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<th>A Remark on Generic Pseudoplanes (Model theory via geometric approach)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2002), 1283: 55-60</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42411">http://hdl.handle.net/2433/42411</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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A Remark on Generic Pseudoplanes

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Abstract

We prove that if δ-generic saturated pseudoplane is strictly stable, then the algebraic closure of a finite set is finite.

1 Generic structures

Let $L$ be a finite relational language and $K$ a class of finite $L$-structures closed under isomorphism and substructures. For any $A, B \in K$ with $A \subseteq B$ let $A \leq B$ be a reflexive and transitive relation which is invariant under isomorphism. In what follows, $K$ satisfies the following set of axioms.

Axiom 1.1 (A1) $A \subseteq B \subseteq C \subseteq K$ and $A \leq C$ implies $A \leq B$;
(A2) $\emptyset \leq A$ for any $A \in K$;
(A3) $A \subseteq B \in K$ and $X \subseteq B$ implies $A \cap X \subseteq X$;
(A4) There are no infinite chains $A_1 \subseteq A_2 \subseteq \ldots$ such that, for each $i < \omega$, $A_i \in K$, $A_i \not\leq A_{i+1}$ and any proper non-empty subset $X$ of $A_{i+1} - A_i$ satisfies $A_i \leq A_i X$.

For an infinite $L$-structure $M$ satisfying $A \in K$ for any finite $A \subseteq M$, define $A \leq M$ if $A \leq B$ for all finite $B$ with $A \subseteq B \subseteq M$.

Note 1.2 Let $M$ satisfy $A \in K$ for all finite $A \subseteq M$. By (A1)–(A4), for a finite $B \subseteq M$ there is a unique smallest superset $B^*$ of $B$ with $B^* \subseteq M$. Such a $B^*$ is called the closure of $B$ in $M$. (in symbol $c_M(B)$).

Definition 1.3 Let $(K, \leq)$ satisfy (A1)–(A4). A structure $M$ is said to be $(K, \leq)$-generic, if
(i) If $A$ is a finite substructure of $M$ then $A \in K$. 
(ii) If $A \leq M$ and $A \leq B \in K$ then there is an $A$-embedding $f : B \to M$ with $f(B) \leq M$. (An $A$-embedding is an embedding fixing $A$ pointwise.)

Whenever we consider a $(K, \leq)$-generic structure, $(K, \leq)$ is supposed to satisfy the above conditions (A1)-(A4). However, even if $(K, \leq)$ satisfies (A1)-(A4), then a $(K, \leq)$-generic structure does not necessarily exist.

**Definition 1.4** $(K, \leq)$ is said to have the amalgamation property if for any $A \leq B \in K$ and $A \leq C \in K$ there is $D \in K$ such that $f(B) \leq D$ and $g(C) \leq D$ for some $A$-embeddings $f : B \to D$ and $g : C \to D$.

**Fact 1.5([1],[2],[5])** If $(K, \leq)$ has the amalgamation property, then there exists a unique $(K, \leq)$-generic structure.

## 2 Theorem and Proof

Let $L$ be a language of bipartite graphs: $L = \{P, Q, R\}$ where $P, Q$ are unary predicates and $R \subset P \times Q$. Let $\alpha$ be a real number. Then

- For a finite $L$-structure $A$, $\delta_\alpha(A) = |P^A| + |Q^A| - \alpha|R^A|$.
- $K_\alpha = \{ A : A$ is a finite $L$-structure, $\forall B \subset A[\delta_\alpha(B) \geq 0] \}$.
- For $A \subset B \in K_\alpha$, $A \leq B$ is defined by $\delta_\alpha(XA) \geq \delta_\alpha(A)$ for any $X \subset B-A$.

**Note 2.1** It is easily checked that $(K_\alpha, \leq)$ satifies (A1)-(A3).

**Definition 2.2** We say that a bipartite graph $M$ is $\delta$-generic, if $M$ is $(K, \leq)$-generic for some $\alpha$ and $K \subset K_\alpha$.

Our goal is to show the following theorem.

**Theorem** Let $M$ be a $\delta$-generic saturated pseudoplane. If $M$ is strictly stable, then the algebraic closure of any finite set is finite.

To prove this theorem, we need some preparations.

In what follows, we assume that $K \subset K_\alpha$ satisfies the amalgamation property, and that $M$ is a $(K, \leq)$-generic saturated pseudoplane.

**Note 2.3([1],[5])** If $\alpha$ is a positive rational number, then $\text{Th}(M)$ is $\omega$-stable.
Definition 2.4 Let $AB$ be a finite bipartite graph. Then
(i) A pair $(B, A)$ is said to be normal, if $A \leq AB \in K$ and $A \cap B = \emptyset$.
(ii) A normal pair $(B, A)$ is said to be small, if there are no normal pairs $(D, C)$ such that $A \subset C, B \subset D$ and $\delta(D/C) < \delta(B/C)$.
(iii) A normal pair $(B, A)$ is said to be minimal, if there are no non-empty proper subsets $C$ of $B$ with $AC \leq AB$.

To simplify our notation, we denote $R(x, y) \lor R(y, x)$ by $S(x, y)$. For any elements $e, a, b$ of a bipartite graph we say a pair $(e, ab)$ is special, if $S(e, a) \land S(e, b)$ holds.

Note 2.5 Suppose that a $(K, \leq)$-generic bipartite graph is a pseudoplane. Let $A$ be a finite bipartite graph with no loops, i.e., for each $n > 2$ there do not exist distinct $a_1, a_2, \ldots, a_n \in A$ with $S(a_1, a_2), S(a_2, a_3), \ldots, S(a_{n-1}, a_n)$ and $S(a_n, a_1)$. Then we can see that $A \in K$. (The proof is by induction.)

Lemma 2.6 $\alpha \leq 1$.

Proof Suppose by way of contradiction that $\alpha > 1$. By genericity there is $a \in M$ with $a \leq M$. Then there are no element $b \in M$ with $S(b, a)$. (If not, then $\delta(b/a) = 1 - \alpha < 0$. This contradicts $a \leq M$.) But this contradicts the definition of pseudoplanes.

Lemma 2.7 $\frac{1}{3} < \alpha$.

Proof Suppose by way of contradiction that $\alpha \leq \frac{1}{3}$. Let $abcd$ be an $L$-structure with the relations $S(d, a), S(d, b), S(d, c)$. By 2.5, we have $abcd \in K$. By $\alpha \leq \frac{1}{3}$, we have $\delta(d/abc) \geq 0$, and so $abc \leq abcd$. By amalgamation property, we can inductively construct a sequence $\{e_i\}_{i<\omega}$ such that (i) $S(e_i, a), S(e_i, b), \neg S(e_i, c)$ for each $i < \omega$, and (ii) $abcde_i \in K$ for each $i < \omega$. In particular we have $S(e_i, a) \land S(e_i, b)$ for each $i < \omega$. This contradicts the definition of pseudoplanes.

Lemma 2.8 Let $\alpha$ be an irrational number with $\frac{n-1}{2n-1} < \alpha \leq \frac{n}{2n+1}$, where $n \geq 2$. Then a special pair is not small.

Proof Let $a_1b_1a_2b_2\ldots a_nb_ncd$ be a finite $L$-structure with the relations $S(a_1, c), S(a_n, d), \{S(a_i, b_i)\}_{i=1,\ldots,n}$ and $\{S(a_i, a_{i+1})\}_{i=1,\ldots,n-1}$. Let $A = \{a_i\}_{i=1,\ldots,n}$ and $B = \{b_i\}_{i=1,\ldots,n}$. By 2.5, we have $ABcd \in K$.

Claim 1: $Bcd \leq ABcd$.

Proof: Take any $X \subset A$. It is easily seen that if $X \neq A$ then $\delta(X/Bcd) \geq |X| - 2|X|\alpha$. So, by $\alpha \leq \frac{n}{2n+1} \leq \frac{1}{3}$ we have $\delta(X/Bcd) \geq 0$. If $X = A$ then $\delta(X/Bcd) = n - (2n+1)\alpha \geq n - (2n+1)\frac{n}{2n+1} = 0$. Hence $Bcd \leq ABcd$. 

\[\text{57} \]
Claim 2: $\delta(a_1/Bcd) > \delta(A/Bcd)$.
Proof: $\delta(A/Bcd) - \delta(a_1/Bcd) = (n-1) - (2n-1)\alpha < (n-1) - (2n-1)\frac{n-1}{2n-1} = 0.$

By claim 1,2, special pair $(a_1, b_1c)$ is not small. This completes the proof of this lemma.

Note 2.9 Let $X = \{a - b\alpha : a, b < \omega, a - b\alpha > 0\}$. Then $\inf X = 0$.

Lemma 2.10 Let $\alpha$ be an irrational number with $\alpha > \frac{1}{2}$. Then any minimal pair is not small.

Proof Let $(B, A)$ be a minimal pair with $\delta(B/A) = m - n\alpha$. By 2.9, there are $p, q < \omega$ such that $m \leq p, n \leq q$ and $0 < p - q\alpha < m - n\alpha$. To show that $(B, A)$ is not small, it is enough to see that there is a normal pair $(D, C)$ such that $A \subset C, B \subset D$ and $\delta(D/C) = p - q\alpha$. Pick a element $b_0 \in B$. Let $k = p - m$ and take $b_1, b_2, ..., b_k$ with the relations $S(b_0, b_1), S(b_1, b_2), ..., S(b_k, b_1)$. Let $l = q - n$ and take $a_1, a_2, ..., a_{l-k}$ with the relations $S(a_i, b_i)$ for $1 \leq i \leq l-k$. Let $C = A a_1 a_2 ... a_{l-k}$ and $D = B b_1 b_2 ... b_k$. By 2.5, $CD \in K$. On the other hand, $\delta(D/C) = \delta(B/A) + k - (k + l - k)\alpha = (m + k) - (n + l)\alpha = p - q\alpha$. Also we can see that $C \leq CD$. (It can be shown as follows: Take any $X \subset D - C$ and let $X_C = X \cap C$ and $X_D = X \cap (D - C)$. Then $\delta(X/C) = \delta(X_B/C) + \delta(X_D/CX_B) = \delta(X_B/A) + \delta(X_D/CX_B)$.) Note that $B \geq A$ and $\alpha > \frac{1}{2}$. Hence $\delta(X/C) \geq 0.$) It follows that $(D, C)$ is normal.

Lemma 2.11 If $\alpha$ is irrational, then any minimal pair is not small.

Proof By 2.6 and 2.7, we have $\alpha \in (\frac{1}{3}, 1]$. If $\alpha > \frac{1}{2}$, then any minimal pair is not small by 2.10. If $\alpha < \frac{1}{2}$, then there is $n < \omega$ with $\alpha \in \left(\frac{n-1}{2n-1}, \frac{n}{2n+1}\right]$, and therefore any minimal pair is not small, by 2.8.

Lemma 2.12 Let $A \leq AB \leq M$. Let $(B, A)$ be a minimal pair. If $tp(B/A)$ is algebraic, then $(B, A)$ is small.

Proof Suppose by way of contradiction that $(B, A)$ is not small. Then there is a normal pair $(D, C)$ such that $A \subset C, B \subset D$ and $\delta(D/C) < \delta(B/C)$. By minimality of $(B, A)$ we can assume that $(D, C)$ is minimal.

Claim 1: There is a sequence $(B_i)_{i<\omega}$ with the following conditions:
(i) $B_i \cong C B_0 ... B_{i-1} B$ for any $i < \omega$;
(ii) $C B_0 ... B_i D \leq C B_0 ... B_i D \in K$ for any $i < \omega$;
(iii) $D, B_0, B_1, B_2, ...$ are pairwise disjoint.

Proof of Claim: We prove by induction. Suppose $(B_i)_{i \leq n}$ has constructed.
By (ii), we have $C B_0 ... B_n \leq C B_0 ... B_n D \in K$, and therefore $C B_0 ... B_n \leq C B_0 ... B_n B \in K$. So, by amalgamation property, we can take $B_{n+1}$ so that
$B_{n+1} \cong_{CB_0} ... B_n$ and $CB_0 ... B_n D, CB_0 ... B_n B_{n+1} \leq CB_0 ... B_n B_{n+1} D \in K$. Thus $B_{n+1}$ satisfies (i) and (ii). For (iii) it is enough to show that $B_{n+1} \cap D = \emptyset$. Suppose that $D' = B_{n+1} \cap D \neq \emptyset$. We have had $CB_0 ... B_n B_{n+1} \leq CB_0 ... B_n B_{n+1} D$, so $CD' \leq CD$. Note that $D' \neq D$. This contradicts minimality of $(D, C)$. Hence $B_{n+1} \cap D = \emptyset$. (End of Proof of Claim 1)

Claim 2: $AB, AB_i \leq AB_0 ... B_i B$ for $j \leq i < \omega$

Proof: We prove by induction on $i$. By (ii) of claim 1, $AB_0 ... B_i B \leq AB_0 ... B_{i+1} B$. By induction hypothesis, we have $AB, AB_j \leq AB_0 ... B_i B$ for $j \leq i$. Hence $AB, AB_i \leq AB_0 ... B_{i+1} B$ for $j \leq i$. So, it is enough to show that $AB_{i+1} \leq AB_0 ... B_{i+1} B$. By induction hypothesis, we have $AB \leq AB_0 ... B_i B$. By (i) of claim 1, we have $AB_{i+1} \leq AB_0 ... B_{i+1} B$. By (ii) of claim 1, $AB_0 ... B_{i+1} B \leq AB_0 ... B_{i+1}$. Hence we have $AB_{i+1} \leq AB_0 ... B_{i+1} B$. (End of Proof of Claim 2)

We show that tp$(B/A)$ is non-algebraic. Fix any $n < \omega$. By claim 2, there are $B_i^*$'s such that $B_0^* ... B_n^* \cong AB B_0 ... B_n$ and $AB \leq ABCB_0^* ... B_n^* \leq M$. Again, by claim 2, $AB_i^* \leq ABCB_0^* ... B_n^* \leq M$ for all $i \leq n$. Therefore we have tp$(B_i^* / A) = tp(B/A)$. By (iii) of claim 1, $B_i^*$'s are pairwise disjoint. Hence tp$(B/A)$ is not algebraic.

Lemma 2.13 If $\alpha$ is irrational, then acl$(X) = cl(X)$ for any finite subset $X$ of $M$.

Proof Take any finite subset $X$ of $M$. Then cl$(X) \subset acl(X)$ is clear. We show acl$(X) \subset cl(X)$. If not, there is $a \in acl(X) - cl(X)$. Let $A = cl(X)$ and $B = cl(aX)$. Take a maximal chain $\{B_i\}_{<\omega}$ with $A = B_0 \leq B_1 \leq ... \leq B_n = B$. Then, for each $i < \omega$, $(B_{i+1} - B_i, B_i)$ is minimal and $A \leq AB_i \leq M$. By 2.11, they are not small, and so tp$(B_{i+1} / B_i)$ is not algebraic. In particular we have $B \not\subset acl(A) = acl(X)$. A contradiction.

Proof of Theorem Let $M$ be a $(K, \leq)$-generic saturated pseudoplane for some $K \subset K_a$. Suppose that $M$ is strictly stable. By 2.3, $\alpha$ is irrational. By 2.13, acl$(X) = cl(X)$ for any finite $X \subset M$. Note that cl$(X)$ is finite by (A4) of Axiom 1.1. Hence acl$(X)$ is finite.

Question Are $\delta$-generic pseudoplanes $\omega$-categorical?

Reference

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