数理解析研究所講究録 1283 巻 2002 年 55-60

A Remark on Generic Pseudoplanes

豊田工業高等専門学校
Toyota National College of Technology
池田宏一郎
Koichiro IKEDA

Abstract

We prove that if δ-generic saturated pseudoplane is strictly stable, then the algebraic closure of a finite set is finite.

1 Generic structures

Let L be a finite relational language and K a class of finite L-structures closed under isomorphism and substructures. For any $A, B \in K$ with $A \subset B$ let $A \leq B$ be a reflexive and transitive relation which is invariant under isomorphism. In what follows, K satisfies the following set of axioms.

Axiom 1.1 (A1) $A \subset B \subset C \in K$ and $A \leq C$ implies $A \leq B$;
(A2) $\emptyset \leq A$ for any $A \in K$;
(A3) $A \leq B \in K$ and $X \subset B$ implies $A \cap X \leq X$;
(A4) There are no infinite chains $A_1 \subset A_2 \subset \ldots$ such that, for each $i < \omega$, $A_i \in K$, $A_i \not\leq A_{i+1}$ and any proper non-empty subset X of $A_{i+1} - A_i$ satisfies $A_i \leq A_i X$.

For an infinite L-structure M satisfying $A \in K$ for any finite $A \subset M$, define $A \leq M$ if $A \leq B$ for all finite B with $A \subset B \subset M$.

Note 1.2 Let M satisfy $A \in K$ for all finite $A \subset M$. By (A1)–(A4), for a finite $B \subset M$ there is a unique smallest superset B^* of B with $B^* \leq M$. Such a B^* is called the *closure* of B in M. (in symbol $\mathrm{cl}_M(B)$).

Definition 1.3 Let $(K, \leq)$ satisfy (A1)–(A4). A structure M is said to be $(K, \leq)$-*generic*, if
(i) If A is a finite substructure of M then $A \in K$.

(ii) If $A \leq M$ and $A \leq B \in K$ then there is an A-embedding $f : B \to M$ with $f(B) \leq M$. (An A-embedding is an embedding fixing A pointwise.)

Whenever we consider a $(K, \leq)$-generic structure, $(K, \leq)$ is supposed to satisfy the above conditions (A1)–(A4). However, even if $(K, \leq)$ satisfies (A1)–(A4), then a $(K, \leq)$-generic structure does not necessarily exist.

Definition 1.4 $(K, \leq)$ is said to have the *amalgamation property* if for any $A \leq B \in K$ and $A \leq C \in K$ there is $D \in K$ such that $f(B) \leq D$ and $g(C) \leq D$ for some A-embeddings $f : B \to D$ and $g : C \to D$.

Fact 1.5([1],[2],[5]) If $(K, \leq)$ has the amalgamation property, then there exists a unique $(K, \leq)$-generic structure.

2 Theorem and Proof

Let L be a language of bipartite graphs: $L = \{P, Q, R\}$ where P, Q are unary predicates and $R \subset P \times Q$. Let α be a real number. Then

- For a finite L-structure A, $\delta_\alpha(A) = |P^A| + |Q^A| - \alpha|R^A|$.
- $K_\alpha = \{A : A \text{ is a finite } L\text{-structure}, \forall B \subset A[\delta_\alpha(B) \geq 0]\}$.
- For $A \subset B \in K_\alpha$, $A \leq B$ is defined by $\delta_\alpha(XA) \geq \delta_\alpha(A)$ for any $X \subset B - A$.

Note 2.1 It is easily checked that $(K_\alpha, \leq)$ satifies (A1)–(A3).

Definition 2.2 We say that a bipartite graph M is *δ-generic*, if M is $(K, \leq)$-generic for some α and $K \subset K_\alpha$.

Our goal is to show the following theorem.

Theorem Let M be a δ-generic saturated pseudoplane. If M is strictly stable, then the algebraic closure of any finite set is finite.

To prove this theorem, we need some preparations.

In what follows, we assume that $K \subset K_\alpha$ satisfies the amalgamation property, and that M is a $(K, \leq)$-generic saturated pseudoplane.

Note 2.3([1],[5]) If α is a positive rational number, then Th(M) is ω-stable.

Definition 2.4 Let AB be a finite bipartite graph. Then
(i) A pair (B, A) is said to be *normal*, if $A \leq AB \in K$ and $A \cap B = \emptyset$.
(ii) A normal pair (B, A) is said to be *small*, if there are no normal pairs (D, C) such that $A \subset C, B \subset D$ and $\delta(D/C) < \delta(B/C)$.
(iii) A normal pair (B, A) is said to be *minimal*, if there are no non-empty proper subsets C of B with $AC \leq AB$.

To simplify our notation, we denote $R(x, y) \vee R(y, x)$ by $S(x, y)$. For any elements e, a, b of a bipartite graph we say a pair (e, ab) is *special*, if $S(e, a) \wedge S(e, b)$ holds.

Note 2.5 Suppose that a $(K, \leq)$-generic bipartite graph is a psuedoplane. Let A be a finite bipartite graph with no loops, i.e., for each $n > 2$ there do not exist dintinct $a_1, a_2, ..., a_n \in A$ with $S(a_1, a_2)$, $S(a_2, a_3)$,..., $S(a_{n-1}, a_n)$ and $S(a_n, a_1)$. Then we can see that $A \in K$. (The proof is by induction.)

Lemma 2.6 $\alpha \leq 1$.

Proof Suppose by way of contradiction that $\alpha > 1$. By genericity there is $a \in M$ with $a \leq M$. Then there are no element $b \in M$ with $S(b, a)$. (If not, then $\delta(b/a) = 1 - \alpha < 0$. This contradicts $a \leq M$.) But this contradicts the definition of pseudoplanes.

Lemma 2.7 $\frac{1}{3} < \alpha$.

Proof Suppose by way of contradiction that $\alpha \leq \frac{1}{3}$. Let $abcd$ be an L-structure with the relations $S(d, a), S(d, b), S(d, c)$. By 2.5, we have $abcd \in K$. By $\alpha \leq \frac{1}{3}$, we have $\delta(d/abc) \geq 0$, and so $abc \leq abcd$. By amalgamation property, we can inductively construct a sequence $\{e_i\}_{i<\omega}$ such that (i) $S(e_i, a), S(e_i, b), \neg S(e_i, c)$ for each $i < \omega$, and (ii) $abcde_1...e_i \in K$ for each $i < \omega$. In particular we have $S(e_i, a) \wedge S(e_i, b)$ for each $i < \omega$. This contradicts the definition of psuedoplane.

Lemma 2.8 Let α be an irrational number with $\frac{n-1}{2n-1} < \alpha \leq \frac{n}{2n+1}$, where $n \geq 2$. Then a special pair is not small.

Proof Let $a_1b_1a_2b_2...a_nb_ncd$ be a finite L-structure with the relations $S(a_1, c)$, $S(a_n, d)$, $\{S(a_i, b_i)\}_{i=1,...,n}$ and $\{S(a_i, a_{i+1})\}_{i=1,...,n-1}$. Let $A = \{a_i\}_{i=1,...,n}$ and $B = \{b_i\}_{i=1,...,n}$. By 2.5, we have $ABcd \in K$.
Claim 1: $Bcd \leq ABcd$.
Proof: Take any $X \subset A$. It is easily seen that if $X \neq A$ then $\delta(X/Bcd) \geq |X| - 2|X|\alpha$. So, by $\alpha \leq \frac{n}{2n+1} \leq \frac{1}{2}$ we have $\delta(X/Bcd) \geq 0$. If $X = A$ then $\delta(X/Bcd) = n - (2n+1)\alpha \geq n - (2n+1)\frac{n}{2n+1} = 0$. Hence $Bcd \leq ABcd$.

Claim 2: $\delta(a_1/Bcd) > \delta(A/Bcd)$.
Proof: $\delta(A/Bcd)-\delta(a_1/Bcd) = (n-1)-(2n-1)\alpha < (n-1)-(2n-1)\frac{n-1}{2n-1} = 0$.

By claim 1,2, special pair (a_1, b_1c) is not small. This completes the proof of this lemma.

Note 2.9 Let $X = \{a - b\alpha : a, b < \omega, a - b\alpha > 0\}$. Then $\inf X = 0$.

Lemma 2.10 Let α be an irrational number with $\alpha > \frac{1}{2}$. Then any minimal pair is not small.

Proof Let (B, A) be a minimal pair with $\delta(B/A) = m - n\alpha$. By 2.9, there are $p, q < \omega$ such that $m \le p, n \le q$ and $0 < p - q\alpha < m - n\alpha$. To show that (B, A) is not small, it is enough to see that there is a normal pair (D, C) such that $A \subset C$, $B \subset D$ and $\delta(D/C) = p - q\alpha$. Pick a element $b_0 \in B$. Let $k = p - m$ and take $b_1, b_2, ..., b_k$ with the relations $S(b_0, b_1), S(b_1, b_2), ..., S(b_{k-1}, b_k)$. Let $l = q-n$ and take $a_1, a_2, ..., a_{l-k}$ with the relations $S(a_i, b_i)$ for $1 \le i \le l-k$. Let $C = Aa_1a_2...a_{l-k}$ and $D = Bb_1b_2...b_k$. By 2.5, $CD \in K$. On the other hand, $\delta(D/C) = \delta(B/A) + k - (k + l - k)\alpha = (m + k) - (n + l)\alpha = p - q\alpha$. Also we can see that $C \le CD$. (It can be shown as follows: Take any $X \subset D - C$ and let $X_C = X \cap C$ and $X_D = X \cap (D-C)$. Then $\delta(X/C) = \delta(X_B/C) + \delta(X_D/CX_B) = \delta(X_B/A) + \delta(X_D/CX_B)$. Note that $B \ge A$ and $\alpha > \frac{1}{2}$. Hence $\delta(X/C) \ge 0$.) It follows that (D, C) is normal.

Lemma 2.11 If α is irrational, then any minimal pair is not small.

Proof By 2.6 and 2.7, we have $\alpha \in \left(\frac{1}{3}, 1\right]$. If $\alpha > \frac{1}{2}$, then any minimal pair is not small by 2.10. If $\alpha < \frac{1}{2}$, then there is $n < \omega$ with $\alpha \in \left(\frac{n-1}{2n-1}, \frac{n}{2n+1}\right]$, and therefore any minimal pair is not small, by 2.8.

Lemma 2.12 Let $A \le AB \le M$. Let (B, A) be a minimal pair. If $\mathrm{tp}(B/A)$ is algebraic, then (B, A) is small.

Proof Suppose by way of contradiction that (B, A) is not small. Then there is a normal pair (D, C) such that $A \subset C, B \subset D$ and $\delta(D/C) < \delta(B/C)$. By minimality of (B, A) we can assume that (D, C) is minimal.
Claim 1: There is a sequence $(B_i)_{i<\omega}$ with the following conditions:
(i) $B_i \cong_{CB_0...B_{i-1}} B$ for any $i < \omega$;
(ii) $CB_0...B_i, CB_0...B_{i-1}D \le CB_0...B_iD \in K$ for any $i < \omega$;
(iii) $D, B_0, B_1, B_2, ...$ are pairwise disjoint.
Proof of Claim: We prove by induction. Suppose $(B_i)_{i\le n}$ has constructed. By (ii), we have $CB_0...B_n \le CB_0...B_nD \in K$, and therefore $CB_0...B_n \le CB_0...B_nB \in K$. So, by amalgamation property, we can take B_{n+1} so that

$B_{n+1} \cong_{CB_0 \ldots B_n} B$ and $CB_0 \ldots B_n D, CB_0 \ldots B_n B_{n+1} \leq CB_0 \ldots B_n B_{n+1} D \in K$. Thus B_{n+1} satisfies (i) and (ii). For (iii) it is enough to show that $B_{n+1} \cap D = \emptyset$. Suppose that $D' = B_{n+1} \cap D \neq \emptyset$. We have had $CB_0 \ldots B_n B_{n+1} \leq CB_0 \ldots B_n B_{n+1} D$, so $CD' \leq CD$. Note that $D' \neq D$. This contradicts minimality of (D, C). Hence $B_{n+1} \cap D = \emptyset$. (End of Proof of Claim 1)
Claim 2: $AB, AB_j \leq AB_0 \ldots B_i B$ for $j \leq i < \omega$
Proof: We prove by induction on i. By (ii) of claim 1, $AB_0 \ldots B_i B \leq AB_0 \ldots B_{i+1} B$. By induction hypothesis, we have $AB, AB_j \leq AB_0 \ldots B_i B$ for $j \leq i$. Hence $AB, AB_j \leq AB_0 \ldots B_{i+1} B$ for $j \leq i$. So, it is enough to show that $AB_{i+1} \leq AB_0 \ldots B_{i+1} B$. By induction hypothesis, we have $AB \leq AB_0 \ldots B_i B$. By (i) of claim 1, we have $AB_{i+1} \leq AB_0 \ldots B_{i+1}$. By (ii) of claim 1, $AB_0 \ldots B_{i+1} \leq AB_0 \ldots B_{i+1} B$. Hence we have $AB_{i+1} \leq AB_0 \ldots B_{i+1} B$. (End of Proof of Claim 2)

We show that $\mathrm{tp}(B/A)$ is non-algebraic. Fix any $n < \omega$. By claim 2, there are B_i^*'s such that $B_0^* \ldots B_n^* \cong_{AB} B_0 \ldots B_n$ and $AB \leq ABB_0^* \ldots B_n^* \leq M$. Again, by claim 2, $AB_i^* \leq ABB_0^* \ldots B_n^* \leq M$ for all $i \leq n$. Therefore we have $\mathrm{tp}(B_i^*/A) = \mathrm{tp}(B/A)$. By (iii) of claim 1, B_i^*'s are pairwise disjoint. Hence $\mathrm{tp}(B/A)$ is not algebraic.

Lemma 2.13 If α is irrational, then $\mathrm{acl}(X) = \mathrm{cl}(X)$ for any finite subset X of M.

Proof Take any finite subset X of M. Then $\mathrm{cl}(X) \subset \mathrm{acl}(X)$ is clear. We show $\mathrm{acl}(X) \subset \mathrm{cl}(X)$. If not, there is $a \in \mathrm{acl}(X) - \mathrm{cl}(X)$. Let $A = \mathrm{cl}(X)$ and $B = \mathrm{cl}(aX)$. Take a maximal chain $\{B_i\}_{i<\omega}$ with $A = B_0 \leq B_1 \leq \ldots \leq B_n = B$. Then, for each $i < \omega$, $(B_{i+1} - B_i, B_i)$ is minimal and $A \leq AB_i \leq M$. By 2.11, they are not small, and so $\mathrm{tp}(B_{i+1}/B_i)$ is not algebraic. In particular we have $B \not\subset \mathrm{acl}(A) = \mathrm{acl}(X)$. A contradiction.

Proof of Theorem Let M be a $(K, \leq)$-generic saturated pseudoplane for some $K \subset K_\alpha$. Suppose that M is strictly stable. By 2.3, α is irrational. By 2.13, $\mathrm{acl}(X) = \mathrm{cl}(X)$ for any finite $X \subset M$. Note that $\mathrm{cl}(X)$ is finite by (A4) of Axiom 1.1. Hence $\mathrm{acl}(X)$ is finite.

Question Are δ-generic pseudoplanes ω-categorical?

Reference

[1] J. T. Baldwin and N. Shi, Stable generic structures, Annals of Pure and Applied Logic 79 (1996) 1–35
[2] J. Goode, Hrushovski's geometries, In Helmut Wolter Bernd Dahn, editor, Proceedings of 7th Easter Conference on Model Theory (1989) 106–118
[3] E. Hrushovski, A stable $\aleph_0$-categorical pseudoplane, preprint, 1988

[4] E. Hrushovski, A new strongly minimal set, Annals of Pure and Applied Logic, 46 (1990) 235–264
[5] F. O. Wagner, Relational structures and dimensions, Kaye, Richard (ed.) et al., Automorphisms of first-order structures. Oxford: Clarendon Press. 153–180 (1994)

Department of Mathematics
Toyota National College of Technology
2-1, Eiseicho,Toyota, 471-8525, JAPAN
ikeda@toyota-ct.ac.jp