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COMPUTATIONS OF SPACES OF SIEGEL MODULAR CUSP FORMS

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ABSTRACT. We survey the known dimensions of S_n^k, the space of Siegel modular forms of weight k and degree n. We mention a few new results for degrees 4, 5 and 6. We obtain our results by combining a Vanishing Theorem and a restriction technique. For a fixed n, k the Vanishing Theorem gives an explicit set of Fourier coefficients which determine S_n^k. The restriction of Siegel modular forms to elliptic modular forms reveals linear relations among these explicit Fourier coefficients. Sometimes we produce enough linear relations to determine $\dim S_n^k$. We discuss conjectures to the effect that $\dim S_n^k$ may always be computed by these means.

§1. Outline.

I. Vanishing Theorem giving upper bounds for $\dim S_n^k$.

II. Restriction to modular curves and examples computing $\dim S_n^k$.

III. Conjectures: will the method in part II always work.

All of the work in this talk is joint work with David Yuen.

§2. Notation.

$e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$
$\mathcal{P}_n(\mathbb{R}) = \{Y \in M_{n\times n}^{\mathrm{sym}}(\mathbb{R}) : Y > 0\}$, the positive definite real matrices.
$\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}) = \{Y \in M_{n\times n}^{\mathrm{sym}}(\mathbb{R}) : Y \geq 0\}$, the positive semi-definite real matrices.
$\mathcal{X}_n = \{T \in \mathcal{P}_n(\mathbb{Q}) : \forall x \in \mathbb{Z}^n, x'Tx \in \mathbb{Z}\}$, the positive semi-integral matrices.
$\mathcal{H}_n = \{\Omega \in M_{n\times n}^{\mathrm{sym}}(\mathbb{C}) : \Im\Omega > 0\}$, the Siegel upper half space of degree n.
$\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$, the modular group.
$\mathcal{F}_n$ = any fundamental domain for Γ_n acting on $\mathcal{H}_n$.
S_n^k = Siegel modular cusp forms of weight k and degree n.

For simplicity of exposition, assume level one and even weights throughout this talk. All the results extend to $S_n^k(\Gamma, \chi)$ for Γ of finite index, χ a character and $k \in \frac{1}{2}\mathbf{Z}^+$.

§3. I. Vanishing Theorem.

"It is a basic and important problem to know how many Fourier coefficients determine a modular form." H. Katsurada [5]

The following Theorem of Siegel [3] gives a finite set of Fourier coefficients that determine the form $f \in S_n^k$. Loosely, f must be zero if its vanishing order is too high.

Theorem (Siegel). *Let $f \in S_n^k$ have Fourier expansion $f(\Omega) = \sum_{T \in \mathcal{X}_n} a(T) e(\mathrm{tr}(T\Omega))$. If $a(T) = 0$ for all T such that $\mathrm{tr}(T) \le \kappa_n \frac{k}{4\pi}$ then we have $f = 0$.*

Here we define
$$\kappa_n = \sup_{\Omega \in \mathcal{F}_n} \mathrm{tr}\left((\Im \Omega)^{-1}\right).$$
The best known upper bound for κ_n is $\kappa_n \le n \frac{2}{\sqrt{3}} \mu_n^n$ where μ_n is Hermite's constant.

The partial order on $\mathcal{X}_n$ given by $A \ge B$ when $A - B$ is semidefinite is natural whereas all linear orders are artificial. How can the vanishing order of f be measured in an intrinsic way without relying on height functions like the trace? We can measure vanishing order by taking the semihull of the support of the Fourier series of f. This set turns out to be a kernel, which we will define. This concept can then be used to formulate an intrinsic vanishing theorem.

Definition. *Let $f \in S_n^k$ have Fourier expansion $f(\Omega) = \sum_{T \in \mathcal{X}_n} a(T) e(\mathrm{tr}(T\Omega))$. Define:*
$$\begin{aligned} \mathrm{supp}(f) &= \{T \in \mathcal{X}_n : a(T) \ne 0\} \subseteq \mathcal{P}_n(\mathbb{R}) \\ \nu(f) &= \mathrm{Closure}\{\mathrm{ConvexHull}\,(\mathbb{R}_{\ge 1}\,\mathrm{supp}(f))\} \subseteq \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}) \\ &= \mathrm{Semihull}(\mathrm{supp}(f)) \end{aligned}$$

Definition. *A* **kernel** *is a closed convex set $K \subseteq \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ satisfying:*

(1) $\mathbb{R}_{\ge 1} K = K$,
(2) $0 \notin K$,
(3) $\mathbb{R}_{>0} K \supseteq \mathcal{P}_n(\mathbb{R})$.

Proposition. *Let $f, g \in S_n$. We have $\nu(fg) = \nu(f) + \nu(g)$.*

Proof. Unpublished.

The operator ν thus behaves like a valuation. The $\nu(f)$ for $f \in S_n$ are all kernels and so the intrinsic vanishing of a Siegel modular cusp form may be measured by kernels. Complete proofs of the Kernel Lemma and the Semihull Theorem may be found in [7].

Kernel Lemma. *If $f \in S_n$ then $\nu(f)$ is a kernel.*

Proof. (sketch) The Koecher Principle tells us that $\nu(f) \subseteq \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$. The proof of the Kernel Lemma uses the same techniques as the proof of the Koecher Principle. The useful added information is item (3). □

The kernel $\nu(f)$ is related to the critical points of the invariant function $\det(Y)^{\frac{k}{2}} |f(\Omega)|$.

Semihull Theorem. *Let $f \in S_n^k$. Write $\Omega = X + iY \in \mathcal{H}_n$. If $\det(Y)^{\frac{k}{2}} |f(\Omega)|$ attains a maximum at $\Omega_0 = X_0 + iY_0$ then $\frac{k}{4\pi} Y_0^{-1} \in \nu(f)$.*

Proof. (sketch) For all $P \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Z})$, $q \in \mathbb{Z}^+$ such that $\inf(\mathrm{tr}(P\nu(f))) \ge q$ apply the maximum modulus principle on $\{\zeta : \Im \zeta \ge -\epsilon\}$, $\epsilon > 0$ to
$$\zeta \mapsto \frac{f(\Omega_0 + \zeta P)}{e(q\zeta)}. \quad \square$$

For every cusp form f we know that $\det(Y)^{\frac{k}{2}}|f(\Omega)|$ attains a maximum in $\mathcal{F}_n$. This makes an intrinsic Vanishing Theorem possible.

Vanishing Theorem (Intrinsic Version). *Let $f \in S_n^k$. If $\frac{k}{4\pi}\left(\Im\mathcal{F}_n\right)^{-1} \cap \nu(f) = \emptyset$ then we have $f = 0$.*

It must be confessed, however, that use of this Intrinsic Version in specific examples requires more information about $\mathcal{F}_n$ than is presently available. $\mathcal{F}_1$ is well known. Gottschling [2] has given a description of $\mathcal{F}_2$ but it is surprisingly complicated: $\mathcal{F}_2$ is bounded by 28 real algebraic hypersurfaces. Although present computations still rely on linear orders, the Intrinsic Version allows us great freedom in the choice of a linear order. Siegel used the trace, $\mathrm{tr}(T)$. Witt used the reduced determinant, $\det(T)^{1/n}$. Eichler used Hermite's function, $m(T)$. We use the dyadic trace, $w(T)$:

$$\text{For } T \in \mathcal{X}_n \text{ define: } w(T) = \inf_{Y \in \mathcal{P}_n(\mathbf{R})} \frac{\mathrm{tr}(YT)}{m(Y)}.$$

The following Theorem, along with techniques for calculating the dyadic trace, can also be found in [7].

Theorem. *Let $f \in S_n^k$ have the Fourier expansion $f(\Omega) = \sum_{T \in \mathcal{X}_n} a(T) e(\mathrm{tr}(T\Omega))$. If $a(T) = 0$ for all T such that $w(T) \leq \frac{2}{\sqrt{3}} n \frac{k}{4\pi}$ then we have $f = 0$.*

Table 1 illustrates for degree 4 how favorably the dyadic trace version compares with Siegel's trace version. Table 1 contains all even $k > 0$ for which $\dim S_4^k$ is presently known.

Table 1. (degree n=4)

k weight	# FC s (trace)	# FC s (dyadic trace)	$\dim S_n^k$ true dim.
2	0	0	0
4	17	0	0
6	334	1	0
8	3285	2	1
10	22635+	10	1 (new)
12	100000+	23	2
14	tables run out	85	3 (new)

Example. Let $J_8 \in S_4^8$ be Schottky's form. Igusa [4] proved the identity:

$$(2^{-2} \cdot 3^2 \cdot 5 \cdot 7)\, J_8 = \vartheta_{E_8 \oplus E_8} - \vartheta_{D_{16}^+}.$$

According to Table 1, we can prove this identity by the relatively easy task of verifying it for 2 Fourier coefficients. In summary, the Vanishing Theorem gives a D such that $\dim S_n^k \leq D$. We can decrease D by studying the restrictions of Siegel modular cusp forms to modular curves. This is the topic of part II.

§4. II. Restriction to Modular Curves.

Let $s \in \mathcal{P}_n(\mathbb{Z})$. Let $\ell \in \mathbb{Z}^+$ such that $\ell s^{-1} \in \mathcal{P}_n(\mathbb{Z})$. Define:

$$\begin{aligned} \phi_s : \mathcal{H}_1 &\to \mathcal{H}_n, \\ \tau &\mapsto s\tau \end{aligned} \qquad\qquad \begin{aligned} \phi_s^* : S_n^k &\longrightarrow S_1^{nk}\left(\Gamma_0(\ell)\right). \\ (\Omega \mapsto f(\Omega)) &\mapsto (\tau \mapsto f(s\tau)) \end{aligned}$$

Casually, if $f(\Omega)$ is a Siegel modular form then $f(s\tau)$ is an elliptic modular form. This is the "Eichler trick." It is usually seen in the context of theta series where ϕ_s^* sends the thetanullwerte of degree n to the theta series for s of degree 1.

The Fourier coefficients of $\phi_s^* f$ *at each cusp* can be worked out in terms of the Fourier coefficients of f. Let $q = e(\tau)$ for $\tau \in \mathcal{H}_1$.

$$(\phi_s^* f)(\tau) = f(s\tau) = \sum_{T \in \mathcal{X}_n} a(T) e(\operatorname{tr}(Ts\tau)) = \sum_{T \in \mathcal{X}_n} a(T) q^{\operatorname{tr}(Ts)} = \sum_{j=1}^{\infty} \left(\sum_{T:\operatorname{tr}(Ts)=j} a(T) \right) q^j.$$

It is essential to make use of similar expansions at the other cusps of $\Gamma_0(\ell)\backslash\mathcal{H}_1$, see [8] for details.

Since $\phi_s^* f$ is modular for $\Gamma_0(\ell)$, the Fourier coefficients of $\phi_s^* f$ for all cusps satisfy linear relations. These induce linear relations on the Fourier coefficients of f and this is the whole point of the method.

Example. $n = 4$; $\ell = 2$; $s = D_4 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$. Also let $H = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}$.
We compute the following expansion:

$$\begin{aligned} \left(\phi_{D_4}^* f\right)(\tau) =& a(D_4) q^4 + \left(16a(D_4) + 48a(A_4)\right) q^5 \\ &+ \left(144a(D_4) + 288a(A_4) + 216a(A_3 \oplus A_1) + 48a(A_2 \oplus A_2) + 12a(H)\right) q^6 + \dots \end{aligned} \tag{4.1}$$

The function $\phi_{D_4}^* f \in S_1^{4k}\left(\Gamma_0(2)\right)$ is invariant under the Fricke involution because D_4^{-1} is equivalent to $\frac{1}{2} D_4$, a helpful lemma. We need information about the ring $M_1\left(\Gamma_0(2)\right)$. In order to fix notation, define $E_{k,d}^{\pm}(\tau) = (E_k(\tau) \pm d^{\frac{k}{2}} E_k(d\tau))/(1 \pm d^{\frac{k}{2}})$ where the $E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$ are the Eisenstein series and the B_k are given by $t/(e^t - 1) = \sum_{k=0}^{\infty} B_k t^k / k!$. We have $E_{k,d}^{\pm} \in M_1^k(\Gamma_0(d))$ except in the case of $E_{2,d}^{+}$. The ring $M_1\left(\Gamma_0(2)\right)$ is generated by $E_{2,2}^{-} \in M_1^2\left(\Gamma_0(2)\right)$ and $E_{4,2}^{-} \in M_1^4\left(\Gamma_0(2)\right)$ and the ring of cusp forms is principally generated by $C_{8,2}^{+} \in S_1^8\left(\Gamma_0(2)\right)$. The $\pm$ superscript indicates an eigenvalue of

± 1 under the Fricke involution. The Fourier expansions of these generators are given by

$$E_{2,2}^{-}(\tau) = 1 + 24\sum_{n=1}^{\infty}(\sigma_1(n) - 2\sigma_1(n/2))\,q^n = 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + \dots$$

$$E_{4,2}^{-}(\tau) = 1 - 80\sum_{n=1}^{\infty}(\sigma_3(n) - 4\sigma_3(n/2))\,q^n = 1 - 80q - 400q^2 - 2240q^3 - 2960q^4 - \dots$$

$$C_{8,2}^{+}(z) = \frac{1}{256}\left(E_{2,2}^{-}(\tau)^4 - E_{4,2}^{-}(\tau)^2\right) = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 - \dots$$

The order of $\phi_{D_4}^{*} f$ at the cusp $[I]$ is at least 4 and the order at the cusp $[J]$ is the same because $\phi_{D_4}^{*} f$ is an eigenfunction of the Fricke involution. Thus we have $(C_{8,2}^{+})^4 | \phi_{D_4}^{*} f$ in $M_1(\Gamma_0(2))$ and we have

$$\phi_{D_4}^{*} f = (C_{8,2}^{+})^4\,(\text{Form of weight } 4k - 32)\,.$$

Let us use this fact, along with column 3 of Table 1, to explain the entries in column 4 of Table 1.

$k = 2, k = 4$. From column 3 of Table 1 we see that the Vanishing Theorem alone proves that $S_4^2 = \{0\}$ and that $S_4^4 = \{0\}$.

$k = 6$. S_4^6 is controlled by one Fourier coefficient, $a(D_4)$. We see that $\phi_{D_4}^{*} f = 0$ and so every coefficient in equation 4.1 gives a homogeneous linear relation; in particular we must have $a(D_4) = 0$ and hence we have $S_4^6 = \{0\}$.

$k = 8$. S_4^8 is controlled by two Fourier coefficients, $a(D_4)$ and $a(A_4)$. For $k = 8$ there is a parameter $c \in \mathbb{C}$ such that

$$\phi_{D_4}^{*} f = c(C_{8,2}^{+})^4 = c\left(q^4 - 32q^5 + 432q^6 - 2944q^7 + 7192q^8 + 39744q^9 - \dots\right).$$

Elimination of the parameter c provides a linear relation for any $f \in S_4^8$:

$$\begin{aligned} a(D_4) &= c, \\ 16a(D_4) + 48a(A_4) &= -32c, \\ \therefore\quad a(D_4) + a(A_4) &= 0. \end{aligned}$$

The relation $a(D_4) + a(A_4) = 0$ implies that $\dim S_4^8 \le 1$.

$k = 10$. S_4^{10} is controlled by 10 Fourier coefficients. For $k = 10$ there are parameters $\alpha, \beta \in \mathbb{C}$ such that $\phi_{D_4}^{*} f = (C_{8,2}^{+})^4\left(\alpha(E_{2,2}^{-})^4 + \beta C_{8,2}^{+}\right)$. The element $(E_{2,2}^{-})^2 E_{4,2}^{-}$ cannot occur in this representation because it has eigenvalue -1 under the Fricke operator. Elimination of the parameters α and β provides two linear relations. Recall the form H, the homomorphism $\phi_H^{*} : S_4^{10} \to S_1^{40}(\Gamma_0(6))$ gives 8 relations on these same ten Fourier coefficients. The span of the two sets of relations is 9 dimensional so that we have $\dim S_4^{10} \le 1$.

We can also show that $\dim S_4^{12} \leq 2$ and $\dim S_4^{14} \leq 3$ by using a lot of different homomorphisms. In every case we can construct this same number of linearly independent cusp forms so that all our upper bounds are in fact equalities. Table 2 gives the state of progress for determining $\dim S_n^k$. The full generating functions are known only for $n \leq 3$; the case $n = 2$ is due to Igusa, the case $n = 3$ to Tsuyumine.

Table 2. $\dim S_n^k$

$n \searrow$ $k \to$	2	4	6	8	10	12	14	16
1	0	0	0	0	0	1	0	1
2	0	0	0	0	1	1	1	2
3	0	0	0	0	0	1	1	3
4		0	0	1	1	2	3	
5		0	0	0	0	2		
6		0	0	0				
7		0	0					
8			0	0				
9			0					
10			0					
11			0					
12								

All the $\dim S_2^k$ were given by Igusa. Igusa also gave $\dim S_3^k$ for $k \leq 10$. Eichler showed that $\dim S_n^4 = 0$ for $n = 4$ and $n = 5$. The theory of singular forms developed by Freitag and Resnikoff showed that $S_n^k = \{0\}$ for $k \leq n/2$. The $S_n^k = \{0\}$ for $k \leq n/2$ have been left blank in Table 2 in order to make it easier to see the so-called *singular* zone. Böcherer showed that the theta-series are onto for $k > 2n$. In view of this result we term the entries for $k > 2n$ the *generic* zone. The middle zone, $n/2 < k \leq 2n$, may be termed the *sporadic* zone. All the $\dim S_3^k$ were given by Tsuyumine, see [9].

Poor-Yuen used divisor methods in [6] to compute $\dim S_4^k$ for $k = 6, 8, 12$. Duke-Imamoglu [1] used explicit formulae and L-functions to compute $\dim S_n^4$ for $4 \leq n \leq 7$ and $\dim S_n^6$ for $4 \leq n \leq 11$ and $\dim S_n^8$ for $n = 4, 8$. Nebe-Venkov computed $\dim S_5^{12}$. The method of this paper adds $\dim S_4^k$ for $k = 10, 14$ and $\dim S_5^k$ for $k = 8, 10$ and $\dim S_6^k$ for $k = 8$.

§5. III. Conjectures.

The previous sections have shown how to give progressively improved upper bounds for $\dim S_n^k$. In order to show equality, one constructs the correct number of linearly independent forms in S_n^k. So far this has not been a problem, at least whenever the upper bound turned out to be the correct dimension. One wonders how good this method for producing upper bounds actually is and whether it might stabilize above the actual dimension. We believe that the method described in this talk will always work. We wish to characterize the Fourier series of Siegel modular cusp forms from among all formal series. The conjectures that follow are an attempt to do this. We write a formal series as $\sum_{T\in\mathcal{X}_n} a(T)q_n^T$, the q_n indicates that the exponent is an $n\times n$ matrix. We define what it means to say that a formal series is of "Koecher Type."

Definition. *Let $n,k\in\mathbb{Z}^+$. A formal series $\sum_{T\in\mathcal{X}_n} a(T)q_n^T$ is of Koecher Type (n,k) when we have $a(v'Tv)=\det(v)^k a(T)$ for all $v\in \mathrm{GL}_n(\mathbb{Z})$.*

More generally, let a set $\mathcal{T}\subseteq\mathcal{X}_n$ be given. A formal series $\sum_{T\in\mathcal{T}} a(T)q_n^T$ is of Koecher Type (n,k) when it can be extended to a formal series $\sum_{T\in\mathcal{X}_n} a(T)q_n^T$ of Koecher Type (n,k).

Conjecture (Theory Version). *Given n, $k\in\mathbb{Z}^+$. Fourier series in S_n^k are characterized among all formal series of Koecher Type (n,k) by the linear relations on the $a(T)$, $T\in\mathcal{X}_n$, induced by the ϕ_s^* homomorphisms at all cusps for all $s\in\mathcal{P}_n(\mathbb{Z})$.*

A second conjecture is formulated with computer applications in mind. By general nonsense, these conjectures are equivalent.

Conjecture (Computer Version). *Given n, $k\in\mathbb{Z}^+$. Given a finite set $\mathcal{T}\subseteq\mathcal{X}_n$. There exists a finite set $\mathcal{S}\subseteq\mathcal{P}_n(\mathbb{Z})$ such that $\mathcal{T}$-partial sums of Fourier series in S_n^k are characterized among all formal series $\sum_{T\in\mathcal{T}} a(T)q_n^T$ of Koecher Type (n,k) by the linear relations on the $a(T)$, $T\in\mathcal{T}$, induced by the ϕ_s^* homomorphisms at all cusps for $s\in\mathcal{S}$.*

A better assertion would be that $\mathcal{S}$ is effectively computable from n, k and $\mathcal{T}$ but we suspect this is more difficult. You probably shouldn't believe either conjecture until you see the following Theorem. The proof of this Theorem is unpublished.

Theorem. *Given n, $k\in\mathbb{Z}^+$. Fourier series in S_n^k are characterized among all convergent series $\sum_{T\in\mathcal{X}_n} a(T)q_n^T$ by the linear relations on the $a(T)$, $T\in\mathcal{X}_n$, induced by the ϕ_s^* homomorphisms at all cusps for all $s\in\mathcal{P}_n(\mathbb{Z})$.*

A counterexample to the conjecture would be a strange creature indeed: a formal series whose coefficients have super-exponential growth such that every time the substitution $q_n^T := q^{\mathrm{tr}(sT)}$ is made for $s\in\mathcal{P}_n(\mathbb{Z})$ we obtain an elliptic modular form of level $\Gamma_0(\ell)$ for the minimal ℓ such that ℓs^{-1} is integral.

At this point we view the Conjecture as a regularity theorem. Recall the original regularity theorem: a weakly harmonic distribution is already C^∞ and hence harmonic. We wish to frame our Conjecture in an analogous manner. Call a formal series *modular* if it is the Fourier series of a Siegel modular form. Call any formal series of Koecher Type that gives an elliptic modular form of level $\Gamma_0(\ell)$ every time the substitution $q_n^T := q^{\mathrm{tr}(sT)}$ is made *weakly modular*. In the above spirit we may rephrase the Conjecture as: a weakly modular formal series is already convergent and hence modular.

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