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# A level set method for a growth of a crystal by screw dislocations

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## 1 Introduction

In this paper we introduce a new level set model for the growth of spirals on the surface of a crystal. Since the conventional method level set method cannot express a spiral curve, we modify the level set method by using a sheet structure function. Since the model equation we obtain is a degenerate parabolic type, we need to consider a notion of weak solution. We shall prove the existence and the uniqueness of the solution for our model in the sence of viscosity solutions.

The theory of spiral crystal growth was proposed by F. C. Frank in 1948(see [BCF1]). He first pointed out that dislocations play an important role in the theory of crystal growth. He especially pointed out the importance of the role of a screw dislocation. In his theory, if a screw dislocation terminates in the exposed surface of a crystal, there is a permanently exposed *cliff* of atoms, say the *step*. The step can grow perpetually *up a spiral staircase*, When one observe the surface from above, one can find spirals drawn by exposed *edge of the step*. He proposed an evolution equation of curves which indicates the location of edges of steps. The equation he proposed is of the form

$$V = C - \kappa, \tag{1.1}$$

where $V$ is a normal velocity of the steps, $\kappa$ is a curvature of the curve corresponding to the edge of steps, and $C$ is the driving force of steps (see [BCF2]). The sign of the curvature is taken so that the problem is parabolic. The curvature term is interpreted as a result of the Gibbs-Thomson effect. We postulate that steps moves under (1.1), and we construct a new mathematical model based on (1.1). The formula (1,1), says the *geometric model*, performs the model of spiral crystal growth for only one screw dislocation. However, it is not enough to handle other situation when there are two or more screw dislocations on the surface of the crystal and curves generated from each screw dislocations may touch each other. We would like to handle such a situation by adjusting the model. There are at least two methods to realize our purpose. One is the Allen–Cahn equation model, and the other is a level set method for geometric model. In this paper we propose a model reflecting a level set method.

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, which denotes the surface of the crystal. For technical reasons we postulate that a screw dislocation is a close disk on the surface. We also assume that all screw dislocations do not touch each other nor the boundary of $\Omega$. We denote by $W$ the complement of all screw dislocations in the surface of the crystal. We denote by $\Gamma_t$ the curve corresponding to edges of steps at time $t$.

In conventional level set approach to (1.1), we denote the evolving curve by the zero-level set of auxiliary function $u$, i.e.,.

$$\Gamma_t = \left\{x \in \overline{W};\ u(t,x) = 0\right\},$$

In this way, however, we cannot distinguish the direction of moving steps. To overcome this difficulty, we recall *sheet structure function* due to R. Kobayashi(See [Ko]).

We postulate that there are $n$ screw dislocations on the crystal surface. Let $a_j$ denote the position of the center of $j$-th screw dislocation. Let $\rho_j$ denotes the radius of $j$-th screw dislocation. We denote by $W$ the complement of all screw dislocations in the surface of the crystal, i.e.,

$$W = \Omega \setminus \Big(\bigcup_{j=1}^{n} \overline{B_{\rho_j}(a_j)}\Big),$$

where $B_\rho(a)$ denotes an open disk of radius $\rho$ centered at $a$. We recall the sheet structure function $\theta$ defined by

$$\theta(x) = \sum_{j=1}^{n} m_j \arg(x - a_j),$$

where $m_j \neq 0$ is an integer such that $|m_j|$ denotes the height of steps and the sign of $m_j$ denotes the direction of steps. We remark that each arguments of $x - a_j$ is multi-valued. We consider an auxiliary function $u = u(t,x)$ defined on $[0,+\infty) \times \overline{W}$. We interpret $\Gamma_t$ as a level set of $u - \theta$ instead of $u$ itself, i.e.,

$$\Gamma_t = \left\{x \in \overline{W};\ u(t,x) - \theta(x) = 0 \mod 2\pi m\right\},$$

where $m$ is the greatest common divisor of $\{|m_j|\}_{j=1}^{n}$.

By the definition of $\Gamma_t$ we formally observe that

$$V = \frac{1}{|\nabla(u-\theta)|}\frac{\partial u}{\partial t},$$

$$\kappa = -\mathrm{div}\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|}.$$

We remark that $\nabla\theta$ is single-valued, so this formula is well-defined. We now obtain the level set model consisting with geometric model of the form

$$\frac{\partial u}{\partial t} - |\nabla(u-\theta)| \left( \operatorname{div} \frac{\nabla(u-\theta)}{|\nabla(u-\theta)|} + C \right) = 0 \quad \text{in } (0,T) \times W, \tag{1.2}$$

To complete the problem we need some boundary condition on $\partial W$. Here we postulate the Neumann boundary condition at the edge of $\Gamma_t$ touching $\partial W$ of the form

$$\langle \vec{\nu}(x), \nabla(u-\theta) \rangle = 0 \quad \text{on } (0,T) \times \partial W. \tag{1.3}$$

where $\vec{\nu}$ denotes a unit normal vector field of $\partial W$, and $\langle \cdot, \cdot \rangle$ is the inner product of $\mathbb{R}^2$. Since the equation (1.2) is degenerate parabolic, we need to consider the solution of these equation in weak sense. We consider the solution in viscosity sense.

Our goal is to prove the comparison principle, existence and uniqueness of a viscosity solution for (1.2)–(1.3). The equation (1.2) has a moving singularity at $\nabla u(t,x) = \nabla\theta(x)$ so it is hard to prove the comparison principle directly. To overcome this difficulty we introduce a *covering spaces* of $\overline{W}$ and $\overline{W} \times \overline{W}$ so that $u - \theta$ and $v - \theta$ respectively be a sub- and supersolution of

$$\frac{\partial u}{\partial t} - |\nabla u| \left\{ \operatorname{div} \frac{\nabla u}{|\nabla u|} + C \right\} = 0 \tag{1.4}$$

$$\langle \vec{\nu}, \nabla u \rangle = 0 \tag{1.5}$$

if $u$ and $v$ respectively be a sub- and supersolution of (1.2)–(1.3). We test $u(t,x) - \theta(x) - (v(t,y) - \theta(y))$ by standard test function by [GS1] but on the covering space. Then we apply the results for (1.4)–(1.5) in [GS1]. Once we obtain the comparison principle for (1.2)–(1.3), then it is easy to see a uniqueness of a viscosity solution for (1.2)–(1.3). It remains to prove the existence of a viscosity solution with a desired initial data. We construct a viscosity sub- and supersolution according to a Perron's method due to H. Ishii(see [I]). Perron's method for a second order equation with Neumann boundary condition is found by [Sa]. So we apply a results of [Sa]. In our problem, however, some difficulties lie in the term of $\theta$. To overcome these difficulties, we first construct sub- and supersolutions on some small neighborhood of each points of $W$. Next we extend their domain of definition to $\overline{W}$ by using *Invariance Lemma*(see [GS2]). We apply the Perron's method.

We take this opportunity to mention somewhat related results. In [GIK] the uniqueness and existence of a *spiral solution* for a geometric model which includes a anisotropy is proved. In [KP] a Allen–Cahn model for spiral crystal growth is introduced. They also showed numerical computations. In [Ko] a Allen–Cahn model including more generalized situations than that in [KP] is introduced He also showed numerical computations. He introduced a *sheet structure function* in this model. We utilize his idea for expressing a edge of steps by level set method. In [NO] a existence of *spiral traveling wave solution* for Kobayashi's model on a annulus is proved. A level set model different from ours are introduced by [Sm]. He expresses a location of edges of steps by using 2 auxiliary function, one denotes a existence of steps, and the other denotes a location of edges of steps. He also showed numerical computation. His model cannot treat a situation of that, for examples, there are 2 screw dislocations and steps generated from each screw dislocations and a height of steps is different from each other. Our model includes such a situation.

Analytic foundation based on the theory of viscosity solution [CIL] has established by [CGG], [ES]. It is extended to the Neumann boundary problem by [GS1] and [Sa]. From technical point of view we use the method developed by [GS1] and [Sa] although it does not apply to our settings directly.

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# 2 Main results

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with $C^2$ boundary $\partial\Omega$. We take $a_1, \dots, a_n \in \Omega$ and $\rho_1, \dots, \rho_n > 0$ satisfying

$$\overline{B_{\rho_j}(a_j)} \subset \Omega \quad \text{for} \quad j = 1, 2, \dots, n, \tag{2.1}$$

$$\overline{B_{\rho_i}(a_i)} \cap \overline{B_{\rho_j}(a_j)} = \emptyset \quad \text{for} \quad i, j = 1, 2, \dots, n, \ i \neq j, \tag{2.2}$$

where $B_{\rho_j}(a_j) = \{x \in \mathbb{R}^2; |x - a_j| < \rho_j\}$ and $\overline{D} \subset \mathbb{R}^k$ denotes the closure of $D$ in $\mathbb{R}^k$. We set

$$W = \Omega \setminus \bigcup_{j=1}^{n} \overline{B_{\rho j}(a_j)}.$$

We introduce a multi–valued function on $\mathbb{R}^2 \setminus \{a_1, \dots, a_n\}$ defined by

$$\theta(x) = \sum_{j=1}^{n} m_j \arg(x - a_j),$$

where $m_j$ is an integer and $\arg(x - a_j)$ is an argument of $x - a_j$, which is regarded as a multi–valued function.

We consider the equation of the form

$$\frac{\partial u}{\partial t} - |\nabla(u-\theta)| \left\{ \operatorname{div} \frac{\nabla(u-\theta)}{|\nabla(u-\theta)|} + C \right\} = 0 \quad \text{in} \quad (0,\infty) \times W, \tag{2.3}$$

$$\langle \vec{\nu}, \nabla(u-\theta) \rangle = 0 \quad \text{on} \quad (0,\infty) \times \partial W, \tag{2.4}$$

where $C$ is a positive constant, and vector field $\vec{\nu}$ is a outer normal unit vector field of $\partial W$ and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^2$. We remark that (2.3) is well–defined on $W$ since $D\theta$ is single-valued.

We consider equations (2.3)–(2.4) in the viscosity sense. For $f\colon D(\subset \mathbb{R}^k) \to \mathbb{R}$ we denote respectively by $f_*$, $f^*$ lower and upper semicontinuous envelope of $f$ defined by

$$\begin{aligned} f_*\colon \quad & \overline{D} \to \mathbb{R} \cup \{\pm\infty\}, \\ & z \mapsto f_*(z) = \lim_{r\downarrow 0} \inf\{f(\omega); |z-\omega| < r\}, \\ f^*\colon \quad & \overline{D} \to \mathbb{R} \cup \{\pm\infty\}, \\ & z \mapsto f^*(z) = \lim_{r\downarrow 0} \sup\{f(\omega); |z-\omega| < r\}. \end{aligned}$$

We are now in position to state our main results.

**Theorem 2.1 (Comparison Principle)**
*Let $u, v\colon (0,T) \times \overline{W} \to \mathbb{R}$ respectively be a viscosity sub- and supersolutions of (2.3)–(2.4) in $(0,T) \times W$ for $T > 0$. If*

$$u^*(0,x) \le v_*(0,x) \quad \textit{for} \quad x \in \overline{W},$$

*then*

$$u^*(t,x) \le v_*(t,x) \quad \textit{for} \quad (t,x) \in (0,T) \times \overline{W}.$$

**Theorem 2.2 (Existence and Uniqueness)**
*For a given $u_0 \in C(\overline{W})$, there exist a unique global viscosity solution $u \in C([0,\infty) \times \overline{W})$ with initial data*

$$u|_{t=0} = u_0 \quad \textit{on} \quad \overline{W}.$$

**Remark 2.3 (Generalization of the equation)**
*The equation (2.3) is written by*

$$\frac{\partial u}{\partial t} + F(\nabla(u-\theta), \nabla^2(u-\theta)) = 0 \tag{2.5}$$

*with* $F\colon (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{S}_2 \to \mathbb{R}$ *defined by*

$$F(p, X) = -tr\left\{\left(I_2 - \frac{p \otimes p}{|p|^2}\right) X\right\} - C\,|p|\,, \tag{2.6}$$

*where* $\mathbb{S}_2$ *is the space of symmetric* $2 \times 2$ *matrices,* $I_k$ *is an identity* $k \times k$ *matrix and* $\otimes$ *denotes a tensor product of vectors in* $\mathbb{R}^2$*. This function* $F$ *satisfies the following property.*

*(F1)* $F\colon (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{S}_2 \to \mathbb{R}$ *is continuous.*

*(F2) (Degenerate elliptic) For all* $\lambda > 0$ *and* $\mu \in \mathbb{R}$*,*

$$F(\lambda p, \lambda X + \mu p \otimes p) = \lambda F(p, X)$$

*holds for all* $p \in \mathbb{R}^2 \setminus \{0\}$ *and* $X \in \mathbb{S}_2$*.*

*(F3)* $-\infty < F_*(0, O) = F^*(0, O) < +\infty$*.*

*(F4) There exists positive constants* $K_1, K_2, K_3$ *and* $K_4$ *such that the following holds;*

*Suppose that* $X, Y \in \mathbb{S}_2$ *and non-negative constants* $\nu_0, \mu, \zeta$ *satisfy*

$$\langle pX, p\rangle + \langle qY, q\rangle \le \nu_0|p - q|^2 + \mu(|p|^2 + |q|^2) + \zeta|p - q|(|p| + |q|)$$

*for all* $p, q \in \mathbb{R}^2$*. Then the following holds;*

$$\begin{aligned} &F(p, X) - F(q, Y) \ge -K_1\nu_0|\bar{p} - \bar{q}|^2 - K_2\mu - K_3\zeta|\bar{p} - \bar{q}| - K_4|p - q| \\ &\text{for all} \quad p, q \in \mathbb{R}^2 \setminus \{0\}, \end{aligned}$$

*where* $\bar{p} = \frac{p}{|p|}$*.*

*Our results extend to general equation* (2.5) *provided that* $F$ *satisfies properties (F1)–(F4). In paticular it applies that anisotropic curvature flow motion of spilrals of the form*

$$b(\mathbf{n})V = -\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\frac{\partial H}{\partial p_j}(\mathbf{n}) + C \quad \text{on } \Gamma_t,$$

*where* $b \in C(\mathbb{R}^2 \setminus \{0\})$ *is positive on* $S^1$ *and* $H \in C^2(\mathbb{R}^2 \setminus \{0\})$ *is positively homogeneous of degree 1. In fact, our results can extend to the equation* (2.5) *for*

$$\begin{gathered} F(p, X) = -\mathrm{tr}\,\{A(\bar{p})X\} + B(p) \\ A(\bar{p}) = \frac{1}{b(-\bar{p})}\nabla^2 H(-\bar{p}), \quad B(p) = \frac{-c\,|p|}{b(-\bar{p})}, \quad \bar{p} = \frac{p}{|p|}. \end{gathered} \tag{2.7}$$

*It is easy to show that* (2.7) *satisfies (F1)–(F4).*

# 3 Comparison principle

As usual, we suppose that

$$\sigma = \max\left\{u^*(t,x) - v_*(t,x); (t,x) \in [0,T] \times \overline{W}\right\} > 0, \tag{3.1}$$

and we lead a contradiction. To lead a contradiction, we use a maximum principle for semicontinuous functions(see [CIL, Theorem 8.3]). However, if we would use that directly to our problems, we would have to handle the problem with moving singularity in $\nabla u$. In fact, the equation is singular at $\nabla u(t,x) = \nabla\theta(x)$ depending on $x$. We are tempting to consider $u - \theta$ instead of $u$, i.e., we are tempting to handle the function

$$\Phi(t,x,y) = u^*(t,x) - \theta(x) - (v_*(t,y) - \theta(y)) - \Psi(t,x,y) \tag{3.2}$$

instead of $\Phi(t,x,y) = u^*(t,x) - v_*(t,x) - \Psi(t,x,y)$. However, this function is multi-valued. So we have to localize a domain of $\Phi$ so that $\Phi$ has a maximum value. To determine a domain of $\Phi$ in a suitable way, we introduce some *covering space* so that $\theta$ is single-valued.

To overcome the difficulty caused by the Neumann boundary condition we choose a *good* test function as in [GS1].

## 3.1 Test function

We shall define a *good* test function as in [GS1] to lead a contradiction.

Since $\partial W$ is $C^2$, there is a positive constant $C_0$ such that

$$\langle \vec{\nu}(x), x - y\rangle \geq -C_0 |x-y|^2 \quad \text{for } x \in \partial W,\ y \in \overline{W}. \tag{3.3}$$

Moreover, for all $\beta > 0$, there exists $\varphi \in C^2(\overline{W})$ satisfying

$$-\frac{\beta}{2} < \varphi < 0 \quad \text{in } W,\ \varphi = 0 \quad \text{on } \partial W, \tag{3.4}$$

$$\vec{\nu} = \frac{\nabla\varphi}{|\nabla\varphi|} \quad \text{on } \partial W, \tag{3.5}$$

We fix $\beta > 0$ and take $\varphi \in C^2(\overline{W})$ satisfying (3.4)–(3.5) and

$$|\nabla\varphi| \geq \max\{8C_0\beta, 1\} \quad \text{on } \partial W. \tag{3.6}$$

For $\varepsilon > 0$, $\delta > 0$ and $\gamma > 0$, we define

$$\begin{aligned}
\Psi(t,x,y) &= \frac{\Xi(x,y)}{\varepsilon} + \delta G(x,y) + \frac{\gamma}{T-t}, && (3.7)\\
\Xi(x,y) &= |x-y|^4 G(x,y), && (3.8)\\
G(x,y) &= \varphi(x) + \varphi(y) + 2\beta. && (3.9)
\end{aligned}$$

See [GS1] to know some properties which $\Psi$ holds.

## 3.2 Covering space

We introduce a covering space so that $u - \theta$ is viewed as a single valued function. We set

$$\mathfrak{X} = \left\{ (x, \xi) \in \overline{W} \times \mathbb{R}^n; \begin{array}{l} \xi = (\xi_1, \xi_2, \ldots, \xi_n), \\ x - a_j = |x - a_j| \, (\cos \xi_j, \sin \xi_j) \ (j = 1, 2, \ldots, n) \end{array} \right\}$$

We define $u_\theta, \ v_\theta \colon [0, T] \times \overline{W} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} u_\theta(t, x, \xi) &= u^*(t, x) - \sum_{j=1}^{n} m_j \xi_j, \\ v_\theta(t, x, \xi) &= v_*(t, x) - \sum_{j=1}^{n} m_j \xi_j. \end{aligned}$$

If we restrict the definition of $u_\theta$ on $[0, T] \times \overline{\mathfrak{X}}$, we can consider $\theta(x)$ formally

$$\theta(x) = u(t, x) - u_\theta(t, x, \xi).$$

We still denote by $u_\theta$ and $v_\theta$ their restriction in $[0, T] \times \overline{\mathfrak{X}}$.

We define $\tilde{\Phi} \colon [0, T) \times \overline{\mathfrak{X}} \times \overline{\mathfrak{X}} \to \mathbb{R}$ by

$$\tilde{\Phi}(t, x, \xi, y, \eta) = u_\theta(t, x, \xi) - v_\theta(t, y, \eta) - \Psi(t, x, y),$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ and $\Psi$ is defined in the previous section. Since $\tilde{\Phi}$ is not bounded because of the term of arguments, we introduce a new covering space $\mathfrak{Y}$ instead of $\mathfrak{X} \times \mathfrak{X}$:

$$\mathfrak{Y} = \left\{ (x, \xi, y, \eta) \in \overline{\mathfrak{X}} \times \overline{\mathfrak{X}}; \ \xi_j - \pi \le \eta_j \le \xi_j + \pi \ (j = 1, 2, \ldots, n) \right\}.$$

We consider $\tilde{\Phi}$ on $[0, T) \times \overline{\mathfrak{Y}}$ rather than on $[0, T) \times \overline{\mathfrak{X}} \times \overline{\mathfrak{X}}$. On this set $\arg(x - a_j)$ and $\arg(y - a_j)$ take same branch of arguments.

We shall prove a existence of maximum value of $\tilde{\Phi}$ on $[0, T) \times \mathfrak{Y}$. We need to consider a subset $\mathfrak{Z} \subset \mathfrak{Y}$ defined by

$$\mathfrak{Z} = \left\{ (x, \xi, y, \eta) \in \mathfrak{Y}; \ 0 \le \xi_j < 2\pi \ (j = 1, 2, \ldots, n) \right\}.$$

**Proposition 3.1**
*The function $\tilde{\Phi}$ has a maximum value on $[0,T) \times \mathfrak{Y}$ and*

$$\max_{[0,T)\times\mathfrak{Y}} \tilde{\Phi} = \max_{[0,T)\times\mathfrak{Z}} \tilde{\Phi}.$$

*Proof.*
It suffices to consider $\tilde{\Phi}$ on $[0,T) \times \overline{\mathfrak{Z}}$. Since $\Psi > 0$ we observe that

$$\begin{aligned}\tilde{\Phi}(t,x,\xi,y,\eta) &\le u_\theta(t,x,\xi) - v_\theta(t,x,\eta) \\ &\le \max_{[0,T]\times\overline{W}} u^* - \min_{[0,T]\times\overline{W}} v_* + \pi\sum_{j=1}^{n}|m_j| < \infty.\end{aligned}$$

Thus $\tilde{\Phi}$ is bounded from above. Then there exists a sequence $\{(t_j,x_j,\xi^j,y_j,\eta^j)\} \subset [0,T) \times \mathfrak{Z}$ satisfying

$$\lim_{j\to\infty} \tilde{\Phi}(t_j,x_j,\xi^j,y_j,\eta^j) = \sup_{[0,T)\times\mathfrak{Z}} \tilde{\Phi}.$$

Since $(t_j,x_j,\xi^j,y_j,\eta^j) \in [0,T) \times \mathfrak{Z} \subset [0,T] \times \overline{\mathfrak{Z}}$, we may assume that

$$t_j \to \hat{t} \in [0,T],\ (x_j,\xi^j,y_j,\eta^j) \to (\hat{x},\hat{\xi},\hat{y},\hat{\eta}) \in \overline{\mathfrak{Z}} \quad \text{as } j \to \infty$$

by taking a subsequence of $(t_j,x_j,\xi^j,y_j,\eta^j)$. If $\hat{\xi}_j = 2\pi$ for some $j$ we can consider $\hat{\xi}_j = 0$ by replacing $\hat{\eta}_j$ with $\hat{\eta}_j - 2\pi$. Therefore it suffices to prove $\hat{t} < T$.

Suppose that $\hat{t} = T$. Then we get

$$\tilde{\Phi}(t_j,x_j,\xi^j,y_j,\eta^j) \le \max_{[0,T]\times\overline{W}} u^* - \min_{[0,T]\times\overline{W}} v_* + \pi\sum_{j=1}^{n}|m_j| - \frac{\gamma}{T-t_j}.$$

Since $\frac{\gamma}{T-t_j} \to -\infty$ as $j \to \infty$, we obtain

$$\lim_{j\to\infty} \tilde{\Phi}(t_j,x_j,\xi^j,y_j,\eta^j) = -\infty.$$

This contradicts $\sup_{[0,T]\times\overline{\mathfrak{Z}}} \tilde{\Phi} > -\infty$. □

We denote by $(\hat{t},\hat{x},\hat{\xi},\hat{y},\hat{\eta}) \in [0,T) \times \mathfrak{Y}$ the maximum point of $\tilde{\Phi}$ over $[0,T) \times \mathfrak{Y}$, i.e.,

$$\Phi(\hat{t},\hat{x},\hat{\xi},\hat{y},\hat{\eta}) = \max_{[0,T)\times\mathfrak{Y}} \Phi. \tag{3.10}$$

The next proposition is standard once we know that $\tilde{\Phi}$ is taken its maximum on $[0,T)\times\mathfrak{Y}$.

**Proposition 3.2**
*Assume that*

$$\sigma = \max_{[0,T]\times\overline{W}} (u^* - v_*) > 0.$$

*Let* $(\hat{t}, \hat{x}, \hat{\xi}, \hat{y}, \hat{\eta}) \in [0,T) \times \mathfrak{Y}$ *be taken as (3.10).*

(i) *There exists constants* $\delta_0 > 0$ *and* $\gamma_0 > 0$ *such that the estimate of the form*

$$\max_{[0,T)\times\mathfrak{Y}} \tilde{\Phi} > \frac{\sigma}{2}$$

*holds for* $0 < \varepsilon < 1$, $0 < \delta < \delta_0$ *and* $0 < \gamma < \gamma_0$

(ii) $|\hat{x} - \hat{y}| \to 0$ *uniformly as* $\varepsilon \to 0$ *on* $0 < \delta < \delta_0$ *and* $0 < \gamma < \gamma_0$.

(iii) $\Xi(\hat{x}, \hat{y})/\varepsilon \to 0$ *uniformly as* $\varepsilon \to 0$ *on* $0 < \delta < \delta_0$ *and* $0 < \gamma < \gamma_0$.

(iv) *Suppose that* $u^*(0,x) \le v_*(0,x)$ *for* $x \in \overline{W}$. *Then there exists a constant* $\varepsilon_0 > 0$ *such that*

$$\hat{t} > 0 \text{ for } 0 < \varepsilon < \varepsilon_0.$$

We can prove Proposition 3.2 by using a standard arguments of the theory of the viscosity solution. But we need to modifty the standard argument to prove Proposition 3.2(iii) because of the term of $\xi_j - \eta_j$.

By Proposition 3.2 (ii) and the compactness of $\overline{W}$ we may assume that

$$\hat{x}(\varepsilon, \delta), \hat{y}(\varepsilon, \delta) \to \bar{x}(\delta) \quad \text{as } \varepsilon \to 0$$

by taking a subsequence of $\varepsilon$. We moreover may assume that

$$\bar{x}(\delta) \to x_0 \in \overline{W} \quad \text{as } \delta \to 0$$

by taking subsequence $\delta$. We set

$$\rho_0 = \min\{\rho_1, \rho_2, \dots, \rho_n\}$$

and

$$U_{\rho_0}(x_0) = B_{\rho_0}(x_0) \cap \overline{W},$$

where $B_{\rho_0}(x_0) = \{x \in \mathbb{R}^2; |x - x_0| < \rho_0\}$. We are now in position to define $\theta(x)$. We now fix

$$\alpha_j \in \{\xi_j + 2k\pi; k \in \mathbb{Z},\ 0 \le \xi < 2\pi,\ x_0 - a_j = |x_0 - a_j|\,(\cos\xi_j, \sin\xi_j)\},$$

and we define $\psi_j \colon [\alpha_j - \frac{\pi}{2}, \alpha_j + \frac{\pi}{2}] \to \mathbb{S}^1$ by

$$\psi_j(\alpha) = (\cos\alpha, \sin\alpha).$$

We define $\theta_j \colon U_{\rho_0}(x_0) \to [\alpha_j - \frac{\pi}{2}, \alpha_j + \frac{\pi}{2}]$ by

$$\theta_j(x) = \psi^{-1}\left(\frac{x - a_j}{|x - a_j|}\right),$$

We note that $\theta_j$ is single-valued and $\theta_j \in C^2(U_{\rho_0}(x_0))$. We define $\theta \colon U_{\rho_0}(x_0) \to \mathbb{R}$ by

$$\theta(x) = \sum_{j=1}^{n} \theta_j(x).$$

We define $\Phi \colon [0, T) \times U_{\rho_0}(x_0) \times U_{\rho_0}(x_0) \to \mathbb{R}$ so that

$$\Phi(t, x, y) = u^*(t, x) - \theta(x) - (v_*(t, x) - \theta(x)) - \Psi(t, x, y) \tag{3.11}$$

for $0 < \varepsilon < \varepsilon_1$, $0 < \delta < \delta_1$ and $0 < \gamma < \gamma_0$, where $\varepsilon_1$, $\delta_1 > 0$ satisfy the following:

$$\hat{x}(\varepsilon, \delta), \hat{y}(\varepsilon, \delta) \in U_{\rho_0}(x_0)$$

for $0 < \varepsilon < \varepsilon_1$ and $0 < \delta < \delta_1$.

**Proposition 3.3**
*The function $\Phi$ attains its maximum on $[0, T) \times U_{\rho_0}(x_0) \times U_{\rho_0}(x_0)$ at $(\hat{t}, \hat{x}, \hat{y})$.*

*Proof.*
This follows from

$$\tilde{\Phi}(\hat{t}, \hat{x}, \hat{\xi}, \hat{y}, \hat{\eta}) = \Phi(\hat{t}, \hat{x}, \hat{y}).\square$$

By the above preparation it sufficies to apply the result in [GS1, Theorem 2.1] to prove Theorem 2.1. But their proof has a small flaw (p. 1224, line 6). They argued that $A \leq B$ implies $A^2 \leq B^2$, but this is not true for matrices. One should replace the righthand of matrix inequality by

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq A + \lambda A^2,$$

where $A = \nabla^2_{x,y}\Psi(\hat{t}, \hat{x}, \hat{y})$. Fortunately the remaining argument is similar.

# 4 Construction of a solution

In this section, we prove the existence of a viscosity solution for the initial-boundary value problem applying Perron's method. For that purpose, we construct a subsolution (denoted by $f(t,x)$) and a supersolution (denoted by $g(t,x)$) satisfying

$$f(t,x) \le g(t,x) \quad \text{for } (t,x) \in (0,T) \times \overline{W}, \tag{4.1}$$

with some positive $T$ independent of $u_0 \in C(\overline{W})$ and satisfying the initial condition, i.e.,

$$f(0,x) = g(0,x) = u_0(x) \in C(\overline{W}) \quad \text{for } x \in \overline{W}, \tag{4.2}$$

with the continuity at time zero:

$$f \text{ and } g \text{ are continuous at } t = 0. \tag{4.3}$$

The solution constructed by Perron's method satisfies the initial condition.

We construct $f$ and $g$ satisfying (4.1), (4.2) and (4.3). The construction of supersolution and subsolution is symmetric, so we only construct the supersolution.

Suppose that $\partial\Omega$ is $C^2$. We recall the exterior ball condition (3.3) and also recall that there exists $\varphi \in C^2(\overline{W})$ satisfying (3.4)–(3.5) with $\beta = 2C_0$. Since the initial value $u_0$ is uniformly continuous on $\overline{W}$, for fixed $\varepsilon > 0$ there exists a positive constant $A_\varepsilon$ such that

$$|u_0(x) - u_0(y)| < A_\varepsilon e^{-C_0}|x-y|^2 + \varepsilon \quad \text{for } \; x, y \in \overline{W}. \tag{4.4}$$

Because the function $\theta$ is Lipschitz continuous if we choose a branch the value of $\theta$, there exists $\delta = \delta(\varepsilon) > 0$ such that the following holds;

$$|\theta(x) - \theta(y)| < \varepsilon \quad \text{if } \; |x-y| < \delta. \tag{4.5}$$

We now fix $y \in \overline{W}$ and set $U_\delta(y) = B_\delta(y) \cap \overline{W}$, and we consider the function $\theta$ on $\overline{U_\delta(y)}$. We fix a branch of the value of $\theta$ on $U_\delta(y)$. We define the function $v_{\varepsilon,y}\colon [0,\infty) \times U_\delta(y) \to \mathbb{R}$ by

$$v_{\varepsilon,y}(t,x) = B_t + A_\varepsilon e^{\varphi(x)}|x-y|^2 + 2\varepsilon + \theta(x) - \theta(y). \tag{4.6}$$

**Proposition 4.1**
*(i) $v_{\varepsilon,y}$ satisfies the boundary condition, i.e.*

$$\langle \vec{\nu}, \nabla(v_{\varepsilon,y} - \theta) \rangle \ge 0 \quad \text{on} \;\; (0,\infty) \times (U_\delta(y) \cap \partial W).$$

*(ii) There exists a constant $B_\varepsilon$ such that the following holds: if $B \geq B_\varepsilon$, then*

$$\frac{\partial v_{\varepsilon,y}}{\partial t}(t,x) + F^*(\nabla(v_{\varepsilon,y}(t,x) - \theta(x)), \nabla^2(v_{\varepsilon,y}(t,x) - \theta(x))) \leq 0$$

*for $(t,x) \in (0,\infty) \times (U_\delta(y) \cap W)$.*

*Proof.*

We calculate derivatives of $v_{\varepsilon,y}$:

$$\begin{aligned}
\frac{\partial v_{\varepsilon,y}}{\partial t}(t,x) &= B, &(4.7)\\
\nabla(v_{\varepsilon,y}(t,x) - \theta(x)) &= A_\varepsilon e^{\varphi(x)}(|x-y|^2\nabla\varphi(x) + 2(x-y)), &(4.8)\\
\nabla^2(v_{\varepsilon,y}(t,x) - \theta(x)) &= A_\varepsilon e^{\varphi(x)}(|x-y|^2\nabla\varphi(x) \otimes \nabla\varphi(x)\\
&\quad +2(\nabla\varphi(x) \otimes (x-y) + (x-y) \otimes \nabla\varphi(x))\\
&\quad +|x-y|^2\nabla^2\varphi(x) + 2I). &(4.9)
\end{aligned}$$

(i) By (3.3), (4.8) and $\nabla\varphi = 2C_0\vec{\nu}$ on $\partial W$ we get

$$\begin{aligned}
\langle\vec{\nu}(x), \nabla(v_{\varepsilon,y}(t-x) - \theta(x))\rangle &= A_\varepsilon e^{\varphi(x)}(|x-y|^2\langle\vec{\nu}, \nabla\varphi(x)\rangle + 2\langle\vec{\nu}, x-y\rangle)\\
&\geq A_\varepsilon e^{\varphi(x)}(2C_0|x-y|^2 - 2C_0|x-y|^2) = 0.
\end{aligned}$$

(ii) We set

$$\begin{aligned}
p = p(x,y) &= e^{\varphi(x)}(|x-y|^2\nabla\varphi(x) + 2(x-y)),\\
X = X(x,y) &= e^{\varphi(x)}(|x-y|^2\nabla\varphi(x) \otimes \nabla\varphi(x)\\
&\quad +2(\nabla\varphi(x) \otimes (x-y) + (x-y) \otimes \nabla\varphi(x))\\
&\quad +|x-y|^2\nabla^2\varphi(x) + 2I);
\end{aligned}$$

in other words,

$$\nabla(v_{\varepsilon,y}(t,x) - \theta(x)) = A_\varepsilon p,$$
$$\nabla^2(v_{\varepsilon,y}(t,x) - \theta(x)) = A_\varepsilon X.$$

By the definition of $p$ and $X$ the set $\{(p(x,y), X(x,y));\ (x,y) \in \overline{W} \times \overline{W}\}$ is bounded in $\mathbb{R}^2 \times \mathbb{S}_2$. So there exists a compact set $K$ such that $K$ is independent of $u_0$ satisfying

$$K \supset \{(p(x,y), X(x,y)); (x,y) \in \overline{W} \times \overline{W}\}.$$

Since $F_*$ is lower semicontinuous on a compact set $K$, $F_*$ has a minimum value on $K$. We set

$$R = -\min\{F_*(p, X); (p, X) \in K\}.$$

By the definition of $F$, we get

$$\begin{aligned}&\frac{\partial v_{\varepsilon,y}}{\partial t}(t,x) + F^*(\nabla(v_{\varepsilon,y}(t,x) - \theta(x)), \nabla^2(v_{\varepsilon,y}(t,x) - \theta(x)) \\ &\geq B + F_*(A_\varepsilon, A_\varepsilon X) \\ &= B + A_\varepsilon F_*(p, X) \\ &\geq B - A_\varepsilon R.\end{aligned}$$

So it is enough to see 2) that we set $B_\varepsilon = A_\varepsilon R$. □

We need to extend the function $v_{\varepsilon,y}$ (resp. $u_{\varepsilon,y}$) on $(0,T) \times \overline{W}$. For this purpose we use *Invariance Lemma*(See [GS2]). We obtain a desired viscosity supersolution to take infimum of supersolutions with respect to $\varepsilon > 0$ and $y \in \overline{W}$.

To construct a subsolution of (2.3)–(2.4), we define $u_{\varepsilon,y}\colon [0,\infty) \times U_\delta(y) \to \mathbb{R}$ by

$$u_{\varepsilon,y}(t,x) = -B't - A_\varepsilon e^{\varphi(x)}|x-y|^2 - 2\varepsilon + \theta(x) - \theta(y), \tag{4.10}$$

where $B'$ is a positive constant. We may assume that $B' = A_\varepsilon R$ by take

$$R = \max\{F^*(p, X); (p, X) \in K\}.$$

We apply the Perron's method to obtain a desired viscosity solution.

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