^{m-1} \sum_{j=i}^{m-1} a_{2i}a_{2j+1} = \sum_{i=1}^{(\alpha-1)/2} \sum_{j=i}^{(\alpha-1)/2} (-1)^{[(2i-1)_{\alpha}]_{\delta}+[2j_{\alpha}]},
\]

To prove the formulas, we use the result of the Casson knot invariant of a 2-bridge knot ([1]).

**Corollary 3.1.** Any type 2 invariant is of the form

\[
d_1 \frac{(A-1)(A-2)}{2} + d_2 A_0(A-2) + d_3 A_1(A-2) + d_4 \sum_{i=1}^{(\alpha-1)/2} \sum_{j=i}^{(\alpha-1)/2} (-1)^{[(2i-1)_{\alpha}]_{\delta}+[2j_{\alpha}]},
\]

\[
+d_5 \sum_{i=1}^{(\alpha-1)/2} \sum_{j=i}^{(\alpha-1)/2} (-1)^{[(2i-1)_{\alpha}]_{\delta}+[2j_{\alpha}]},
\]

\[
+d_6 \frac{A_0(A_0-1)}{2} + d_7 \frac{A_1(A_1-1)}{2},
\]

where \(d_i\)'s are constants.

Detail will appear elsewhere.

**REFERENCES**
