Relatively Simple Chain Complexes

城西大学・理学部　山崎　正之 (Masayuki Yamasaki)
Faculty of Science,
Josai University

Introduction

The purpose of this short note is to give a characterization of geometric module chain complexes representing the trivial element in the relative $p_X^{-1}(\epsilon)$-controlled Whitehead group $Wh(X, Y; p_X, n, \epsilon)$ of a pair (X, Y) of metric spaces. These groups were introduced in [5]. Here $p_X : W \to X$ is a given control map, and n is the restriction on the dimensions of chain complexes. For different n's, $Wh(X, Y; p_X, n, \epsilon)$ are in general different abelian groups, but they are all the same after stabilization (Propositions 4.6, 4.7, and the comment after 4.7 of [5]).

Main Theorem. *There exist a positive constant α such that the following holds: For any chain complex C representing the trivial element of $Wh(X, Y, p_X, n, \epsilon)$, there exist n-dimensional trivial chain complexes T, T' and an n-dimensional free $\alpha\epsilon$ chain complex F on $p_X|Y^{\alpha\epsilon}$ such that $C \oplus T$ and $F \oplus T'$ are $\alpha\epsilon$-simple isomorphic. Actually $\alpha = 500$ works.*

In the first section, we review some facts from [5] and give the proof of the main theorem. The main ingredient of the proof is the restriction operation of simple isomorphisms described in [3]. In the second section we discuss how this can be used in the theroy of controlled L-theory.

1. Proof of the Main Theorem

We first review the definition of the controlled Whitehead group of a pair. Let (X, Y) be a pair of metric spaces, and $p_X : W \to X$ be a continuous map. $Wh(X, Y; p_X, n, \epsilon)$ is the set of equivalence classes of n-dimensional free ϵ chain complexes on p_X which are strongly ϵ contractible over $X - Y$. The equivalence relation is generated by n-stable 40ϵ-simple equivalences away from $Y^{20\epsilon}$. Here the term "stable" means that we allow taking sums with trivial chain complexes

$$\ldots \longrightarrow 0 \longrightarrow F \overset{1}{\longrightarrow} F \longrightarrow 0 \longrightarrow \ldots \;,$$

the term "away from $Y^{20\epsilon}$" means that we allow taking sums with free chain complexes lying over the specified set, the prefix n indicates that all the complexes involoved should be strictly n-dimensional (*i.e.* $C_i = 0$ for $i < 0$ and $i > n$), and the prefix 40ϵ indicates that everything involved should have appropriate size control. The following is immediate from 4.1 of [5].

Proposition 1. *If* $[C] = 0 \in Wh(X, Y, p_X, n, \epsilon)$, *then there exist trivial chain complexes* T, T' *on* p_X, *free* 86ϵ *chain complexes* D, D' *on* $p_X | Y^{20\epsilon}$, *and an* 86ϵ *equivalence* $f : C \oplus D \oplus T \to D' \oplus T'$.

For each i, the 86ϵ isomorphism $f_i : C_i \oplus D_i \oplus T_i \to D'_i \oplus T'_i$ is the composition of an 86ϵ deformation

$$C_i \oplus D_i \oplus T_i = G_0 \xrightarrow{g_1} G_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} G_m = D'_i \oplus T'_i \ .$$

Each g_j is either

(1) an elementary automorphism of the form $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} : \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2] \to \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2]$, or

(2) a geometric isomorphism $\mathbb{Z}[S] \to \mathbb{Z}[S']$ made up of paths with coefficient ± 1 which give a one-to-one correspondence of the basis elements S and S'.

Make a new 86ϵ deformation $G_0 \xrightarrow{g'_1} G_1 \xrightarrow{g'_2} \cdots \xrightarrow{g'_m} G_m$ as follows. Firstly, if g_j is of type (1) above, then define g'_j by the matrix $\begin{pmatrix} 1 & h' \\ 0 & 1 \end{pmatrix}$, where $h' : \mathbb{Z}[S_2] \to \mathbb{Z}[S_1]$ is obtained from h by deleting paths whose starting points are in $p_X^{-1}(Y^{(20+86\times2)\epsilon})$. Then $g'_j = 1$ over $Y^{192\epsilon}$. Secondly, if g_j is of type (2), then let $g'_j = g_j$. Then the new deformation is geometric over $Y^{106\epsilon}$ and defines a geometric isomorphism of D with a geometric submodule of $D'_i \oplus T'_i$ lying over $Y^{106\epsilon}$. We can delete D_i and the corresponding submodule to get an 86ϵ-simple isomorphism

$$\bar{f}_i : C_i \oplus T_i \to E_i \ ,$$

where E_i is the submodule of $D'_i \oplus T'_i$ generated by the basis elements corresponding to the basis elements of $C_i \oplus T_i$. We define the boundary map $d_E : E_i \to E_{i-1}$ by the 173ϵ morphism $\bar{f}_{i-1} \circ (d_C \oplus d_T) \circ \bar{f}_i^{-1}$. The composition d_E^2 is $(86 \times 3 + 2)\epsilon$ homotopic to 0, and therefore (E, d_E) is a 173ϵ chain complex. Note that $f'_i = f_i$ outside of $Y^{(20+86\times3)\epsilon}$; therefore, d_E is equal to $f_{i-1} \circ (d_C \oplus d_T) \circ f_i$ outside of $Y^{(20+86\times4)\epsilon}$, and it is 172ϵ homotopic to the boundary map of T' there, since f is an 86ϵ chain map. Replace the boundary map d_E by the boundary map of T' outside of $Y^{(20+86\times4)\epsilon}$. Now E splits as the sum of a free chain complex $F = \{E_i(Y^{(20+86\times5)\epsilon})\}$ and a trivial chain complex $T'' = \{E_i(X - Y^{(20+86\times5)\epsilon})\}$, and $\bar{f}$ is a $(1 + 86 \times 3)\epsilon$-simple isomorphism between $C \oplus T$ and $F \oplus T''$. This completes the proof. $\square$

2. Controlled L-theory

In [2], Pedersen, Quinn, and Ranicki established the controlled surgery exact sequence when the local fndamental group of the control map is trivial. One of the key ingredients was the splitting of quadratic Poincaré complexes. In this section we discuss an obstruction for splitting in the general case.

Fix a control map p_B on a metric space B and fix $n \geq 3$. Let (D, ψ) be an n-dimensional quadratic Poincaré complex on p_B of radius $< \delta$ ([4][6]), and let W be a subset of B. One can construct pairs $(C \to D')$, $(C \to D'')$ such that D' and D'' lie over $B - W$ and W^γ respectively and the union $D' \cup_C D''$ is equivalent to D. C is Ranicki's algebraic boundary of D''. As it stands, it has two flaws:

(1) It may lie all over $B - W$.

(2) It may be non-trivial in degrees n and -1.

The second flaw is easy to remedy. Homologically, C is $(n - 1)$-dimensional, and one can use the usual folding argument to make it into a strictly $(n - 1)$-dimensional complex. Here we need $n \geq 3$. The first flaw is harder to remedy, but C is strongly contractible away from $V = W^\gamma \cap (B - W)$, and defines an element $\xi \in Wh(B - W, V; p, n-1, \gamma)$, where $p = p_B|(B-W)$. Recall from [5] that there is a constant $\kappa > 1$ and a homomorphism

$$\partial : Wh(B - W, V; p, n - 1, \gamma) \longrightarrow \widetilde{K}_0(V^{\kappa\gamma}, p|V^{\kappa\gamma}, n - 1, \kappa\gamma) .$$

The image $\partial(\xi)$ is the obstruction for splitting. Roughly speaking C is equivalent to a projective chain complex P lying over $V^{\kappa\gamma}$, and this represents $\partial(\xi)$. If this element is 0, then C is equivalent to a free chain complex lying over $V^{\kappa\gamma}$, and we get the desired splitting.

When the local fundamental group of the control map is trivial, the absolute Wh groups and $\widetilde{K}_0$ groups are stably trivial (see 8.1 and 8.2 of [5] when p=1). Therefore there is no obstruction. In fact, since the sequence

$$\ldots \longrightarrow Wh(B - W; p, n - 1, \gamma) \longrightarrow Wh(B - W, V; p, n - 1, \gamma) \longrightarrow \widetilde{K}_0(V^{\kappa\gamma}, p|V^{\kappa\gamma}, n - 1, \kappa\gamma)$$

is stably exact (5.3 of [5]), the relative Whitehead group also vanishes stably, and hence $[C] = 0 \in Wh(B - W, V^{\lambda\gamma}; p, n - 1, \lambda\gamma)$ for some $\lambda > 0$. By the main theorem, there exist $(n - 1)$-dimensional trivial chain complexes T and T' and an $(n-1)$-dimensional free $\alpha\lambda\gamma$ chain complex F on $p_B|V^{(1+\alpha)\lambda\gamma}$ such that $C \oplus T$ and $F \oplus T'$ are $\alpha\lambda\gamma$-simple isomorphic. This F can be used as the center piece for the splitting. This is essentially the proof for splitting in [2]. And from this the stability (squeezing) of controlled L-theory follows.

References

[1] E. K. Pedersen, Controlled algebraic K-theory, a survey, *Geometry and topology:Aarhus (1998)*, 351 – 368, Contemp. Math., **258**, Amer.math. Soc., Providence, RI, 2000.

[2] E. Pedersen, F. Quinn and A. Ranicki, Controlled surgery with trivial local fundamental groups, (preprint).

[3] F. Quinn, Ends of maps I., Annals of Maths. **110**, 275 – 331 (1979).

[4] A. Ranicki and M. Yamasaki, Symmetric and quadratic complexes with geometric control, Proc. of TGRC-KOSEF vol.3, 139 – 152 (1993), available electronically on WWW from http://math.josai.ac.jp/~yamasaki/.

[5] A. Ranicki and M. Yamasaki, Controlled K-theory, Topology Appl. **61**, 1 – 59 (1995).

[6] M. Yamasaki, L-groups of crystallographic groups, Invent. Math. **88**, 571 – 602 (1987).

Dept. of Mathematics
Josai University
Sakado, Saitama 350-0295, Japan
yamasaki@math.josai.ac.jp