<table>
<thead>
<tr>
<th>Title</th>
<th>Constructing Saturated Quasi-minimal Structures (Model theory via geometric approach)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Tsuboi, Akito</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2002 (1283), 33-41</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42408">http://hdl.handle.net/2433/42408</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
Constructing Saturated Quasi-minimal Structures

坪井明人 (Akito TSUBOI)
筑波大学数学科 (Institute of Mathematics, University of Tsukuba)

2002年3月18日-20日 数理解析研究所講演会

1 Properties with stability assumptions

An uncountable structure $M$ is said to be quasi-minimal, if there is no uncountable definable set $A \subset M \setminus A$ also uncountable. We study the properties of quasi-minimal structure $M$ with stability theoretic assumption on $\mathrm{Th}(M)$. First we show that the exchange axiom is true if $\mathrm{Th}(M)$ is stable. Secondly we show that the cardinality of $M$ must be $\aleph_1$ if $M$ is quasi-minimal, homogeneous and 1-unstable. Thirdly we construct saturated quasi-minimal models when $\mathrm{Th}(M)$ is $\omega$-stable.

1.1 Stability and the exchange property

Definition 1 1. Let $M$ be quasi-minimal. Then $p(x)$ defined by

$$ p(x) = \{ \varphi(x) \in L(M) : |\varphi^M| \geq \omega_1 \} $$

is a complete type in $S(M)$. The type $p(x)$ will be called the main type of $M$.

2. Let $A \subset M$. The $n$-th countable closure $\mathrm{ccl}_n(A)$ of $A$ is inductively defined as follows:

$$ \mathrm{ccl}_0(A) = A $$

$$ \mathrm{ccl}_{n+1}(A) = \bigcup \{ \varphi^M : \varphi(x) \in L(\mathrm{ccl}_n(A)), \ \varphi^M \text{ is countable} \}. $$

We put $\mathrm{ccl}(A) = \bigcup_{n \in \omega} \mathrm{ccl}_n(A)$ (the countable closure of $A$).

Remark 2 In 2.1 we have shown that if $M$ is homogeneous, $\mathrm{ccl}(A) = \mathrm{ccl}_1(A)$ (cf. Proposition 6).

The following lemma is easy.

Lemma 3 Let $M$ be a quasi-minimal structure of power $\kappa$. Let $\alpha < \kappa$. For each $i < \alpha$, let $\varphi_i(x)$ be a formula with uncountably many solutions. Then $\{ \varphi_i(x) : i < \alpha \}$ is realized in $M$.

Lemma 4 Let $M$ be a quasi-minimal structure with $\mathrm{Th}(M)$ stable. Then

1. For all $\varphi(x, \bar{y}) \in L$, there is a formula $\theta(\bar{y}) \in L(M)$ such that for all $\bar{b} \in M$,

$$ M \models \theta(\bar{b}) \iff M \text{ has uncountably many solutions of } \varphi(x, \bar{b}). $$
2. After naming countably many appropriate elements of $M$, we have $\operatorname{ccl}(A) = \operatorname{ccl}_1(A)$, for all $A \subset M$. Moreover, if $\operatorname{Th}(M)$ is $\omega$-stable, then the number of necessary elements is finite.

Proof: 1. An easy application of definability of types. Let $p(x)$ be the main type of $M$. Let $\theta(\bar{y})$ be the defining formula of $\varphi(x, \bar{y})$ in $p(x)$. Then we have

$$M \models \theta(\bar{b}) \iff \varphi(x, \bar{b}) \in p(x) \iff |\varphi(x, \bar{b})|^M \geq \omega_1.$$ 

2. By adding countably many constants to $L$, we can assume that $\theta(x, \bar{y})$ obtained in part 1 is an $L$-formula. (If $\operatorname{Th}(M)$ is $\omega$-stable, the type $p(x)$ defined in part 1 has a finite base $D \subset M$. So $\theta$ is an $L(D)$-formula.) It is sufficient to show that $\operatorname{ccl}_2(A) \subset \operatorname{ccl}_1(A)$. Let $a \in \operatorname{ccl}_2(A)$. Choose $b_1, \ldots, b_m \in \operatorname{ccl}_1(A)$ with $a \in \operatorname{ccl}_1(b_1, \ldots, b_m)$. Then choose formulas $\varphi(x, y_1, \ldots, y_m) \in \text{tp}(a, b_1, \ldots, b_m)$ and $\psi_i(y_i) \in \text{tp}(b_i/A) (i = 1, \ldots, m)$ such that

- each $\psi(y_i)$ has only countably many solutions, and
- $\varphi(x, b_1, \ldots, b_m)$ has only countably many solutions.

By part 1, we can assume that $\varphi(x, b'_1, \ldots, b'_m)^M$ is countable whenever $b'_1, \ldots, b'_m \in M$. So if we put $\varphi^*(x) = \exists y_1, \ldots, y_m[\varphi(x, y_1, \ldots, y_m) \land \psi_1(y_1) \land \cdots \land \psi_m(y_m)]$, then $\varphi^*(x) \in \text{tp}(a/A)$ has only countably many solutions.

Example 5 Let $E$ be an equivalence relation on an uncountable set $M_0$ with exactly two equivalence classes such that one class, say $A$, is countable and the other class, say $B$, is uncountable. Let $M$ be the structure $(M_0 \cup \{A, B\}, E)$, where $A$ and $B$ are treated as eq-elements. Clearly $A \in \operatorname{acl}(\emptyset)$. Moreover, the countable closure of the point $A \in M$ includes $A$ as a subset of $M$. So $A \subset \operatorname{ccl}(\emptyset)$. However, $A$ is not included in $\operatorname{ccl}_1(\emptyset)$.

Proposition 6 Let $M$ be quasi-minimal. If $\operatorname{Th}(M)$ is stable, then after naming countably many elements, the countable closure satisfies the exchange axiom.

Proof: By way of a contradiction, we assume that there are $a, b \in M$ with $a \in \operatorname{ccl}(Ab) - \operatorname{ccl}(A)$ and $b \notin \operatorname{ccl}(Aa)$. We may assume $A = \emptyset$. Two elements $a$ and $b$ have the same type over $\emptyset$. Using Lemma 4 choose formulas $\varphi(x, y) \in \text{tp}(a, b)$ and $\theta(x) \in \text{tp}(a)$ such that

1. $\varphi(x, b')$ has only countably many solutions, for every $b' \in M$, and
2. $\varphi(a', y)$ has uncountably many solutions, whenever $\theta(a')$ holds.

Using induction on $i < \omega_1$, we shall construct a sequence $\{a_i\}_{i < \omega_1} \subset \theta^M$ such that for all $i < j < \omega_1$,

$$M \models \varphi(a_i, a_j) \land \neg \varphi(a_j, a_i).$$

Let $a_0 = a$ and suppose that we have found $a_j$'s for $j < i$. Let us consider the following set:

$$\Gamma(x) = \{\theta(x)\} \cup \{\varphi(a_j, x) \land \neg \varphi(x, a_j) : j < i\}.$$ 

By properties 1 and 2 above, $\Gamma(x)$ consists of formulas with uncountably many solutions, so by Lemma 3, it has a solution $a_i \in M$. This $a_i$ satisfies the required condition. Then the sequence $\{a_i\}_{i < \omega_1}$ is totally ordered by the formula $\varphi(x, y)$, hence $\operatorname{Th}(M)$ must be unstable. A contradiction.
Remark 7 Mention Maesono’s result: homogeneous + without strict order property implies the exchange property of ccl??

1.2 1-unstability and the exchange property

Definition 8 We will say that a structure $M$ is 1-unstable if there is an $L(M)$-formula $\varphi(x, y)$ and an uncountable set $I \subset M$ such that $\{(a, b) \in I^2 : M \models \varphi(a, b)\}$ is a total order on $I$.

If $M$ is 1-unstable, then $Th(M)$ is unstable. But the converse does not hold in general.

Lemma 9 Let $M$ be homogeneous and quasiminimal. Suppose that $a_1, a_2 \in M$ have the same type over $A \subset M$. Then for any $b_1 \in M - ccl(A, a_1)$ and $b_2 \in M - ccl(A, b)$, we have $tp(a_1b_1 / A) = tp(a_2b_2 / A)$.

Proof: Suppose otherwise. We can choose a formula $\varphi(x, y) \in L(A)$ such that

$$ M \models \varphi(a_1, b_1) \land \neg \varphi(a_2, b_2) $$

By homogeneity of $M$, there is an automorphism $\sigma$ with $\sigma(a_2) = a_1$ such that $\sigma$ fixes the parameters $A_0$ of $\varphi$. So we have

$$ M \models \varphi(a_1, b_1) \land \neg \varphi(a_1, \sigma(b_2)) $$

Since $M$ is quasi-minimal (and $b_1 \notin ccl(Aa_1)$), $\neg \varphi(a_1, x)$ has only countably many solutions. So $\sigma(b_2) \in ccl(A_0a_1)$. Hence $b_2 \in ccl(A_0, a_2)$, a contradiction.

Proposition 10 Let $M$ be a homogeneous quasi-minimal structure. If the countable closure does not satisfy the exchange axiom, then $M$ is 1-unstable.

Proof: We assume that there are $a, b \in M$ with $a \in ccl( Ab) - ccl(A)$ and $b \notin ccl( Aa)$. We may assume $A = \emptyset$. Then $a$ and $b$ have the same type over $\emptyset$, say $p$. From Lemma 9, we know that if both $c$ and $d$ realize $p$ and $d \notin ccl(c)$, then $tp(cd) = tp(ab)$. By induction on $i < |M|$ we can easily find realizations $a_i \in M$ of $p$ such that $a_i \notin ccl(\{a_j : j < i\})$. Then $I = \{a_i : i < |M|\}$ forms a 2-indiscernible sequence. Choose a formula $\varphi(x, b) \in tp(a/b)$ with only countably many solutions in $M$. Then we have

- $\varphi(a_i, a_j)$ for all $i < j < |M|$, and
- $\neg \varphi(a_i, a_j)$ for all $j < i < |M|$.

Hence $M$ is 1-unstable.

Proposition 11 Let $M$ be a homogeneous quasi-minimal structure. If $M$ is 1-unstable, then $|M| \leq \omega_1$.

Proof: By way of a contradiction, assume that $M$ is 1-unstable and $|M| \geq \omega_2$. Let $\varphi(x, y) \in L(M)$ and $I$ witness the 1-instability of $M$. Namely, $I$ is an uncountable sequence totally ordered by $\varphi(x, y)$. We assume that $I$ is maximal among such sequences. Let us write $a < b$ if $\varphi(a, b) \land \neg \varphi(b, a)$ holds.

Define two sets:
• $I_\neg = \{a \in I : |\{b \in I : b < a\}| \leq \omega\}$;
• $I_\neg\neg = \{a \in I : |\{b \in I : a < b\}| \leq \omega\}$.

By the quasi-minimality, the pair $(I_\neg, I_\neg\neg)$ clearly defines a Dedekind cut of $I$.

Claim A Both $I_\neg$ and $I_\neg\neg$ are uncountable.

Suppose otherwise. We may assume that $I_\neg\neg$ is countable. For any $a \in I_\neg$, $x < a$ has only countably many solutions in $I$, and $a < x$ has uncountably many solutions in $I_\neg$ (and hence in $M$). Then an easy argument shows that $I_\neg$ can be written as a union of $\omega_1$-many countable sets. So we have $|I_\neg| = \omega_1$. Then by lemma 3 there is an element in $M$ realizing \{a < x : a \in I_\neg\} \cup \{x < b : b \in I_\neg\neg\}. This contradicts the maximality of $I$.

By shrinking $I$, we assume both $I_\neg$ and $I_\neg\neg$ have cardinality $\omega_1$. By lemma 3, we have a realization $a^* \in M$ of

$\Gamma(x) = \{a < x < b : a \in I_\neg, b \in I_\neg\neg\}$.

Then $x < a^*$ and $\neg(x < a^*)$ divide $I$ into two uncountable sets, contradicting the quasi-minimality of $M$.

Remark 12 The rationals $Q$ is definable in the structure $(C, +, \cdot, \exp, 0, 1)$ by

$\varphi(x) \overset{\text{def}}{=} \exists \alpha \exists \beta(\exp(\alpha) = \exp(\beta) = 1 \land \alpha \neq 0 \land x = \beta/\alpha)$.

It is also easy to see that both $Z$ and $N$ are definable in $(C, +, \cdot, \exp, 0, 1)$. It follows that the theory of $(C, +, \cdot, \exp, 0, 1)$ is unstable. It is interesting to see that there seems no uncountable total order definable in $(C, +, \cdot, \exp, 0, 1)$.

1.3 Strong independence property and quasi-minimality.

In 2.2 we showed that there are no quasi-minimal random graphs. Here we discuss the reason of non-existence as a consequence of the strong independency.

Definition 13 $T$ is said to have the strong independence property if there are formulas $R(x, y)$ and $D(x)$ such that for all distinct elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in D$ there is $c$ with $R(a_i, c)$ and $\neg R(b_i, c)$ ($i = 1, \ldots, n$).

Proposition 14 Suppose that $T$ has the strong independence property. Let $R$ and $D$ witness the property. Then $T$ does not have a quasi-minimal model with $D$ uncountable.

Proof: By way of a contradiction, assume that $M$ is a quasi-minimal structure with $D$ uncountable. Let $A \subset M$ be a countable set with $\text{ccl}(A) = A$. By taking ccl and Skolem hull repeatedly, we may assume that $A$ is a model. Let $a \in D \cap A$. (Since $A$ is a model, $D \cap A$ is non-empty.) Notice that

(*) any two elements from $M - A$ have the same type over $A$.

So we may assume that
\textbf{Corollary 15} There are no quasi-minimal random graphs.

1.4 Construction of saturated quasi-minimal structures.

In model theory it is often very convenient to work in saturated models. As we noted in the introduction, however, the technique of adding realizations of types to the original structure in order to construct a saturated model may not work in the study of quasi-minimal structures. Thus the question of the existence of saturated models attracts some attention. Consider the following question.

\textbf{Question.} Suppose that $M$ is \omega-stable quasi-minimal model. Is there a quasi-minimal $\aleph_0$-saturated model of Th(M)?

Before giving a positive answer to the above question it is worth mentioning that there is a counterexample for superstable theories: Let $M_0 = (2^{\omega}, E_i(i < \omega))$ such that $E_i(z, y) \iff x(i) = y(i)$ for $x, y \in 2^{\omega}$. Let $M_1 < M_0$ be a countable model of Th($M_0$) and fix $a \in M_1$. Let $M_2 = (M_1 \cup B, E_i(i < \omega))$ where $|B| > \aleph_0$ and $tp(b) = tp(a)$ for all $b \in B$. Then $M_2$ is quasi-minimal. But any $\aleph_0$-saturated model of Th($M_2$) includes $M_0$. Hence it is not quasi-minimal. Note that $M_2$ is not homogeneous.

We now explain how to construct a saturated quasi-minimal structures. From now on in this subsection $M$ denotes an $\omega$-stable quasi-minimal structure. Let $p(x) \in S(M)$ always denote the main type of $M$. Then each $\varphi(x) \in p(x)$ has uncountably many solutions in $M$. We may assume that $p(x)$ is strongly based on $\emptyset$. (i.e., $p(x)$ is the unique nonforking extension of $p|\emptyset$ over $M$.) The nonforking extension of $p$ to the domain $A$ is denoted by $p|A$. The prime model over $A$ is unique up to isomorphism over $A$. The prime model over the set $NA$ is denoted by $N(A)$, where $N$ is a model. $N(A)(B)$ is an abbreviation of $(N(A))(B)$.

We work in a big saturated elementary extension $\mathcal{M}$ of $M$. First we show:

\textbf{Lemma 16} There is a countable model $M_0 < M$ and an uncountable Morley sequence $I = \{a_i\}_{i < \omega_1}$ of $p|M_0$ such that $I$ dominates $M$ over $M_0$.

\textbf{Proof:} Let $N < M$ be any countable model. First we choose an uncountable Morley sequence $I \subseteq M$ of $p|N$, using induction. Suppose that we have chosen $a_i$'s for $j < i < \omega_1$. Let $A_i = N \cup \{a_j\}_{j<i}$. Note that each formula $\varphi(x)$ in $p|A_i$ has uncountably many solutions. Since $p|A_i$ is a countable set, by lemma 3, $p|A_i$ has a solution $a_i$ in $M$.

We assume that $I$ chosen above is maximal among such. Let $M_0 < M$ be a maximal model such that

\begin{align*}
N &< M_0 \
M_0 &\nleq N I.
\end{align*}
I is clearly a Morley sequence of $p|M_0$.

**Claim A** $M_0$ is countable.

Otherwise, there is a type $q(x) \in S(N)$ with $q|M_0$ uncountable. By the maximality of $I$, $q \neq p|N$. This contradicts the quasi-minimality of $M$.

**Claim B** $I$ dominates $M$ over $M_0$.

Extend $M_0 I$ to a maximal set $M' \subset M$ such that $M'$ is dominated by $I$ over $M_0$. Clearly $M'$ is an elementary submodel of $M$. Suppose that $\text{tp}(M/M')$ is not orthogonal to $M_0$. Then, by the three model theorem for an $\omega$-stable theory, we have an element $b \in M - M'$ such that $\text{tp}(b/M')$ does not fork over $M_0$. Then we have

$$M_0(b) \downarrow I.$$ 

Since $I$ and $M_0$ are independent over $N$, we must have $M_0(b) \downarrow_N I$, contradicting the maximality of $M_0$. Thus $\text{tp}(M/M')$ is orthogonal to $M_0$. Hence, by the maximality of $M'$, we have $M = M'$.

**Remark 17** In the above lemma, $M_0$ can be chosen arbitrarily large: For any finite subset $A \subset M$, we can choose $M_0$ above so that $M_0$ contains $A$.

$M_0$ having the properties in the above lemma will be called a base model of $M$. In what follows, $M_0$ will be used to denote a base model of a given quasi-minimal model.

**Lemma 18** Let $M_0$ be a base model of $M$. Let $r(x) \in S(M_0)$ be a type with $r \neq p|M_0$. Then we have:

1. $p$ is orthogonal to $r$;
2. $r|M$ is the unique extension of $r$ to the domain $M$;
3. Any element $d \in M - M_0$ realizes $p|M_0$.

**Proof:** Statements 2 and 3 follow from 1 and lemma 16. We prove 1. Suppose otherwise. Then there is a consistent formula $\varphi(x, a_0) \in L(M_0a_0)$ with the following properties:

- $\varphi(x, a_0)$ forks over $M_0$, and
- Any solution of $\varphi(x, a_0)$ does not realize $p|M_0$;

Choose the maximum $n < \omega$ such that $\{\varphi(x, a_j) : i \leq j < \omega_1\}$ is $n$-consistent (i.e., any subset of cardinality $n$ is consistent). For each $j < \omega_1$, choose $b_j \in M$ satisfying $\bigwedge_{nj \leq k < n(j+1)} \varphi(x, a_k)$. Then the $b_j$'s are distinct elements not realizing $p|M_0$, by the above two properties. This contradicts the quasi-minimality of $M$. 

1
Lemma 19 Let $a \in \mathcal{M}$ be any element with $\text{tp}(a/M) \neq p(x)$. Then $M(a)$ is quasi-minimal.

Proof: Choose a base model $M_0$ such that

- $a$ and $M$ are independent over $M_0$;
- $\text{tp}(a/M_0) \neq p|M_0$.

Then choose a Morley sequence $I$ of $p|M_0$ such that $I$ dominates $M$ over $M_0$. We prove the present lemma by a series of claims. Notice that $M(a)$ is prime and atomic model over $M_0(a)M$.

**Claim A** Any element $b$ from $M(a) - M_0(a)$ realizes the type $p|M_0$. A more precise statement is the following: If $\text{tp}(b/Ma)$ is isolated but $\text{tp}(b/M_0a)$ is not isolated, then $b$ realizes $p|M_0$.

By way of a contradiction, we assume that some element $b \in M(a) - M_0(a)$ does not realize $p|M_0$. Notice that $\text{tp}(b/M_0(a)M)$ is an isolated type. Then we have

$$ab \not\models \varphi \overline{d} \text{M}_0$$

since otherwise the open mapping theorem would imply that $\text{tp}(b/M_0(a))$ is an isolated type. Since $I$ dominates $M$ over $M_0$, we have $ab \not\models \varphi \overline{d} I$. Choose $\varphi(xy, \overline{d}) \in \text{tp}(ab/M_0I)$ ($\overline{d}$ is from $I$) such that

1. $\varphi(xy, \overline{d})$ forks over $M_0$ and
2. $\varphi(xy, \overline{d})$ implies $\neg \theta(y)$,

where $\theta(x)$ is a formula with $\theta(x) \in p|M_0$. Since $\text{tp}(a/M_0I)$ does not fork over $M_0$, $(\exists y)\varphi(x, y, \overline{d})$ has a solution $a' \in M_0$. For notational simplicity, we assume $\overline{d}$ is a single element, say $a_0$. Then by condition 1,

$$\Gamma(y) = \{\varphi(a', y, a_j) : j < \omega_1\}$$

must be inconsistent, since $I$ is a Morley sequence. By the indiscernibility of $I$ over $M_0$, $\Gamma(y)$ is $n$-inconsistent, for some $n \in \omega$. So by exactly the same argument as in lemma 18, we have uncountably many solutions of $\neg \theta(x)$ in $M$. This contradicts the quasi-minimality of $M$.

**Claim B** $M(a)$ is quasi-minimal.

Suppose otherwise and choose a formula $\varphi(x) \in L(M(a))$ witnessing the non-quasi-minimality of $M(a)$. We may assume that the parameters of $\varphi$ are in $M_0(a)$. (Otherwise extend $M_0$ a little bit more.) Since $M_0(a)$ is countable, we can choose $b, c \in M(a) - M_0(a)$ such that $M \models \varphi(b) \land \neg \varphi(c)$. By claim A, both $b$ and $c$ realize $p|M_0$. Since $\text{tp}(a/M_0)$ and $p|M_0$ are almost orthogonal, both $\text{tp}(b/M_0(a))$ and $\text{tp}(c/M_0(a))$ are non-forking extensions of a stationary type $p|M_0$. So we have $\text{tp}(b/M_0(a)) = \text{tp}(c/M_0(a))$, and hence we must have $M \models \varphi(b) \iff \varphi(c)$, a contradiction.
Remark 20  1. $p(M(a))$ is the main type of $M(a)$: It is sufficient to show that if 
$\varphi(x, \bar{d}) \in L(M(a))$ has uncountably many solutions in $M(a)$, then $\varphi(x, \bar{d})$ does not 
fork over $M$. Choose a countable model $M_0 < M$ such that $d \in M_0(a)$. We can 
assume that $M_0$ is a base model of $M$ and that $tp(a/M_0) \neq p|M$. Since $M_0(a)$ is 
a countable set, we can choose $b \in M(a) - M_0(a)$ satisfying $\varphi(x, \bar{d})$. By claim A, 
tp(b/M_0) = p|M_0. By the almost orthogonality of $tp(a/M_0)$ and $p|M_0$, we have 
b \downarrow_{M_0} M_0(a). Hence $\varphi(x, \bar{d})$ does not fork over $M_0 \subset M$.

2. Let $\{d_i : i < \omega\}$ be a countable sequence of elements with $tp(d_i/M) \neq p(x)$. Then 
$M(\{d_i\}_{i<\omega})$ is quasi-minimal. This can be shown by the iterated use of lemma 16.

Theorem 21 Let $M$ be a quasi-minimals model of an $\omega$-stable theory. Then there is an 
extension $M^* \succ M$ such that

1. $M^*$ is still quasi-minimal, and
2. $M^*$ realizes all types $q(x)$ with $\text{dom}(q) \subset_{\text{finite}} M$ and $q \neq p|\text{dom}(q)$.

Proof: For notational simplicity, we assume $|M| = \omega_1$. Let $\{q_i(x)\}_{i<\alpha}$ be a maximal 
set of regular types with the following properties: For $i < j < \alpha$,

- $q_i(x)$ is orthogonal to $p(x)$;
- $q_i(x)$ and $q_j(x)$ are orthogonal;
- $\text{dom}(q_i)$ is a finite subset of $M$.

For each $i < \alpha$, choose a countable Morley sequence $J_i$ of $q_i|M$. For each subset $X$ of $\omega_1$, 
let $J_X$ denote the set $\bigcup_{i \in X} J_i$. We put $M^* = M(\{J_i\}_{i<\alpha})$. By decomposing each type 
to regular types, we can easily show that $M^*$ realizes any type over a finite subset of $M$. It remains to show that $M^*$ is quasi-minimal. If $\alpha = \omega$, then $M^*$ is a quasiminimal 
extension of $M$, by lemma 19 and remark after it. So we can assume that $\alpha = \omega_1$. Now 
we can forget the maximality of $\{q_i(x)\}_{i<\alpha}$. Only assumption we need is that they are 
orthogonal.

By way of a contradiction, we now assume that $M^*$ is not quasi-minimal. Choose a 
type $q(x) \in S(A)(q \neq p|A)$ having uncountably many realizations in $M^*$. The set $A$ can be 
assumed to be a finite subset of $M^*$. So we can choose a finite subset $F \subset \omega_1$ with 
$A \subset M(J_F)$. Then $M^*$ can be decomposed in the form 

$$M^* = \bigcup_X M(J_X),$$

where $X$ ranges over all finite subsets of $\omega_1$ with $X \supset F$. By lemma 19, each $M(J_X)$ is 
quasi-minimal. So each $M(J_X)$ has only countably many realizations of $q(x)$. Hence there are 
uncountably many distinct finite subsets $X_i (i < \omega_1)$ of $\omega_1$ such that each $M(J_{X_i})$ has 
a realization, say $d_i$, of $q(x)$.

By the $\Delta$-system lemma (cf: p. 49 [4]), taking a subsequence of $\{X_i : i < \omega_1\}$, we may 
assume that there is $Y \subset \omega_1$ such that for any $i \neq i'$, 

$$X_i \cap X_{i'} = Y.$$
Now for each $i < \omega_1$ we have

$$d_i \in M(J_Y)(J_{X_i-Y}) - M(J_Y). \quad (1)$$

Recall that $M(J_Y)$ includes $A$. Let $M_0 \supset A$ be a base model of $M(J_Y)$. Notice that $q(x) \in S(A)$ has only countably many extensions in $S(M_0)$. So, by replacing $q$ by its suitable extension, and by taking subsequence of the $d_i$'s, we can assume that $q \in S(M_0)$ and that all $d_i$'s realize $q(x)$. Moreover, by lemma 18, $q((M(J_Y))$ is the unique extension of $q$ to the domain $M(J_Y)$. So each $d_i$ realizes $q((M(J_Y))$. In other words,

$$\text{tp}(d_i/M(J_Y)) = q((M(J_Y)). \quad (2)$$

By the $\omega$-stability, discarding countably many $J_i$'s, we can assume that each $J_i$ is a Morley sequence of $q|M(A_{J_Y})$. This together with (1) and (2) shows that for all $i < \omega_1$, we can find a type $r_i \in \{q_j : j \in X_i - Y\}$ which is nonorthogonal to $q((M(J_Y))$. But $r_i$'s are orthogonal. This contradicts our assumption that $T$ is $\omega$-stable.

**Corollary 22** Any quasi-minimal model of an $\omega$-stable theory can be elementarily extended to an $\omega$-saturated quasi-minimal model.

**Proof**: Using theorem 21, we can construct a chain $N_i$ ($i < \omega$) of quasi-minimal models such that

1. $M = N_0 \prec N_1 \prec \cdots \prec N_i \prec N_{i+1} \prec \cdots$

2. $N_{i+1}$ realizes any type $q$ with $\text{dom}(q) \subseteq \text{finite} N_i$ and $q \neq p|\text{dom}(q)$.

We show that $M^* = \bigcup_{i \in \omega} N_i$ has the required properties. Since $M^*$ is a union of quasi-minimal models, it is clearly quasi-minimal. So it remains to show that any type $q(x) \in S(A)$ with $A \subseteq \text{finite} M^*$ is realized in $M^*$. Choose $M_n$ with $A \subseteq M_n$. If $q(x) \neq p|M$, then it has a realization in $M_n$ by the property 1. So we can assume that $q(x) = p|M$. Let $I \subseteq M$ be a Morley sequence of $p|M_0$ such that $I$ dominates $M$ over $M_0$, where $M_0$ is a base model of $M$ (lemma 16). Since $I$ is (uncountable) infinite, there is $a \in I$ with $a \downarrow_{M_0} A$. Then $a$ realizes $p|M_0 A$. Hence $a$ realizes $q(x)$.

Akito Tsuboi
Institute of Mathematics,
University of Tsukuba,
Tsukuba, Ibaraki 305-8571
E-mail : tsuboi@math.tsukuba.ac.jp

**References.**