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Motion of a graph by $R$-curvature

北海道大学・理学研究科 三上 敏夫 (Toshio Mikami)
Department of Mathematics
Hokkaido University

1. Introduction.

In this talk we introduce our recent result:

H. Ishii and T. Mikami, Motion of a graph by $R$-vurvature, Hokkaido mathematical preprint series, No. 340.

Let us first introduce two definitions.

**Definition 1 ($R$-curvature)** Let $R \in L^1(\mathbb{R}^d : [0, \infty), dx)$. For $u \in C(\mathbb{R}^d : \mathbb{R})$, we define the $R$-curvature of $u$ as the finite Borel measure $w(R, u, dx)$ on $\mathbb{R}^d$ given by

$$w(R, u, A) \equiv \int_{\cup_{x \in \partial u(x)}} R(y)dy \quad \text{for all Borel } A \subset \mathbb{R}^d. \tag{0.1}$$

**Definition 2 (Motion by $R$-curvature)** The graph of $u \in C([0, \infty) \times \mathbb{R}^d : \mathbb{R})$ is called the motion by $R$-curvature if the following holds: for any $\varphi \in C_\circ(\mathbb{R}^d : \mathbb{R})$ and any $t \geq 0$, 

\[ \text{not shown in image} \]
\begin{equation}
\int_{\mathbb{R}^d} \varphi(x)u(t, x)dx - \int_{\mathbb{R}^d} \varphi(x)u(0, x)dx = \int_0^t ds \int_{\mathbb{R}^d} \varphi(x)w(R, u(s, \cdot), dx).
\end{equation}

By the continuum limit of a class of infinite particle systems, we first show the existence of the motion by \( R \)-curvature, and then the uniqueness by the comparison theorem. We also show that the motion by \( R \)-curvature is a viscosity solution to

\[(PDE) \quad \partial u(t, x)/\partial t = \chi(u, Du(t, x), t, x)\text{Det}_+ (D^2 u(t, x)) R(Du(t, x)),\]

where \( Du(t, x) \equiv (\partial u(t, x)/\partial x_i)_{i=1}^d \), \( D^2 u(t, x) \equiv (\partial^2 u(t, x)/\partial x_i \partial x_j)_{i,j=1}^d \),

\[\chi(u, p, t, x) \equiv \begin{cases} 1 & \text{if } p \in \partial u(t, x), \\ 0 & \text{otherwise}, \end{cases}\]

\( \partial u(t, x) \) denotes the subdifferential of the function \( x \mapsto u(t, x) \), and for a real \( d \times d \)-symmetric matrix \( X \),

\[\text{Det}_+ X \equiv \begin{cases} \text{Det}X & \text{if } X \text{ is nonnegative definite,} \\ 0 & \text{otherwise}. \end{cases}\]

We introduce the definition of the viscosity solution to (PDE).

**Definition 3 (Viscosity solution)** (Viscosity subsolution) \( u \in C((0, \infty) \times \mathbb{R}^d : \mathbb{R}) \) is a viscosity subsolution of (PDE) if whenever \( \varphi \in C^2((0, \infty) \times \mathbb{R}^d : \mathbb{R}) \) and \( u - \varphi \leq (u - \varphi)(t_o, x_o) \),
\[ \partial \varphi(t_o, x_o)/\partial t \leq \chi(u, D\varphi(t_o, x_o), t_o, x_o) \text{Det}_+ (D^2 \varphi(t_o, x_o)) R(D\varphi(t_o, x_o)). \]

(Viscosity supersolution) \( u \in C((0, \infty) \times \mathbb{R}^d : \mathbb{R}) \) is a viscosity supersolution of (PDE) if whenever \( \varphi \in C^2((0, \infty) \times \mathbb{R}^d : \mathbb{R}) \) and \( u - \varphi \geq (u - \varphi)(t_o, x_o), \)

\[ \partial \varphi(t_o, x_o)/\partial t \geq \chi^{-}(u, D\varphi(t_o, x_o), t_o, x_o) \text{Det}_+ (D^2 \varphi(t_o, x_o)) R(D\varphi(t_o, x_o)). \]

Here \( \chi^{-}(v, p, t, x) = 1 \) if

\[ v(t, y) > v(t, x) + <p, y - x> \quad (y \neq x) \]

and if there exists \( \epsilon > 0 \) such that for all \( (s, y) \in (0, \infty) \times \mathbb{R}^d \) satisfying \(|y| > \epsilon^{-1}\) and \(|s - t| < \epsilon|y| > \epsilon^{-1}\) and \(|s - t| < \epsilon|y|\),

\[ v(s, y) > p \cdot y + \epsilon|y|, \]

and \( \chi^{-}(v, p, t, x) = 0 \), otherwise.

**Remark 1** If \( \chi^{-}(v, p, t, x) = 1 \) and \( s \) is close to \( t \), then \( p \in \partial v(s, y) \) for some \( y \).

Finally we discuss under what condition the viscosity solution to (PDE) is the motion by \( R \)-curvature.

2. **Infinite particle systems and the motion by \( R \)-curvature.**

In this section we construct the motion by \( R \)-curvature by the continuum limit of infinite particle systems.
Fix $\epsilon_n \downarrow 0$ as $n \to \infty$, and put

(A.1.): $\|R\|_{L^1} \equiv \int_{\mathbb{R}^d} R(y)dy > 0$, $R \geq 0$, $h \in C(\mathbb{R}^d : \mathbb{R})$,

(A.2.): $|\partial h(\mathbb{R}^d)(\equiv \cup_{z \in \mathbb{R}^d} \partial h(x))| > 0$,

$$S_n \equiv \{ v : \mathbb{Z}^d / n \mapsto \mathbb{R} | \sum_{z \in \mathbb{Z}^d / n} (v(z) - h(z)) < \infty,$$

$$ (v(z) - h(z)) / \epsilon_n \in \mathbb{N} \cup \{0\} \text{ for all } z \in \mathbb{Z}^d / n \}.$$  

Let $\{Y_n(k, \cdot)\}_{0 \leq k}$ be a Markov chain on $S_n$ such that $Y_n(0, \cdot) = h(\cdot)$, and that

$$P(Y_n(k + 1, \cdot) = v_{n,z} | Y_n(k, \cdot) = v) = w(R, \hat{v}, \{z\}) / w(R, \hat{Y}_n(0, \cdot), \mathbb{R}^d),$$

where

$$v_{n,z}(x) \equiv \begin{cases} 
 v(x) + \epsilon_n & \text{if } x = z, \\
 v(x) & \text{if } x \in (\mathbb{Z}^d / n) \setminus \{z\}.
 \end{cases}$$

Let $p_n(t)$ be a Poisson process, with parameter $n^d \epsilon_n^{-1} w(\hat{R}, \hat{Y}_n(0, \cdot), \mathbb{R}^d)$, which is independent of $Y_n$. Put

$$Z_n(t, z) \equiv Y_n(p_n(t), z),$$

$$X_n(t, x) \equiv \max(\hat{Z}_n(t, x), h(x)).$$

For $f$ and $g \in C(\mathbb{R}^d : \mathbb{R})$, we put

$$d_{C(\mathbb{R}^d : \mathbb{R})}(f, g) \equiv \sum_{m \geq 1} 2^{-m} \min(\sup_{|x| \leq m} |f(x) - g(x)|, 1).$$

Then we show that $X_n(t, x)$ converges to the motion by $R$-curvature under the following additional conditions.
(A.3). The closure of the set \( \{ x \in \mathbb{R}^d : \hat{h}(x) < h(x) \} \) does not contain any line which is unbounded in two different directions.

(A.4). For any \( p \notin \partial h(\mathbb{R}^d) \) and \( C \in \mathbb{R} \),

\[
\int_{\mathbb{R}^d} \max(<p, x> + C - h(x), 0) dx = \infty.
\]

**Theorem 1** Suppose that (A.1) and (A.3)-(A.4) hold. Then there exists a unique continuous solution \( u \) to (1.2) with \( u(0, \cdot) = h \). Suppose in addition that (A.2) holds. Then the following holds: for any \( \gamma > 0 \) and \( T > 0 \),

\[
\lim_{n \to \infty} P(\sup_{0 \leq t \leq T} d_{C(\mathbb{R}^d; \mathbb{R})}(X_n(t, \cdot), u(t, \cdot)) \geq \gamma) = 0.
\]

**Remark 2** (A.3) holds when \( d = 1 \). If \( h \) is convex, then (A.4) holds.

We give the properties of the motion by \( R \)-curvature.

**Theorem 2** Suppose that (A.1) holds. Let \( u \in C([0, \infty) \times \mathbb{R}^d : \mathbb{R}) \) be the solution to (1.2) with \( u(0, \cdot) = h \). Then:

(a) \( t \mapsto u(t, x) \) is nondecreasing.

(b) \( u = \max(\hat{u}, h) \)

(c) \( u(t, x) - \hat{u}(t, x) \leq h(x) - \hat{h}(x) \). In particular, if \( h(x) = \hat{h}(x) \), then \( u(t, x) = \hat{u}(t, x) \).

Suppose in addition that (A.4) holds. Then:

(d) For any \( t > 0 \), \( \partial u(t, \mathbb{R}^d) = \partial h(\mathbb{R}^d) \).

\[
\int_{\mathbb{R}^d} (u(t, x) - h(x)) dx = t \cdot w(R, h, \mathbb{R}^d).
\]
(e) Let \( \overline{u} \in C([0, \infty) \times \mathbb{R}^d : \mathbb{R}) \) be the solution to (1.2) with \( u(0, \cdot) = \hat{h} \). If \( u(s, \cdot) - \hat{u}(s, \cdot) \neq h - \hat{h} \) for some \( s \in (0, \infty) \), then \( \overline{u}(t, \cdot) - \hat{u}(t, \cdot) \neq 0 \) for all \( t \geq s \).

According to the above theorem, (a) any graph moves upward by \( R \)-curvature, (b) those points on any graph moving by \( R \)-curvature do not move as far as they stay in its cavities, (c) the height between any graph moving by \( R \)-curvature and its convex envelope is nonincreasing as it evolves, (d) any graph moving by \( R \)-curvature sweeps in time \( t > 0 \) a region with volume given by \( t \cdot w(R, h, \mathbb{R}^d) \), and (e) for the motion of a graph by \( R \)-curvature, taking its convex envelope at time \( t > 0 \) and evolving up to time \( t \) starting with the convex envelope of the initial graph give different graphs in general, if the initial graph is not convex.

3. Motion by \( R \)-curvature and the viscosity solution.

In this section we discuss the relation between the motion by \( R \)-curvature and the viscosity solution to (PDE).

\[ \text{(A.5). } R \in C(\mathbb{R}^d : [0, \infty)). \]

**Theorem 3** Suppose that (A.1) and (A.5) hold. Then a continuous solution \( u \) to (1.2) with \( u(0, \cdot) = h \) is a viscosity solution to (PDE).

Theorem 3 means that the motion by \( R \)-curvature is the viscosity solution to (PDE). To discuss under what condition the reverse is true, we discuss the uniqueness of the viscosity solution to (PDE).

\[ \text{(A.6). } R(x) \geq R(rx) \text{ for any } r \geq 1 \text{ and } x \in \mathbb{R}^d. \]

\[ \text{(A.7). } \inf_{x \neq 0} h(x)/|x| > 0. \]
(A.8). There exists a constant $C > 0$ such that $h(x+y) + h(x-y) - 2h(x) \leq C$ for all $(x, y) \in \mathbb{R}^d \times U_1(0)$, where $U_1(0) \equiv \{y \in \mathbb{R}^d : |y| < 1\}$.

**Theorem 4** Suppose that (A.1) and (A.3)-(A.8) hold. Then there exists a unique continuous viscosity solution $u$ to (PDE) with $u(0, \cdot) = h$ in the space of continuous functions $v$ for which

$$
\sup\{|v(t, x) - h(x)| : (t, x) \in [0, T] \times \mathbb{R}^d\} < \infty \text{ for all } T > 0.
$$

$u$ is also a unique continuous solution to (1.2) with $u(0, \cdot) = h$.

We restrict our attention to Gauss curvature flow and give a finer result. For $A \subset \mathbb{R}^d$ and $v : A \mapsto \mathbb{R}$, put

$$
\text{epi}(v) = \{(x, y) : x \in A, \ y \geq v(x)\}.
$$

For $r > 0$, put

$$
h^r(x) = \inf\{y \in \mathbb{R} | U_r((x, y)) \subset \text{epi}(h)\} \quad (x \in \mathbb{R}^d).
$$

Under the following condition, we give the comparison theorem for the continuous viscosity solution to (PDE).

(A.1)' $R(y) = (1 + |y|^2)^{-(d+1)/2}$ and $h \in C(\mathbb{R}^d : \mathbb{R})$.

(A.2)'

$$
\lim_{\theta \downarrow 1} \liminf_{r \to \infty} \lim_{|x| \to \infty} (h(\theta x) - h^r(x)) > 0,
$$
Theorem 5 Suppose that \((A.1)'-(A.2)'\) hold. Then for any viscosity sub-solution \(u\) and supersolution \(v\), of \((PDE)\) in the space \(C([0,\infty) \times \mathbb{R}^d : \mathbb{R})\), such that \(u(0,\cdot) \leq h \leq v(0,\cdot), u \leq v\).

Remark 3 \((A.2)'\) holds if there exists a convex function \(h_0 : \mathbb{R}^d \mapsto \mathbb{R}\) such that \(h_0(x) \to \infty\) as \(|x| \to \infty\) and that

\[
\lim_{|x| \to \infty} [h(x) - h_0(x)] = 0.
\]

In fact, the following holds:

\[
\lim_{|x| \to \infty} [h(\theta x) - h^r(x)] = \infty \quad \text{for all } \theta > 1, r > 0,
\]

\[
\lim_{\theta \downarrow 1} \{ \sup_{x \in \mathbb{R}^d} [h(x) - h(\theta x)] \} = 0.
\]

The following corollary is better than Theorem 4 in that we can consider the viscosity solution in the entire space \(C(\mathbb{R}^d : \mathbb{R})\).

Corollary 1 Suppose that \((A.1)'-(A.2)'\) and \((A.3)-(A.4)\) hold. Then there exists a unique continuous viscosity solution \(u\) to \((PDE)\) with \(u(0,\cdot) = h\). \(u\) is also a unique continuous solution to \((1.2)\) with \(u(0,\cdot) = h\).

Acknowledgement: We would like to thank Prof. K. Ishii for informing us the following paper:
G. Barles, S. Biton and O. Ley, Quelque résultats d'unicité pour l'équation du mouvement par courbure moyenne dans $\mathbb{R}^N$, preprint, Theorem 4.1,

where they studied a similar result to Theorem 5 for the mean curvature flow with a convex coercive initial function.