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Harmonic Relations between Green’s Functions and Green’s Matrices for Boundary Value Problems II

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1 Introduction

In previous papers [3], [4], we remarked that there is a harmonic relation between the Green functions $G(x, \xi)$ for

$$
\begin{align*}
-\frac{d}{dx}(p(x)\frac{du}{dx}) &= f(x), \quad a < x < b \\
u(a) &= u(b) = 0, \quad p(x) > 0 \text{ in } [a, b]
\end{align*}
$$

(1.1)

and the Green matrix $A_0^{-1} = (g_{ij})$ for the discretized system

$$
\begin{align*}
a &= x_0 < x_1 < \cdots < x_n < x_{n+1} = b, \quad x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1}) \\
p_{i+\frac{1}{2}} \frac{U_{i+1} - U_i}{h_{i+1}} - p_{i-\frac{1}{2}} \frac{U_i - U_{i-1}}{h_i} &= f_i, \quad h_i = x_i - x_{i-1}, \quad i = 1, 2, \cdots, n \\
U_0 &= U_{n+1} = 0
\end{align*}
$$

(1.2) (1.3)

or

$$
HA_0 U = f
$$

with

$$
H = \begin{pmatrix}
\frac{2}{h_1 + h_2} & & \\
& \ddots & \\
& & \frac{2}{h_n + h_{n+1}}
\end{pmatrix},
$$
\[ A_0 = \begin{pmatrix} a_1 + a_2 & -a_2 & -a_3 & \cdots & -a_n \\ -a_2 & a_2 + a_3 & -a_4 & \cdots & -a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & a_n + a_{n+1} & \cdots & \cdots & -a_{n-1} \end{pmatrix}, \quad a_i = \frac{1}{h_i} p_{i-\frac{1}{2}} \quad (1.4) \]

\[ U = (U_1, \cdots, U_n)^t, \quad f = (f_1, \cdots, f_n)^t \]

In fact, we have

\[ G(x_i, x_j) = \begin{cases} \left( \int_a^x \frac{ds}{p(s)} \right)^{-1} \int_a^x \frac{ds}{p(s)} \int_x^b \frac{ds}{p(s)} \left( \int_a^b \frac{ds}{p(s)} \right)^{-1} \int_a^x \frac{ds}{p(s)} \int_x^b \frac{ds}{p(s)} & (x \leq \xi) \\ \left( \int_a^x \frac{ds}{p(s)} \right)^{-1} \int_a^x \frac{ds}{p(s)} \int_x^b \frac{ds}{p(s)} \left( \int_a^b \frac{ds}{p(s)} \right)^{-1} \int_a^x \frac{ds}{p(s)} \int_x^b \frac{ds}{p(s)} & (x \geq \xi) \end{cases} \]

and

\[ g_{ij} = \begin{cases} \left( \sum_{k=1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}} \right)^{-1} \left( \sum_{k=1}^{i} \frac{h_k}{p_{k-\frac{1}{2}}} \right) \left( \sum_{k=j+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}} \right) & i \leq j \\ \left( \sum_{k=1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}} \right)^{-1} \left( \sum_{k=1}^{j} \frac{h_k}{p_{k-\frac{1}{2}}} \right) \left( \sum_{k=i+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}} \right) & i \geq j \end{cases} \]

Hence,

\[ G(x_i, x_j) = g_{ij} + O(h^2) \quad \forall i, j, \quad h = \max_i h_i, \]

if \( p \in C^{1,1}[a,b] \).

On the other hand, the finite element approximation \( v_n = \sum_{i=1}^n \phi_i \) with piecewise linear polynomials is determined by solving

\[ \sum_{j=1}^n \left( \int_a^b p(x) \phi_i'(x) \phi_j'(x) dx \right) \hat{U}_j = \int_a^b f(x) \phi_i(x) dx, \quad i = 1, 2, \cdots, n \quad (1.5) \]

with respect to \( \hat{U}_j \), where \( \phi_i, i = 1, 2, \cdots, n \) are piecewise linear polynomials satisfying \( \phi_i(x_j) = \delta_{ij} \). The equations(1.5) can be written in the matrix-vector form

\[ \hat{A} \hat{U} = \hat{f}, \]

where \( \hat{A} = \hat{A}_0 \) is obtained by replacing \( a_i \) in (1.4) by

\[ \hat{a}_i = \frac{1}{h_i} \rho_i, \quad \rho_i = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} p(x) dx, \]

\[ \hat{f} = (\hat{f}_1, \cdots, \hat{f}_n), \quad \hat{f}_i = \int_{x_{i-1}}^{x_{i+1}} f(x) \phi_i(x) dx. \]
Then it can also be shown that $A^{-1} = (g_{ij})$ satisfies

$$
g_{ij} = \begin{cases} 
\left(\sum_{k=1}^{n+1} \frac{h_k}{\rho_k}\right)^{-1} \left(\sum_{k=1}^{i} \frac{h_k}{\rho_k}\right) \left(\sum_{k=i+1}^{n+1} \frac{h_k}{\rho_k}\right) & i \leq j \\
\left(\sum_{k=1}^{n+1} \frac{h_k}{\rho_k}\right)^{-1} \left(\sum_{k=1}^{j} \frac{h_k}{\rho_k}\right) \left(\sum_{k=j+1}^{n+1} \frac{h_k}{\rho_k}\right) & i \geq j,
\end{cases}
$$

which indicates a similar harmony between the Green function $G(x, \xi)$ and the corresponding discrete Green function:

$$G(x_i, x_j) = g_{ij} + O(h^2) \quad \forall i, j,$$

if $p \in C^{1,1}[a, b]$.

The purpose of this paper is to establish a similar relation for the Green function $G(x, \xi)$ for

$$Lu \equiv -\frac{d}{dx}(p(x)\frac{du}{dx}) + q(x)\frac{du}{dx} + r(x)u = f(x), \quad a < x < b \quad (1.6)$$

$$u(a) = u(b) = 0$$

and the discrete Green function $G_h(x_i, x_j)$ (Green matrix) for the discretized system

$$\begin{cases} 
L_h U \equiv -\frac{p_{i+\frac{1}{2}}U_{i+1} - U_i}{h_{i+\frac{1}{2}}} - \frac{p_{i-\frac{1}{2}}U_{i} - U_{i-1}}{h_{i-\frac{1}{2}}} + q_i \frac{U_i - U_{i-1}}{h_i} + r_i U_i = f_i, \quad i = 1, 2, \ldots, n \quad (1.7) \\
U_0 = U_{n+1} = 0,
\end{cases}
$$

provided that $p(x) \in C^{3,1}$, $q(x), r(x) \in C^{1,1}[a, b]$, $p(x) > 0, r(x) \geq 0$ in $[a, b]$

### 2 Results

The discrete Green function $G_h(x_i, x_j)$ for the operator $L_h$ is defined as the solution of the linear system

$$\begin{cases} 
L_h G_h(x_i, x_j) = \frac{2}{h_{j+1} + h_j} \delta_{ij}, \quad i, j = 1, 2, \ldots, n \\
G_h(x_i, x_j) = 0, i = 0, n + 1, \quad 1 \leq j \leq n,
\end{cases}$$
where $\delta_{ij}$ stands for the Kronecker symbol. This means that the $n \times n$ matrix $(G_h(x_i, x_j))$ is the inverse of the matrix $A = A_0 + Q + D$, where $A_0$ is defined by (1.3),

\[
Q = \begin{pmatrix}
0 & 0 & \cdots & 0 & \frac{q_2}{2} \\
-\frac{q_2}{2} & 0 & \cdots & \frac{q_2}{2} \\
0 & \frac{q_2}{2} & \cdots & \frac{q_{n-1}}{2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{q_2}{2} & \frac{q_2}{2} & \cdots & 0
\end{pmatrix}, \quad D = \begin{pmatrix}
\frac{r_1(h_1 + h_2)}{2} & \frac{r_2(h_3 + h_4)}{2} & \cdots \\
\frac{r_2(h_3 + h_4)}{2} & \cdots & \frac{r_{n}(h_n + h_{n+1})}{2}
\end{pmatrix}
\]

We first prove the following lemma:

**Lemma 2.1.** Given positive integers $N_a$ and $N_b$, we have

\[
\sum_{j=1}^{n} G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} = \left\{ \begin{array}{ll}
O(h) & \text{if } i \leq N_a \text{ or } i \geq n+1-N_b, \\
O(1) & \text{otherwise.}
\end{array} \right.
\]

**Proof.** Let $\phi(x) \in C^2[a, b]$ be the solution of the problem $Lu = 1$ in $(a, b)$ and $u(a) = u(b) = 0$. Then we have

\[
\sum_{j=1}^{n} G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} \leq 2\phi(x_i) \quad \forall i
\]

(cf. Matsunaga-Yamamoto[2]), which proves Lemma 2.1. \hfill \Box

Then we have the following result.

**Theorem 2.2.** If $p \in C^{3,1}[a, b]$, $q(x), r(x) \in C^{1,1}[a, b]$, $p(x) > 0$, $r(x) \geq 0$ in $[a, b]$, then

\[
G_h(x_i, x_j) - G(x_i, x_j) = \left\{ \begin{array}{ll}
O(h^3) & (i \leq N_a \text{ or } i \geq n+1-N_b), \\
O(h^2) & (\text{otherwise}).
\end{array} \right.
\]

**Proof.** Let $\{V_i\}$ be any mesh function defined on $I = \{x_0, x_1, \cdots, x_n, x_{n+1}\}$ such that $V_0 = V_{n+1} = 0$. Then it is easy to see

\[
V_i = \sum_{j=1}^{n} G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} L_h V_j, \quad i = 1, 2, \cdots, n
\]

Hence

\[
G(x_i, x_j) = \sum_{k=1}^{n} G_h(x_i, x_k) \frac{h_k + h_{k+1}}{2} L_h G(x_k, x_j), \quad i, j = 1, 2, \cdots, n \tag{2.1}
\]
Furthermore, a careful computation leads to

\[
L_h G(x_k, x_j) = \begin{cases} 
\frac{2}{h_k + h_{k+1}} [(h_k^2 - h_{k+1}^2) \phi_{kj} + O(h_k^3 + h_{k+1}^3)] & (k \neq j) \\
\frac{2}{h_j + h_{j+1}} [1 + (h_{j+1}^2 \phi_j^+ - h_j^2 \phi_j^-) + O(h_{j+1}^3 + h_j^3)] & (k = j),
\end{cases}
\]

where

\[
\phi_{kj} = \frac{1}{6} p_k \frac{\partial^3 G(x_k, x_j)}{\partial x^3} + \frac{1}{4} (p_k' - q_k) \frac{\partial^2 G(x_k, x_j)}{\partial x^2} + \frac{1}{8} p_k' \frac{\partial G(x_k, x_j)}{\partial x},
\]

\[
\phi_j^+ = \frac{1}{4} q_j \frac{\partial^2 G(x_j + 0, x_j)}{\partial x^2} - \frac{1}{8} p_j' \frac{\partial G(x_j + 0, x_j)}{\partial x} - \frac{1}{4} p_j' \frac{\partial^2 G(x_j + 0, x_j)}{\partial x^2} - \frac{1}{6} p_j \frac{\partial^3 G(x_j + 0, x_j)}{\partial x^3},
\]

\[
\phi_j^- = \frac{1}{4} q_j \frac{\partial^2 G(x_j - 0, x_j)}{\partial x^2} - \frac{1}{8} p_j' \frac{\partial G(x_j - 0, x_j)}{\partial x} - \frac{1}{4} p_j' \frac{\partial^2 G(x_j - 0, x_j)}{\partial x^2} - \frac{1}{6} p_j \frac{\partial^3 G(x_j - 0, x_j)}{\partial x^3}.
\]

Substituting this relation into (2.1) yields

\[
G(x_i, x_j) = \sum_{k \neq j, k=1}^{n} G_h (x_i, x_k) \{(h_k^2 - h_{k+1}^2) \phi_{kj} + O(h_{k+1}^3 + h_k^3)\}
\]

\[
+ G_h (x_i, x_j) \{1 + (\phi_j^+ h_{j+1}^2 - \phi_j^- h_j^2) + O(h_{j+1}^3 + h_j^3)\}
\]

or

\[
G(x_i, x_j) - G_h (x_i, x_j) = \sum_{k \neq j} G_h (x_i, x_j) \{(h_k^2 - h_{k+1}^2) \phi_{kj} + O(h_{k+1}^3 + h_k^3)\}
\]

\[
+ G_h (x_i, x_j) \{(\phi_j^+ h_{j+1}^2 - \phi_j^- h_j^2) + O(h_{j+1}^3 + h_j^3)\}
\]

Hence there exists a constant $C_1 > 0$ such that

\[
|G(x_i, x_j) - G_h (x_i, x_j)| \leq C_1 h \sum_{k \neq j, k=1}^{n} G_h (x_i, x_k) (h_k + h_{k+1}) + O(h^2),
\]

and, by Lemma 2.1, we have

\[
G_h (x_i, x_j) = G(x_i, x_j) + O(h).
\]
Substituting this into (2.2) we have

\[G(x_i, x_j) - G_h(x_i, x_j) = \sum_{k \neq j} G(x_i, x_k) (h_k^2 - h_{k+1}^2) \phi_{kj} + G(x_i, x_j) (\phi_j^+ h_{j+1}^2 - \phi_j^- h_j^2) + O(h^2)\]

\[= h_1^2 G(x_i, x_1) \phi_{1j} + \sum_{k=1}^{j-2} h_{k+1}^2 [G(x_i, x_{k+1}) \phi_{k+1j} - G(x_i, x_k) \phi_{kj}] - h_j^2 [G(x_i, x_{j-1}) \phi_{j-1j} + G(x_i, x_j) \phi_j^-] + h_{j+1}^2 [G(x_i, x_j) \phi_j^+ + G(x_i, x_{j+1}) \phi_{j+1j}] + \sum_{k=j+1}^{n-1} h_{k+1}^2 [G(x_i, x_{k+1}) \phi_{k+1j} - G(x_i, x_k) \phi_{kj}] - h_{n+1}^2 [G(x_i, x_n) \phi_{nj}] + O(h^2)\]

\[= O(h^2) \quad \text{(an improvement of (2.3))}\]

since

\[G(x_i, x_{k+1}) \phi_{k+1j} - G(x_i, x_k) \phi_{kj} = [G(x_i, x_{k+1}) - G(x_i, x_k)] \phi_{k+1j} + G(x_i, x_k) [\phi_{k+1j} - \phi_{kj}] = O(h_{k+1}) \phi_{k+1j} + G(x_i, x_k) h_{k+1} = O(h_{k+1}), \quad \text{etc.}\]

Replacing \(O(h)\) in (2.3) by \(O(h^2)\) and repeating similar argument as above, we obtain for \(i \leq N_a\) or \(i \geq n + 1 - N_b\)

\[G(x_i, x_j) - G_h(x_i, x_j) = O(h^3).\]

This proves Theorem 2.2. \(\square\)

We can apply Theorem 2.2 to derive the superconvergence of the Shortley-Weller approximation applied to the semilinear problem

\[
\begin{cases}
- \frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x) \frac{du}{dx} + f(x, u) = 0, & a < x < b \\
u(a) = \alpha, \quad u(b) = \beta
\end{cases}
\]

with any nodes (1.2):

**Theorem 2.3.** In addition to the assumptions of Theorem 2.2, assume that \(f\) is continuous on \(\mathcal{R} : a \leq x \leq b, \quad -\infty < u < +\infty\). Furthermore, assume that \(f\) is continuously differentiable with respect to \(u\) on \(\mathcal{R}\) and \(f_u \geq 0\). Then the finite difference method

\[
\begin{cases}
- \frac{p_{i+\frac{1}{2}} U_{i+1} - U_i}{h_{i+1}} - \frac{p_{i-\frac{1}{2}} U_i - U_{i-1}}{h_i} + q_i \frac{U_{i+1} - U_{i-1}}{h_{i+1} + h_i} + f(x_i, U_i) = 0, & i = 1, 2, \ldots, n, (2.6) \\
U_0 = \alpha, \quad U_{n+1} = \beta
\end{cases}
\]
for solving (2.4)–(2.5) is superconvergent with any nodes (1.2):

\[
    u_i - U_i = \begin{cases} 
        O(h^3), & i \in \Gamma = \{1, 2, \cdots, N_a, n - N_b + 1, n - N_b + 2, \cdots, n\} \\
        O(h^2), & i \notin \Gamma 
    \end{cases}
\]

as \( h \to 0 \), where \( N_a \) and \( N_b \) are arbitrary given positive integers.

Remark. If the boundary conditions (2.5) are replaced by

\[
    \alpha_1 u(a) + \alpha_2 u'(a) = \alpha \quad \text{and} \quad \beta_1 u(b) + \beta_2 u'(b) = \beta,
\]

where \( \alpha_2 \beta_2 \neq 0 \), \( \alpha_1 \alpha_2 \geq 0 \) and \( \beta_1 \beta_2 \geq 0 \), then it can be shown that the corresponding Shortley-Weller approximation (2.6) is quadratic convergent with any nodes (1.2):

\[
    u_i - U_i = O(h^2) \quad \forall i
\]

as \( h \to 0 \). However, superconvergence can not be expected in general.

References


