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Constructing Saturated Quasi-minimal Structures

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1 Properties with stability assumptions

An uncountable structure M is said to be quasi-minimal, if there is no uncountable definable set $A \subset$ with $M \setminus A$ also uncoutable. We study the properties of quasi-minimal structur M with stability theoretic assumption on $\mathrm{Th}(M)$. First we show that the exchange axiom is true if $\mathrm{Th}(M)$ is stable. Secondly we show that the cardinality of M must be $\aleph_1$ if M is quasi-minimal, homogeneous and *1-unstable*. Thirdly we construct saturated quasi-minimal models when $\mathrm{Th}(M)$ is ω-stable.

1.1 Stability and the exchange property

Definition 1 1. Let M be quasi-minimal. Then $p(x)$ defined by

$$p(x) = \{\psi(x) \in L(M) : |\psi^M| \geq \omega_1\}$$

is a complete type in $\mathrm{S}(M)$.The type $p(x)$ will be called the main type of M.

2. Let $A \subset M$. The n-th countable closure $\mathrm{ccl}_n(A)$ of A is inductively defined as follows: $\mathrm{ccl}_0(A) = A$ and

$$\mathrm{ccl}_{n+1}(A) = \bigcup \{\varphi^M : \varphi(x) \in L(\mathrm{ccl}_n(A)),\ \varphi^M \text{ is countable}\}.$$

We put $\mathrm{ccl}(A) = \bigcup_{n \in \omega} \mathrm{ccl}_n(A)$ (the countable closure of A).

Remark 2 In 2.1 we have shown that if M is homogeneous, $\mathrm{ccl}(A) = \mathrm{ccl}_1(A)$ (cf. Proposition 6).

The following lemma is easy.

Lemma 3 *Let M be a quasi-minimal structure of power κ. Let $\alpha < \kappa$. For each $i < \alpha$, let $\varphi_i(x)$ be a formula with uncountably many solutions. Then $\{\varphi_i(x) : i < \alpha\}$ is realized in M.*

Lemma 4 *Let M be a quasi-minimal structure with $Th(M)$ stable. Then*

1. *For all $\varphi(x, \bar{y}) \in L$, there is a formula $\theta(\bar{y}) \in L(M)$ such that for all $\bar{b} \in M$,*

$$M \models \theta(\bar{b}) \iff M \text{ has uncountably many solutions of } \varphi(x, \bar{b}).$$

2. *After naming countably many appropriate elements of M, we have* $\mathrm{ccl}(A) = \mathrm{ccl}_1(A)$, *for all* $A \subset M$. *Moreover, if* $Th(M)$ *is* ω*-stable, then the number of necessary elements is finite.*

Proof: 1. An easy application of definability of types. Let $p(x)$ be the main type of M. Let $\theta(\bar{y})$ be the defining formula of $\varphi(x,\bar{y})$ in $p(x)$. Then we have

$$\begin{aligned} M \models \theta(\bar{b}) &\iff \varphi(x,\bar{b}) \in p(x) \\ &\iff |\varphi(x,\bar{b})^M| \geq \omega_1. \end{aligned}$$

2. By adding countably many contants to L, we can assume that $\theta(x,\bar{y})$ obtained in part 1 is an L-formula. (If $Th(M)$ is ω-stable, the type $p(x)$ defined in part 1 has a finite base $D \subset M$. So θ is an $L(D)$-formula.) It is sufficent to show that $\mathrm{ccl}_2(A) \subset \mathrm{ccl}_1(A)$. Let $a \in \mathrm{ccl}_2(A)$. Choose $b_1, ..., b_m \in \mathrm{ccl}_1(A)$ with $a \in \mathrm{ccl}_1(b_1, ..., b_m)$. Then choose formulas $\varphi(x, y_1, ..., y_m) \in \mathrm{tp}(a, b_1, ..., b_m)$ and $\psi_i(y_i) \in \mathrm{tp}(b_i/A)$ $(i = 1, ..., m)$ such that

- each $\psi(y_i)$ has only countably many solutions, and
- $\varphi(x, b_1, ..., b_m)$ has only countably many solutions.

By part 1, we can assume that $\varphi(x, b'_1, ..., b'_m)^M$ is countable whenever $b'_1, ..., b'_m \in M$. So if we put $\varphi^*(x) = \exists y_1, ..., y_m[\varphi(x, y_1, ..., y_m) \wedge \psi_1(y_1) \wedge \cdots \wedge \psi_m(y_m)]$, then $\varphi^*(x) \in \mathrm{tp}(a/A)$ has only coutably many solutions.

∎

Example 5 Let E be an equivalence relation on an uncountable set M_0 with exactly two equivalence classes such that one class, say A, is countable and the other class, say B, is uncountable. Let M be the structure $(M_0 \cup \{A, B\}, E)$, where A and B are treated as eq-elements. Clearly $A \in \mathrm{acl}(\emptyset)$. Moreover, the countable closure of the point $A \in M$ includes A as a subset of M. So $A \subset \mathrm{ccl}(\emptyset)$. However, A is not included in $\mathrm{ccl}_1(\emptyset)$.

Proposition 6 *Let M be quasi-minimal. If $Th(M)$ is stable, then after naming countably many elements, the countable closure satisfies the exchange axiom.*

Proof: By way of a contradiction, we assume that there are $a, b \in M$ with $a \in \mathrm{ccl}(Ab) - \mathrm{ccl}(A)$ and $b \notin \mathrm{ccl}(Aa)$. We may assume $A = \emptyset$. Two elements a and b have the same type over $\emptyset$. Using Lemma 4 choose formulas $\varphi(x, y) \in \mathrm{tp}(a, b)$ and $\theta(x) \in \mathrm{tp}(a)$ such that

1. $\varphi(x, b')$ has only countably many solutions, for every $b' \in M$, and
2. $\varphi(a', y)$ has uncountably many solutions, whenever $\theta(a')$ holds.

Using induction on $i < \omega_1$, we shall construct a sequence $\{a_i\}_{i<\omega_1} \subset \theta^M$ such that for all $i < j < \omega_1$,

$$M \models \varphi(a_i, a_j) \wedge \neg\varphi(a_j, a_i).$$

Let $a_0 = a$ and suppose that we have found a_j's for $j < i$. Let us consider the following set:

$$\Gamma(x) = \{\theta(x)\} \cup \{\varphi(a_j, x) \wedge \neg\varphi(x, a_j) : j < i\}.$$

By properties 1 and 2 above, $\Gamma(x)$ consists of formulas with uncountably many solutions, so by Lemma 3, it has a solution $a_i \in M$. This a_i satisfies the required condition. Then the sequence $\{a_i\}_{i<\omega_1}$ is totally ordered by the formula $\varphi(x, y)$, hence $Th(M)$ must be unstable. A contradiction.

Remark 7 Mention Maesono's result: homogeneous + without strict order porperty implies the exchange property of ccl??

1.2 1-unstability and the exchange property

Definition 8 We will say that a structure M is 1-unstable if there is an $L(M)$-formula $\varphi(x, y)$ and an uncountable set $I \subset M$ such that $\{(a, b) \in I^2 : M \models \varphi(a, b)\}$ is a total order on I.

If M is 1-unstable, then $Th(M)$ is unstable. But the converse does not hold in general.

Lemma 9 *Let M be homogeneous and quasiminimal. Suppose that $a_1, a_2 \in M$ have the same type over $A \subset M$. Then for any $b_1 \in M - \mathrm{ccl}(A, a_1)$ and $b_2 \in M - \mathrm{ccl}(A, b))$, we have $\mathrm{tp}(a_1 b_1/A) = \mathrm{tp}(a_2 b_2/A)$.*

Proof: Suppse otherwise. We can choose a formula $\varphi(x, y) \in L(A)$ such that

$$M \models \varphi(a_1, b_1) \land \neg\varphi(a_2, b_2)$$

By homogeneity of M, there is an automorphism σ with $\sigma(a_2) = a_1$ such that σ fixes the parameters A_0 of φ. So we have

$$M \models \varphi(a_1, b_1) \land \neg\varphi(a_1, \sigma(b_2))$$

Since M is quasi-minimal (and $b_1 \notin \mathrm{ccl}(Aa_1)$), $\neg\varphi(a_1, x)$ has only countably many solutions. So $\sigma(b_2) \in \mathrm{ccl}(A_0 a_1)$. Hence $b_2 \in \mathrm{ccl}(A_0, a_2)$, a contradiction. ∎

Proposition 10 *Let M be a homogeneous quasi-minimal structure. If the countable closure does not satisfy the exchange axiom, then M is 1-unstable.*

Proof: We assume that there are $a, b \in M$ with $a \in \mathrm{ccl}(Ab) - \mathrm{ccl}(A)$ and $b \notin \mathrm{ccl}(Aa)$. We may assume $A = \emptyset$. Then a and b have the same type over $\emptyset$, say p. From Lemma 9, we know that if both c and d realize p and $d \notin \mathrm{ccl}(c)$, then $\mathrm{tp}(cd) = \mathrm{tp}(ab)$. By induction on $i < |M|$ we can easily find realizations $a_i \in M$ of p such that $a_i \notin \mathrm{ccl}(\{a_j : j < i\})$. Then $I = \{a_i : i < |M|\}$ forms a 2-indiscernible sequence. Choose a formula $\varphi(x, b) \in \mathrm{tp}(a/b)$ with only countably many solutions in M. Then we have

- $\varphi(a_i, a_j)$ for all $i < j < |M|$, and
- $\neg\varphi(a_i, a_j)$ for all $j < i < |M|$.

Hence M is 1-unstable. ∎

Proposition 11 *Let M be a homogeneous quasi-minimal structure. If M is 1-unstable, then $|M| \leq \omega_1$.*

Proof: By way of a contradiction, assume that M is 1-unstable and $|M| \geq \omega_2$. Let $\varphi(x, y) \in L(M)$ and I witness the 1-instability of M. Namely, I is an uncountable sequence totally ordered by $\varphi(x, y)$. We assume that I is maximal among such sequences. Let us write $a < b$ if $\varphi(a, b) \land \neg\varphi(b, a)$ holds.

Define two sets:

- $I_- = \{a \in I : |\{b \in I : b < a\}| \leq \omega\}$;
- $I_+ = \{a \in I : |\{b \in I : a < b\}| \leq \omega\}$.

By the quasi-minimality, the pair (I_-, I_+) clearly defines a Dedekind cut of I.

Claim A *Both I_- and I_+ are uncountable.*

Suppose otherwise. We may assume that I_+ is countable. For any $a \in I_-$, $x < a$ has only countably many solutions in I, and $a < x$ has uncountably many solutions in I_- (and hence in M). Then an easy argument shows that I_- can be written as a union of ω_1-many countable sets. So we have $|I_-| = \omega_1$. Then by lemma 3 there is an element in M realizing $\{a < x : a \in I_-\} \cup \{x < b : b \in I_+\}$. This contradicts the maximality of I.

By shrinking I, we assume both I_- and I_+ have cardinality ω_1. By lemma 3, we have a realization $a^* \in M$ of

$$\Gamma(x) = \{a < x < b : a \in I_-, b \in I_+\}.$$

Then $x < a^*$ and $\neg(x < a^*)$ divide I into two uncountable sets, contradicting the quasi-minimality of M. ∎

Remark 12 The rationals **Q** is definable in the structure $(\mathbf{C}, +, \cdot, \exp, 0, 1)$ by

$$\varphi(x) \stackrel{\text{def}}{=} \exists\alpha\exists\beta(\exp(\alpha) = \exp(\beta) = 1 \wedge \alpha \neq 0 \wedge x = \beta/\alpha).$$

It is also easy to see that both **Z** and **N** are definable in $(\mathbf{C}, +, \cdot, \exp, 0, 1)$. It follows that the theory of $(\mathbf{C}, +, \cdot, \exp, 0, 1)$ is unstable. It is interesting to see that there seems no uncountable total order definable in $(\mathbf{C}, +, \cdot, \exp, 0, 1)$.

1.3 Strong independence property and quasi-minimality.

In 2.2 we showed that there are no quasi-minimal random graphs. Here we discuss the reason of non-existence as a consequence of the strong independency.

Definition 13 T is said to have the *strong independence property* if there are formulas $R(x, y)$ and $D(x)$ such that for all distinct elements $a_1, ..., a_n, b_1, ..., b_n \in D$ there is c with $R(a_i, c)$ and $\neg R(b_i, c)$ $(i = 1, ..., n)$.

Proposition 14 *Suppose that T has the strong independence property. Let R and D witness the property. Then T does not have a quasi-minimal model with D uncountable.*

Proof: By way of a contradiction, assume that M is a quasi-minimal structure with D uncoutable. Let $A \subset M$ be a countable set with $\mathrm{ccl}(A) = A$. By taking ccl and Skolem hull repeatedly, we may assume that A is a model. Let $a \in D \cap A$. (Since A is a model, $D \cap A$ is non-empty.) Notice that

(*) any two elements from $M - A$ have the same type over A.

So we may assume that

(**) $\neg R(a,y)$ for all $y \in M - A$.

$D - A$ is an uncountable set. So we can choose two distinct elemets $b, c \in D - A$. Now consider the formula

$$R(a,y) \wedge R(b,y) \wedge (\neg R(c,y)).$$

By the strong independence property, there is a solution d. If d falls into A, then we have $R(x,d) \in tp(b/A)$ and $\neg R(x,d) \in tp(c/A)$, contradicting (*). On the other hand, if $d \in M - A$, then $R(a,d)$ contradicts (**) ∎

Corollary 15 *There are no quasi-minimal random graphs.*

1.4 Construction of saturated quasi-minimal structures.

In model theory it is often very convenient to work in saturated models. As we noted in the introduction, however, the technique of adding realizations of types to the original structure in order to construct a saturated model may not work in the study of quasi-minimal structures. Thus the question of the existence of saturated models attracts some attension. Consider the following question.

Question. Suppose that M is ω-stable quasi-minimal model. Is there a quasi-minimal $\aleph_0$-saturated model of $\mathrm{Th}(M)$?

Before giving a positive answer to the above question it is worth mentioning that there is a counterexample for superstable theories: Let $M_0 = (2^\omega, E_i(i < \omega))$ such that $E_i(x,y) \iff x(i) = y(i)$ for $x, y \in 2^\omega$. Let $M_1 \prec M_0$ be a countable model of $\mathrm{Th}(M_0)$ and fix $a \in M_1$. Let $M_2 = (M_1 \cup B, E_i(i < \omega))$ where $|B| > \aleph_0$ and $\mathrm{tp}(b) = \mathrm{tp}(a)$ for all $b \in B$. Then M_2 is quasi-minimal. But any $\aleph_0$-saturated model of $\mathrm{Th}(M_2)$ includes M_0. Hence it is not quasi-minimal. Note that M_2 is not homogeneous.

We now explain how to construc a saturated quasi-minimal structures. From now on in this subsection M denotes an ω-stable quasi-minimal structure. Let $p(x) \in S(M)$always denote the main type of M. Then each $\varphi(x) \in p(x)$ has uncountably many solutions in M. We may assume that $p(x)$ is strongly based on $\emptyset$. (i.e., $p(x)$ is the unique nonforking extension of $p|\emptyset$ over M.) The nonforking extension of p to the domain A is denoted by $p|A$. The prime model over A is unique up to isomorphism over A. The prime model over the set NA is denoted by $N(A)$, where N is a model. $N(A)(B)$ is an abbreviation of $(N(A))(B)$.

We work in a big saturated elementary extension $\mathcal{M}$ of M. First we show:

Lemma 16 *There is a countable model $M_0 \prec M$ and an uncountable Morley sequence $I = \{a_i\}_{i<\omega_1}$ of $p|M_0$ such that I dominates M over M_0.*

Proof: Let $N \prec M$ be any countable model. First we choose an uncountable Morley sequence $I \subset M$ of $p|N$, using induction. Suppose that we have chosen a_j's for $j < i < \omega_1$. Let $A_i = N \cup \{a_j\}_{j<i}$. Note that each formula $\varphi(x)$ in $p|A_i$ has uncountably many solutions. Since $p|A_i$ is a countable set, by lemma 3, $p|A_i$ has a solution a_i in M.

We assume that I chosen above is maximal among such. Let $M_0 \prec M$ be a maximal model such that

$$N \subset M_0 \quad \text{and} \quad M_0 \mathop{\perp}_{N} I.$$

I is claerly a Morley sequence of $p|M_0$.

Claim A *M_0 is countable.*

Otherwise, there is a type $q(x) \in S(N)$with q^{M_0} uncountable. By the maximality of I, $q \neq p|N$. This contradicts the quasi-minimality of M.

Claim B *I dominates M over M_0.*

Extend $M_0 I$ to a maximal set $M' \subset M$ such that M' is dominated by I over M_0. Clearly M' is an elementary submodel of M. Suppose that $\mathrm{tp}(M/M')$ is not orthogonal to M_0. Then, by the three model theorem for an ω-stable theory, we have an element $b \in M - M'$ such that $\mathrm{tp}(b/M')$ does not fork over M_0. Then we have

$$M_0(b) \underset{M_0}{\perp} I.$$

Since I and M_0 are independent over N, we must have $M_0(b) \perp_N I$, contradicting the maximality of M_0. Thus $\mathrm{tp}(M/M')$ is orthogonal to M_0. Hence, by the maximality of M', we have $M = M'$. ∎

Remark 17 *In the above lemma, M_0 can be chosen arbitrarily larage : For any finite subset $A \subset M$, we can choose M_0 above so that M_0 contains A.*

M_0 having the properies in the above lemma will be called a *base model* of M. In what follows, M_0 will be used to denote a base model of a given quasi-minimal model.

Lemma 18 *Let M_0 be a base model of M. Let $r(x) \in S(M_0)$be a type with $r \neq p|M_0$. Then we have:*

1. *p is orthogonal to r;*

2. *$r|M$ is the unique extension of r to the domain M;*

3. *Any element $d \in M - M_0$ realizes $p|M_0$.*

Proof: Statements 2 and 3 follow from 1 and lemma 16. We prove 1. Suppose otherwise. Then there is a consistent formula $\varphi(x, a_0) \in L(M_0 a_0)$ with the following properties:

- $\varphi(x, a_0)$ forks over M_0, and

- Any solution of $\varphi(x, a_0)$ does not realize $p|M_0$;

Choose the maximum $n < \omega$ such that $\{\varphi(x, a_j) : i \leq j < \omega_1\}$ is n-consistent (i.e., any subset of cardinality n is consistent). For each $j < \omega_1$, choose $b_j \in M$ satisfying $\bigwedge_{nj \leq k < n(j+1)} \varphi(x, a_k)$. Then the b_j's are distinct elements not realizing $p|M_0$, by the above two properties. This contradicts the quasi-minimality of M.

Lemma 19 *Let $a \in \mathcal{M}$ be any element with* $\mathrm{tp}(a/M) \neq p(x)$. *Then $M(a)$ is quasi-minimal.*

Proof : Choose a base model M_0 such that

- a and M are independent over M_0;
- $\mathrm{tp}(a/M_0) \neq p|M_0$.

Then choose a Morley sequence I of $p|M_0$ such that I dominates M over M_0. We prove the present lemma by a series of claims. Notice that $M(a)$ is prime and atomic model over $M_0(a)M$.

Claim A *Any element b from $M(a) - M_0(a)$ realizes the type $p|M_0$. A more precise statement is the following: If* $\mathrm{tp}(b/Ma)$ *is isolated but* $\mathrm{tp}(b/M_0a)$ *is not isolated, then b realizes $p|M_0$.*

By way of a contradiction, we assume that some element $b \in M(a) - M_0(a)$ does not realize $p|M_0$. Notice that $\mathrm{tp}(b/M_0(a)M)$ is an isolated type. Then we have

$$ab \underset{M_0}{\not\perp} M,$$

since otherwise the open mapping theorem would imply that $\mathrm{tp}(b/M_0(a))$ is an isolated type. Since I dominates M over M_0, we have $ab \not\perp_{M_0} I$. Choose $\varphi(xy, \bar{d}) \in \mathrm{tp}(ab/M_0I)$ ($\bar{d}$ is from I) such that

1. $\varphi(xy, \bar{d})$ forks over M_0 and
2. $\varphi(xy, \bar{d})$ implies $\neg\theta(y)$,

where $\theta(x)$ is a formula with $\theta(x) \in p|M_0$. Since $\mathrm{tp}(a/M_0I)$ does not fork over M_0, $(\exists y)\varphi(x, y, \bar{d})$ has a solution $a' \in M_0$. For notational simplicity, we assume $\bar{d}$ is a single element, say a_0. Then by condition 1,

$$\Gamma(y) = \{\varphi(a', y, a_j) : j < \omega_1\}$$

must be inconsistent, since I is a Morley sequence. By the indiscernibility of I over M_0, $\Gamma(y)$ is n-inconsistent, for some $n \in \omega$. So by exactly the same argument as in lemma 18, we have uncountably many solutions of $\neg\theta(x)$ in M. This contradicts the quasi-minimality of M.

Claim B $M(a)$ *is quasi-minimal.*

Suppose otherwise and choose a formula $\varphi(x) \in L(M(a))$ witnessing the non-quasi-minimality of $M(a)$. We may assume that the parameters of φ are in $M_0(a)$. (Otherwise extend M_0 a little bit more.) Since $M_0(a)$ is countable, we can choose $b, c \in M(a) - M_0(a)$ such that $M \models \varphi(b) \wedge \neg\varphi(c)$. By claim A, both b and c realize $p|M_0$. Since $\mathrm{tp}(a/M_0)$ and $p|M_0$ are almost orthogonal, both $\mathrm{tp}(b/M_0(a))$ and $\mathrm{tp}(c/M_0(a))$ are non-forking extensions of a stationary type $p|M_0$. So we have $\mathrm{tp}(b/M_0(a)) = \mathrm{tp}(c/M_0(a))$, and hence we must have $M \models \varphi(b) \leftrightarrow \varphi(c)$, a contradiction.

Remark 20 1. $p|(M(a))$ is the main type of $M(a)$: It is sufficient to show that if $\varphi(x,\bar{d}) \in L(M(a))$ has uncountably many solutions in $M(a)$, then $\varphi(x,\bar{d})$ does not fork over M. Choose a countable model $M_0 \prec M$ such that $\bar{d} \in M_0(a)$. We can assume that M_0 is a base model of M and that $\mathrm{tp}(a/M_0) \neq p|M$. Since $M_0(a)$ is a countable set, we can chose $b \in M(a) - M_0(a)$ satisfying $\varphi(x,\bar{d})$. By claim A, $\mathrm{tp}(b/M_0) = p|M_0$. By the almost orthogonality of $\mathrm{tp}(a/M_0)$ and $p|M_0$, we have $b \mathop{\smile\!\!\!\!\!\!\;|}_{M_0} M_0(a)$. Hence $\varphi(x,\bar{d})$ does not fork over $M_0 \subset M$.

2. Let $\{d_i : i < \omega\}$ be a countable sequence of elements with $\mathrm{tp}(d_i/M) \neq p(x)$. Then $M(\{d_i\}_{i<\omega})$ is quasi-minimal. This can be shown by the iterated use of lemma 16.

Theorem 21 *Let M be a quasi-minimals model of an ω-stable theory. Then there is an extension $M^* \succ M$ such that*

1. M^* is still quasi-minimal, and

2. M^* realizes all types $q(x)$ with $\mathrm{dom}(q) \subset_{\mathrm{finite}} M$ and $q \neq p|\mathrm{dom}(q)$.

Proof: For notational simplicity, we assume $|M| = \omega_1$. Let $\{q_i(x)\}_{i<\alpha}$ be a maximal set of regular types with the following properties: For $i < j < \alpha$,

- $q_i(x)$ is orthogonal to $p(x)$;
- $q_i(x)$ and $q_j(x)$ are orthogonal;
- $\mathrm{dom}(q_i)$ is a finite subset of M.

For each $i < \alpha$, choose a countable Morley sequence J_i of $q_i|M$. For each subset X of ω_1, let J_X denote the set $\bigcup_{i\in X} J_i$. We put $M^* = M(\{J_i\}_{i<\alpha})$. By decomposing each type into regular types, we can easily show that M^* realizes any type over a finite subset of M. It remains to show that M^* is quasi-minimal. If $\alpha = \omega$, then M^* is a quasiminimal extension of M, by lemma 19 and remark after it. So we can assume that $\alpha = \omega_1$. Now we can forget the maximality of $\{q_i(x)\}_{i<\alpha}$. Only assumption we need is that they are orthogonal.

By way of a contradiction, we now assume that M^* is not quasi-minimal. Choose a type $q(x) \in S(A)(q \neq p|A)$ having uncountably many realizations in M^*. The set A can be assumed to be a finite subset of M^*. So we can choose a finite subset $F \subset \omega_1$ with $A \subset M(J_F)$. Then M^* can be decomosed in the form

$$M^* = \bigcup_X M(J_X),$$

where X ranges over all finite subsets of ω_1 with $X \supset F$. By lemma 19, each $M(J_X)$ is quasi-minimal. So each $M(J_X)$ has only countably many realizations of $q(x)$. Hence there are uncountably many distinct finite subsets X_i $(i < \omega_1)$ of ω_1 such that each $M(J_{X_i})$ has a realization, say d_i, of $q(x)$.

By the Δ-system lemma (cf: p. 49 [4]), taking a subsequence of $\{X_i : i < \omega_1\}$, we may assume that there is $Y \subset \omega_1$ such that for any $i \neq i'$,

$$X_i \cap X_{i'} = Y.$$

Now for each $i < \omega_1$ we have

$$d_i \in M(J_Y)(J_{X_i - Y}) - M(J_Y). \tag{1}$$

Recall that $M(J_Y)$ includes A. Let $M_0 \supset A$ be a base model of $M(J_Y)$. Notice that $q(x) \in \mathrm{S}(A)$has only countably many extensions in $\mathrm{S}(M_0)$.So, by replacing q by its suitable extension, and by taking subsequence of the d_i's, we can assume that $q \in \mathrm{S}(M_0)$and that all d_i's realize $q(x)$. Moreover, by lemma 18, $q|(M(J_Y))$ is the unique extension of q to the domain $M(J_Y)$. So each d_i realizes $q|(M(J_Y))$. In other words,

$$\mathrm{tp}(d_i/M(J_Y)) = q|(M(J_Y)). \tag{2}$$

By the ω-stability, discarding countably many J_i's, we can assume that each J_i is a Morley sequence of $q_i|M(AJ_Y)$. This together with (1) and (2) shows that for all $i < \omega_1$, we can find a type $r_i \in \{q_j : j \in X_i - Y\}$ which is nonorthogonal to $q|(M(J_Y))$. But r_i's are orhthogonal. This contradicts our assumption that T is ω-stable. ∎

Corollary 22 *Any quasi-minimal model of an ω-stable theory can be elementarily extended to an ω-saturated quasi-minimal model.*

Proof: Using theorem 21, we can construct a chain N_i $(i < \omega)$ of quasi-minimal models such that

1. $M = N_0 \prec N_1 \prec \cdots \prec N_i \prec N_{i+1} \prec \cdots$;
2. N_{i+1} realizes any type q with $\mathrm{dom}(q) \subset_{\mathrm{finite}} N_i$ and $q \neq p|\mathrm{dom}(q)$.

We show that $N^* = \bigcup_{i \in \omega} N_i$ has the required properties. Since N^* is a union of quasi-minimal models, it is clearly quasi-minimal. So it remains to show that any type $q(x) \in \mathrm{S}(A)$with $A \subset_{\mathrm{finite}} M^*$ is realzed in M^*. Choose M_n with $A \subset M_n$. If $q(x) \neq p|M$, then it has a realization in M_n by the property 1. So we can assume that $q(x) = p|A$. Let $I \subset M$ be a Morley sequence of $p|M_0$ such that I dominates M over M_0, where M_0 is a base model of M (lemma 16). Since I is (uncountable) infinite, there is $a \in I$ with $a \perp_{M_0} A$. Then a realizes $p|M_0A$. Hence a realizes $q(x)$. ∎

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