数理解析研究所講究録 1287 巻 2002 年 27-34

DYNAMIC BOUNDARY CONDITIONS FOR HAMILTON-JACOBI EQUATIONS*

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Abstract. A non standard dynamic boundary condition for a Hamilton-Jacobi equation in one space dimension is studied in the context of viscosity solutions. A comparison principle and, hence, uniqueness is proved by consideration of an equivalent notion of viscosity solution for an alternative formulation of the boundary condition. The relationship with a Neumann condition is established. Global existence is obtained by consideration of a related parabolic approximation with a dynamic boundary condition. The problem is motivated by applications in superconductivity and interface evolution.

1. Introduction. We consider the first order equation

$$u_t - F\left(u_x^2+\gamma^2\right)^{1/2} = 0 \quad \text{in } \Omega \times (0,\infty) \tag{1.1}$$

supplemented with the dynamic boundary condition

$$u_t - F\alpha = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.2}$$

where Ω is a bounded open interval. The function F and α are given continuous functions on $\overline{\Omega} \times [0,\infty)$, $\partial\Omega \times [0,\infty)$ respectively and $\gamma \geq 0$ is a constant.

A source of this problem is found in the mean field theory of superconductivity. Consider the mean field vortex density model in a cylinder $D \times \mathbb{R}$ ($D \subset \mathbb{R}^2$) when the magnetic field $\vec{H}$ is orthogonal to the axis of the cylinder; Chapman [3]. The vorticity field $\vec{\omega} = (\nabla^{\perp}\psi, 0)$, $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$ is required to satisfy the conservation of vorticity

$$\vec{\omega}_t + \operatorname{curl}(\vec{\omega} \times \vec{v}) = 0.$$

If the velocity field $\vec{v}$ is of the form $\vec{v} = \operatorname{curl}\vec{H} \times \vec{\omega}/|\vec{\omega}|$ and $\vec{H}$ is given, then the conservation of vorticity yields

$$\psi_t = |\nabla\psi| F,$$

where F is a given function. Our equation (1.1) is derived by assuming that $\partial_{x_2}\psi = \gamma$ is a constant on $D = \Omega \times \mathbb{R}$ if we set $u(x_1, t) = \psi(x_1, x_2, t) - \gamma x_2$.

*This is a joint work with Charles M. Elliott (University of Sussex) and Yoshikazu Giga (Hokkaido University).

The quantity $-\psi_t$ on the boundary corresponds to the flux $\vec{n} \times (\vec{\omega} \times \vec{v})$ on $\partial D \times \mathbb{R}$. The condition $\psi_t = F\alpha$ is considered as a special case of assigning the value of flux and we obtain (1.1), (1.2). A full system with a different boundary condition $\vec{\omega} \cdot \vec{n} = 0$ is studied by Elliott, Schätzle and Stoth [6].

Our goal is to study the unique global-in-time solvability of (1.1), (1.2) for a given initial data. Since the problem is of first order, it is convenient to handle this problem in the realm of viscosity solutions; see e.g. G. Barles [2]. We establish the comparison principle for (1.1) and (1.2) by deriving an equivalent definition of solutions. Although the dynamic boundary value problem is studied in [2, P.102 (4.23)], it is essentially of Neumann type and does not include (1.2).

We further prove that the solution of (1.1) and (1.2) solves the Neumann problem for (1.1) with

$$\partial u/\partial \nu = (\mathrm{Sign}F)\left\{(\alpha-\gamma)_+(\alpha+\gamma)\right\}^{1/2} \tag{1.3}$$

in the viscosity sense, where β_+ denotes the positive part of β and Sign$F$ denotes the sign of F i.e. Sign$F = \pm 1$ if $F \gtrless 0$ and Sign$F = 0$ if $F = 0$. It might be possible to prove the comparison principle for (1.1) with the inhomogeneous data $\partial u/\partial \nu = p(t)$ when p is continuous; see J. Claisse [4]. However, our comparison principle for (1.1) and (1.2) still holds when F changes sign in which case the Neumann data in (1.3) is discontinuous and hence it is not included in the literature. Moreover, our proof is more direct and does not use (1.3). Our comparison principle yields the uniqueness of viscosity solutions for (1.1) and (1.2).

We also prove the global existence of a solution for (1.1), (1.2) when the initial data a is a Lipschitz function in $\overline{\Omega}$, by using the approximate equation

$$u_t - \varepsilon u_{xx} - F\left(u_x^2+\gamma^2\right)^{1/2} = 0 \quad \text{in } \Omega \times (0,\infty) \tag{1.4}$$

with the dynamic boundary condition

$$u_t - F\alpha + \varepsilon \partial u/\partial \nu = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.5}$$

where ε is a positive parameter. The dynamic boundary condition for uniformly parabolic equations is well studied, for example by Hinterman [10] and Escher [7, 8]. Their results may be applied to (1.4) and (1.5) in order to yield at least a local solution. However, the global existence of solutions of (1.4), (1.5) is easy to show, directly. By the maximum principle we derive a priori bounds for the sup norms of u_t^ε, u_x^ε, u^ε in $\overline{\Omega} \times [0,T]$ for solutions of (1.4), (1.5) independent of $\varepsilon \in (0,1)$. This yields the solution of (1.1), (1.2) as a limit as $\varepsilon \to 0$. The presence of the term $\varepsilon\partial u/\partial \nu$ in (1.5) is crucial in order to obtain the a priori bound.

Finally, we remark that the boundary condition (1.2) cannot be replaced by a formally equivalent Dirichlet boundary condition

$$u(x,t) = \int_0^t F(x,\tau)\alpha(x,\tau)d\tau + a(x) \tag{1.6}$$

even in the viscosity sense. We give in Section 5 an explicit solution of (1.1) which solves (1.2) (resp. (1.6)) but does not solve (1.6) (resp. (1.2)) when $\alpha \equiv 1$, $F \equiv 1$ and $\alpha > \gamma$.

2. Definitions and Equivalent Notions of Solutions. Let Ω be a bounded interval $(0, L) \subset \mathbb{R}$ and $T > 0$ be a constant. For brevity we set $Q = \Omega \times (0,T)$, $\hat{Q} = \overline{\Omega} \times (0,T)$ and their closure $\overline{Q} = \overline{\Omega} \times [0,T]$. Given a mapping $k := k(x,t,\tau,p) : \hat{Q} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ we recall the following definitions of viscosity sub- and supersolutions $u \in C(\hat{Q})$ for k.

DEFINITION 2.1. *A function u is said to be a viscosity subsolution of k (in $\hat{Q}$) provided for any $(\hat{x}, \hat{t}, \phi) \in \hat{Q} \times C^1(\hat{Q})$ such that*

$$(u - \phi)(\hat{x}, \hat{t}) = \sup_{\hat{Q}}(u - \phi)$$

then the inequality $k(\hat{x}, \hat{t}, \tau, p) \leq 0$ holds where $\tau = \phi_t(\hat{x}, \hat{t})$ and $p = \phi_x(\hat{x}, \hat{t})$.

DEFINITION 2.2. *A function u is said to be a viscosity supersolution of k (in $\hat{Q}$) provided for any $(\hat{x}, \hat{t}, \phi) \in \hat{Q} \times C^1(\hat{Q})$ such that*

$$(u - \phi)(\hat{x}, \hat{t}) = \inf_{\hat{Q}}(u - \phi)$$

then the inequality $k(\hat{x}, \hat{t}, \tau, p) \geq 0$ holds where $\tau = \phi_t(\hat{x}, \hat{t})$ and $p = \phi_x(\hat{x}, \hat{t})$.

Let F and α be given functions in $C(\overline{Q})$, $C(\partial\Omega \times [0,T])$ respectively and $\gamma \geq 0$ be a given constant. We use the notation, since $\partial\Omega = \{0, L\}$, that $\partial/\partial\nu = \nu\partial/\partial x$ on $\partial\Omega$ with $\nu = -1$ for $x = 0$ and $\nu = +1$ for $x = L$. The initial boundary value problem is

$$\begin{cases} u_t - F\left(u_x^2 + \gamma^2\right)^{1/2} = 0 & \text{in} \quad Q, \\ u_t - F\alpha = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u\big|_{t=0} = a & \text{on} \quad \Omega. \end{cases} \tag{2.1}$$

In order to formulate the definition of a viscosity solution to (2.1) we define, for $(x,t,\tau,p) \in \hat{Q} \times \mathbb{R} \times \mathbb{R}$

$$E(x,t,\tau,p) := \tau - F(x,t)\left(p^2 + \gamma^2\right)^{1/2},$$

$$F_{\min}(x,t,\tau,p) := \begin{cases} E(x,t,\tau,p) & \text{if} \quad x \in \Omega, \\ \min\{\tau - F(x,t)\alpha(x,t), E(x,t,\tau,p)\} & \text{if} \quad x \in \partial\Omega, \end{cases}$$

$$F_{\max}(x,t,\tau,p) := \begin{cases} E(x,t,\tau,p) & \text{if} \quad x \in \Omega, \\ \max\{\tau - F(x,t)\alpha(x,t), E(x,t,\tau,p)\} & \text{if} \quad x \in \partial\Omega. \end{cases}$$

DEFINITION 2.3. *We say that $u \in C(\overline{Q})$ is a viscosity solution of (2.1) provided $u(x,0) = a(x)$, $x \in \overline{\Omega}$, u is a viscosity subsolution for $F_{\min}$ and a viscosity supersolution for $F_{\max}$.*

This is the usual notion of viscosity solution for boundary value problems (cf. [5]). We give an equivalent notion of solution by introducing, for $(x,t,\tau,p) \in \hat{Q} \times \mathbb{R} \times \mathbb{R}$

$$G(x,t,\tau,p) := \begin{cases} E(x,t,\tau,p) & \text{if } \ x \in \Omega, \\ \tau - F(x,t) \max \left\{ \alpha(x,t), (([p\nu \mathrm{Sign} F]_-)^2 + \gamma^2)^{1/2} \right\} & \text{if } \ x \in \partial\Omega, \end{cases}$$

where f_- is the negative part of f.

The main result of this section is the following proposition:

PROPOSITION 2.4. *A function u is a viscosity solution of (2.1) if and only if $u \in C(\overline{Q})$, $u(x,0) = a(x)$, $x \in \Omega$, and u is both a viscosity subsolution and a viscosity supersolution for G.*

3. Comparison Principle. We have the comparison principle for (1.1) and (1.2) by driving the equivalent definition of solutions.

THEOREM 3.1. *Assume that $F \in C(\overline{Q})$, $\alpha \in C(\partial\Omega \times [0,T])$ and*

$$|F(x,t) - F(y,t)| \leq C|x-y| \quad \textit{for all } (x,t),(y,t) \in \overline{Q}$$

holds for some constant $C > 0$ independent of t. Let u and $-v$ be bounded upper semicontinuous functions on $\overline{\Omega} \times [0,T)$. Let u be a viscosity subsolution for G in $\hat{Q}$ and v be a viscosity supersolution for G in $\hat{Q}$. If $u(\cdot,0) \leq v(\cdot,0)$ in $\overline{\Omega}$, then $u \leq v$ in $\hat{Q}$.

4. Existence Theorem. Our goal is to show the existence of viscosity solutions of the dynamic boundary problem (2.1).

THEOREM 4.1. *Assume that $F \in C^1(\overline{Q})$ and $\alpha \in C(\partial\Omega \times [0,T])$. Assume that a is a Lipschitz function over $\overline{\Omega}$. Then there exists a function $u \in C(\overline{Q})$ which is a unique viscosity solution of (2.1). Moreover, $|u_x|$ is bounded in $\overline{Q}$.*

Let $\varepsilon > 0$. First, we shall prove a priori estimates for a classical solution u^ε for the approximate problem

$$\begin{cases} u_t^\varepsilon - \varepsilon u_{xx}^\varepsilon = F^\varepsilon \left((u_x^\varepsilon)^2 + \gamma^2 \right)^{1/2} & \text{in } \ Q, \\ u_t^\varepsilon + \varepsilon \nu u_x^\varepsilon = F^\varepsilon \max \left\{ \alpha^\varepsilon, (([\nu u_x^\varepsilon \mathrm{Sign} F^\varepsilon]_-)^2 + \gamma^2)^{1/2} \right\} & \text{on } \ \partial\Omega \times (0,T), \\ u^\varepsilon|_{t=0} = a^\varepsilon & \text{on } \ \Omega, \end{cases}$$

where ν denotes the outer unit normal of $\partial\Omega$. The existence of a solution of (4.1) is omitted here.

PROPOSITION 4.2. *Assume that* $F^\varepsilon \in C^1(\overline{Q}) \cap C^\infty(Q)$ *and* $\alpha^\varepsilon \in C^1(\partial\Omega \times [0,T])$. *Assume that* a^ε *is a* C^3 *function over* $\overline{\Omega}$ *and* $\varepsilon a^\varepsilon_{xx}$ *is bounded on* $\overline{\Omega}$ *uniformly for* ε. *Let* u^ε *be a classical solution of (4.1). Then the estimate holds*

$$\max_{\overline{Q}} (|u^\varepsilon| + |u^\varepsilon_x| + |u^\varepsilon_t|) \leq C \tag{4.2}$$

with some constant $C > 0$ *depending only on* T, γ, $|a^\varepsilon|_{C^1(\overline{\Omega})}$, $|\varepsilon a^\varepsilon_{xx}|_{C(\overline{\Omega})}$, $|F^\varepsilon|_{C^1(\overline{Q})}$ *and* $|\alpha^\varepsilon|_{C(\partial\Omega\times[0,T])}$.

Proof of Theorem 4.1. For a given Lipschitz function a there is a sequence $a^\varepsilon \in C^\infty(\overline{\Omega})$ such that $a^\varepsilon \to a$ uniformly and that $|a^\varepsilon_x|_{C(\overline{\Omega})}$ and $|\varepsilon a^\varepsilon_{xx}|_{C(\overline{\Omega})}$ are bounded. For a given $F \in C^1(\overline{Q})$ and $\alpha \in C(\partial\Omega \times [0,T])$ there is a sequence $\{F^\varepsilon, \alpha^\varepsilon\}$ with $F^\varepsilon \in C^1(\overline{Q}) \cap C^\infty(Q)$, $\alpha^\varepsilon \in C^1(\partial\Omega \times [0,T])$ such that $F^\varepsilon \to F$ uniformly in $\overline{Q}$ and $\alpha^\varepsilon \to \alpha$ uniformly in $\partial\Omega \times [0,T]$ and that $|F^\varepsilon|_{C^1(\overline{Q})}$ and $|\alpha^\varepsilon|_{C(\partial\Omega\times[0,T])}$ are bounded as $\varepsilon \to 0$.

By the uniform estimate (4.2) the Arzela-Ascoli theorem implies that there exists a function u such that

$$u^\varepsilon \to u \quad \text{uniformly on } \overline{Q}.$$

We shall show that u is the viscosity solution of the original dynamic boundary problem (2.1). Since the proof for viscosity supersolutions is symmetric, we only prove that u is a viscosity subsolution for G. To do this, let $\phi \in C^2(\overline{Q})$ be a test function and $(\hat{x}, \hat{t}) \in \hat{Q}$ be the maximum point of $u - \phi$. We may assume that $(\hat{x}, \hat{t})$ is a strict maximum of $u - \phi$. Then there exists $(x_\varepsilon, t_\varepsilon)$ such that $(x_\varepsilon, t_\varepsilon) \to (\hat{x}, \hat{t})$ and $\sup_{\hat{Q}}(u^\varepsilon - \phi) = (u^\varepsilon - \phi)(x_\varepsilon, t_\varepsilon)$.

By the standard argument we see that u is a viscosity solution of (2.1) and it is unique by the comparison principle. The Lipschitz continuity of u in x follows from the estimate for u^ε_x. □

5. Relation to Other Boundary Conditions. We shall relate an inhomogeneous Neumann boundary value problem for

$$u_t - F(u_x^2 + \gamma^2)^{1/2} = 0 \tag{5.1}$$

supplemented with the dynamic boundary

$$u_t - F\alpha = 0. \tag{5.2}$$

Formally, (5.1) and (5.2) yields

$$F(u_x^2+\gamma^2)^{1/2}=F\alpha.$$

If F is not zero, this implies $u_x^2+\gamma^2=\alpha^2$. Thus we obtain

$$\partial u/\partial\nu=u_x\nu=\pm(\alpha^2-\gamma^2)^{1/2} \tag{5.3}$$

on the boundary. The Neumann data in (5.3) needs more explanation since both its sign and its value for $\alpha^2<\gamma^2$ are unclear. We shall clarify these points and prove that a solution of (5.1), (5.2) solves an inhomogeneous Neumann problem in the viscosity sense.

THEOREM 5.1. *Assume that F and α are continuous on $\overline{Q}$ and $\partial\Omega\times[0,T]$, respectively. Assume that u is a viscosity subsolution (resp. supersolution) for G in $\hat{Q}$. Then u is a viscosity subsolution (resp. supersolution) of the Neumann problem of (5.1) in $\hat{Q}$ with*

$$\partial u/\partial\nu=\mathrm{Sign}F\left\{(\alpha-\gamma)_+(\alpha+\gamma)\right\}^{1/2}.$$

Here β_+ is the plus part of β defined by $\beta_+=\max(\beta,0)$.

When we are asked to solve (5.1) and (5.2), we are tempted to integrate (5.2) in order to obtain the Dirichlet condition:

$$u(x,t)=\int_0^t F(x,\tau)\alpha(x,\tau)d\tau+a(x),\quad x\in\partial\Omega. \tag{5.4}$$

However, unfortunately, (5.1) with the Dirichlet condition (5.4) is not equivalent to (5.1), (5.2).

We shall give a counterexample to show that the problem (5.1), (5.2) is different from the Dirichlet problem (5.1), (5.4) in the viscosity sense. We suppress the word viscosity.

We shall give two different functions u and v which initially agree with each other but u solves (5.1), (5.2) while v solves (5.1), (5.4) when $\alpha\equiv1$, $F\equiv1$, $\alpha>\gamma$ and $\Omega=(0,\infty)$. Although it is not difficult to give such functions for $\Omega=(0,L)$ with more general α and F, we keep such assumptions to clarify the argument. Let β be a constant strictly greater than $\sigma=(1-\gamma^2)^{1/2}$ so that $\eta=(\beta^2+\gamma^2)^{1/2}>1$. We set

$$w(x,t)=\min\{\beta+\gamma t,\beta x+\eta t,-\sigma x+\sigma+\beta+t\},\quad x\in\overline{\Omega}. \tag{5.5}$$

This function is nondecreasing in t and

$$w(x,0)=\min\{\beta x,-\sigma x+\sigma+\beta\}$$

so that $w(x,0)$ is linear except at $x=1$. At time $t_0=\beta(\eta-\gamma)^{-1}$

$$w(x,t_0)=\min\{\beta+\gamma t_0,-\sigma x+\sigma+\beta+t_0\}.$$

Since $\beta\geq\sigma$, it is easy to see that

$$\phi_t-\left(\phi_x^2+\gamma^2\right)^{1/2}\leq 0 \quad \text{at } (\hat{x},\hat{t})$$

if $w-\phi$ attains its maximum at $(\hat{x},\hat{t})$ over $\overline{\Omega}\times(0,t_0]$ even if $\hat{x}\in\partial\Omega$. So w is a subsolution of $\overline{\Omega}\times(0,t_0]$ of (5.1), (5.2) and (5.1), (5.4). It is easy to see that w is a supersolution of (5.1), (5.2) and (5.1), (5.4) in $\overline{\Omega}\times(0,t_0]$ since $w_t\geq 1$, $w\geq t_0$ on the boundary. We now set

$$u(x,t)=v(x,t)=w(x,t) \quad \text{for } t\leq t_0, x\in\overline{\Omega} \tag{5.6}$$

and

$$v(x,t)=\min\{\beta+\gamma t,-\sigma x+\sigma+\beta+t\} \quad \text{for } t\geq t_0, x\in\overline{\Omega}, \tag{5.7}$$

$$u(x,t)=\max\{\beta+(\gamma-1)t_0+t-\sigma x, v(x,t)\} \quad \text{for } t\geq t_0, x\in\overline{\Omega}. \tag{5.8}$$

As for w it is easy to see that v is a subsolution of both the dynamic (5.1), (5.2) and the Dirichlet problem (5.1), (5.4) in $\overline{\Omega}\times(0,\infty)$. Since $\eta>1$ so that $t_1=\beta(1-\gamma)^{-1}>t_0$, and since $v(0,t)>t$ for $t<t_1$, v is a supersolution of the Dirichlet problem in $\overline{\Omega}\times(0,t_1)$. However, v is not a supersolution in $\overline{\Omega}\times(0,t_1)$ of (5.1), (5.2) since at the boundary $v_t<1$ with $v_x=0$.

Since $u_t=1$ on the boundary and since it is easy to see that u is a solution of (5.1) in $\Omega\times(0,\infty)$, we conclude that u is a solution of (5.1), (5.2) in $\overline{\Omega}\times(0,\infty)$. This is not a subsolution of (5.1), (5.4) in $\overline{\Omega}\times(0,\infty)$ since $u(0,t)>t$ by $\eta>1$ and

$$\phi_t-\left(\phi_x^2+\gamma^2\right)^{1/2}>0 \quad \text{at } (0,\hat{t})$$

if $u-\phi$ attains its maximum on $\overline{\Omega}\times(0,\infty)$ and $\hat{t}>t_0$. (The function u is a supersolution of (5.1), (5.4) since $u(0,t)>t$.) We summarize our results.

PROPOSITION 5.2. *Assume that $\alpha\equiv F\equiv 1$ and $\gamma<1$. Let $\beta>\sigma=(1-\gamma^2)^{1/2}$. For $\Omega=(0,\infty)$, let u and v be functions defined by (5.5)–(5.8). Then u is a solution of the dynamic boundary problem (5.1), (5.2) in $\overline{\Omega}\times(0,\infty)$ while v is a solution of the Dirichlet problem (5.1), (5.4) in $\overline{\Omega}\times(0,t_1)$ with $t_1=\beta(1-\gamma)^{-1}$. However, u is not a subsolution of (5.1), (5.4) in $\overline{\Omega}\times(0,T)$, $T>t_0$ while u is a supersolution of (5.1), (5.4) in $\overline{\Omega}\times(0,\infty)$. The function v is not a supersolution of (5.1), (5.2) while it is a subsolution of (5.1), (5.2).*

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