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Continuous graphs and crossed products of Cuntz algebras.

Takeshi Katsura (勝良 健史)
Department of Mathematical Sciences
University of Tokyo, Komaba, Tokyo, 153-8914, JAPAN
e-mail: katsu@ms.u-tokyo.ac.jp

0 Introduction

In [Ka1, Ka2, Ka3], the author examined the structure of crossed products of Cuntz-algebras by so-called quasi-free actions of abelian groups. Recently, he introduced a new class of C^*-algebras which are arising from *continuous graphs* [Ka4]. These C^*-algebras are generalization of graph algebras [KPRR, KPR, FLR] and homeomorphism C^*-algebras [T3, T4]. The above crossed products are examples of C^*-algebras arising from continuous graphs. From this point of view, some of results in [Ka1] and [Ka3] can be considered as a continuous counterpart of ones in [BHRS] and [HS]. This observation is further studied in [Ka5] for more general settings.

In this short article, we give a definition of continuous graphs and C^*-algebras associated with them, and then discuss how the results in [Ka1] and [Ka3] can be interpreted in terms of continuous graphs.

1 C^*-algebras arising from continuous graphs

Definition 1.1 Let E^0 and E^1 be locally compact (Hausdorff) spaces. A map $d: E^1 \to E^0$ is said to be *locally homeomorphic* if for any $e \in E^1$, there exists a neighborhood U of e such that the restriction of d on U is a homeomorphism onto $d(U)$ and that $d(U)$ is a neighborhood of $d(e)$.

Every local homeomorphisms are continuous and open.

Definition 1.2 ([Ka4, Definition 2.1]) A *continuous graph* $E = (E^0, E^1, d, r)$ consists of two locally compact spaces E^0, E^1, a local homeomorphism $d : E^1 \to E^0$, and a continuous map $r : E^1 \to E^0$.

Note that $d, r : E^1 \to E^0$ are not necessarily surjective nor injective. We think that E^0 is a set of vertices and E^1 is a set of edges and that an edge $e \in E^1$ is directed from its domain $d(e) \in E^0$ to its range $r(e) \in E^0$. From a homeomorphism σ on a locally compact space X, we can define a continuous graph $E = (E^0, E^1, d, r)$ by $E^0 = E^1 = X$, $d = \mathrm{id}$ and $r = \sigma$. In this sense, a continuous graph can be considered as a generalization of dynamical systems.

Let us denote by $C_d(E^1)$ the set of continuous functions ξ of E^1 such that $\langle \xi \,|\, \xi \rangle(v) = \sum_{e \in d^{-1}(v)} |\xi(e)|^2 < \infty$ for any $v \in E^0$ and $\langle \xi \,|\, \xi \rangle \in C_0(E^0)$. For $\xi, \eta \in C_d(E^1)$ and $f \in C_0(E^0)$, we define $\xi f \in C_d(E^1)$ and $\langle \xi \,|\, \eta \rangle \in C_0(E^0)$ by

$$(\xi f)(e) = \xi(e) f(d(e)) \quad \text{for } e \in E^1,$$

$$\langle \xi \,|\, \eta \rangle(v) = \sum_{e \in d^{-1}(v)} \overline{\xi(e)} \eta(e) \quad \text{for } v \in E^0.$$

With these operations, $C_d(E^1)$ is a (right) Hilbert $C_0(E^0)$-module ([Ka4, Proposition 1.10]). We define a left action π_r of $C_0(E^0)$ on $C_d(E^1)$ by $(\pi_r(f)\xi)(e) = f(r(e))\xi(e)$ for $e \in E^1$, $\xi \in C_d(E^1)$ and $f \in C_0(E^0)$. Thus we get a Hilbert $C_0(E^0)$-bimodule $C_d(E^1)$.

Definition 1.3 Let $E = (E^0, E^1, d, r)$ be a continuous graph. A *Toeplitz E-pair* on a C^*-algebra A is a pair of maps $T = (T^0, T^1)$ where $T^0 : C_0(E^0) \to A$ is a $*$-homomorphism and $T^1 : C_d(E^1) \to A$ is a linear map satisfying that

(i) $T^1(\xi)^* T^1(\eta) = T^0(\langle \xi \,|\, \eta \rangle)$ for $\xi, \eta \in C_d(E^1)$,

(ii) $T^0(f) T^1(\xi) = T^1(\pi_r(f)\xi)$ for $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$.

For $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$, the equation $T^1(\xi)T^0(f) = T^1(\xi f)$ holds automatically from the condition (i). For a Toeplitz E-pair $T = (T^0, T^1)$, we write $C^*(T)$ for denoting the C^*-algebra generated by the images of the maps T^0 and T^1. We can define a $*$-homomorphism $\Phi^1 : \mathcal{K}(C_d(E^1)) \to C^*(T)$ by $\Phi^1(\theta_{\xi,\eta}) = T^1(\xi)T^1(\eta)^*$ for $\xi, \eta \in C_d(E^1)$ where $\theta_{\xi,\eta} \in \mathcal{K}(C_d(E^1))$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi\langle \eta \,|\, \zeta \rangle$ for $\zeta \in C_d(E^1)$.

Definition 1.4 Let $E = (E^0, E^1, d, r)$ be a continuous graph. We define three open subsets $E^0_{\mathrm{sce}}, E^0_{\mathrm{fin}}$ and E^0_{rg} of E^0 by $E^0_{\mathrm{sce}} = E^0 \setminus \overline{r(E^1)}$,

$$E^0_{\mathrm{fin}} = \{v \in E^0 \mid \text{there exists a neighborhood } V \text{ of } v \text{ such that } r^{-1}(V) \subset E^1 \text{ is compact}\},$$

and $E^0_{\mathrm{rg}} = E^0_{\mathrm{fin}} \setminus \overline{E^0_{\mathrm{sce}}}$. We define two closed subsets E^0_{inf} and E^0_{sg} of E^0 by $E^0_{\mathrm{inf}} = E^0 \setminus E^0_{\mathrm{fin}}$ and $E^0_{\mathrm{sg}} = E^0 \setminus E^0_{\mathrm{rg}}$.

A vertex in E^0_{sce} is called a *source*. When E is a discrete graph, E^0_{fin} is the set of vertices which receive finitely many edges, while E^0_{inf} is the set of vertices which receive infinitely many edges. A vertex in E^0_{rg} is said to be *regular*, and a vertex in E^0_{sg} is said to be *singular*. Clearly we have that $E^0_{\mathrm{sce}} \subset E^0_{\mathrm{fin}}$ and $E^0_{\mathrm{sg}} = \overline{E^0_{\mathrm{sce}}} \cup E^0_{\mathrm{inf}}$. We have that $\ker \pi_r = C_0(E^0_{\mathrm{sce}})$ and $\pi_r^{-1}(\mathcal{K}(C_d(E^1))) = C_0(E^0_{\mathrm{fin}})$ ([Ka4, Proposition 1.24]). Hence the restriction of π_r on $C_0(E^0_{\mathrm{rg}})$ is an injection into $\mathcal{K}(C_d(E^1))$.

Definition 1.5 Let $E = (E^0, E^1, d, r)$ be a continuous graph. A Toeplitz E-pair $T = (T^0, T^1)$ is called a *Cuntz-Krieger E-pair* if $T^0(f) = \Phi^1(\pi_r(f))$ for any $f \in C_0(E^0_{\mathrm{rg}})$.

We denote by $\mathcal{O}(E)$ the universal C^*-algebra generated by a Cuntz-Krieger E-pair

When E is a discrete graph, $\mathcal{O}(E)$ is isomorphic to the graph algebra of the opposite graph of E. When a continuous graph E is defined by a homeomorphism σ on a locally compact space X, $\mathcal{O}(E)$ is isomorphic to the homeomorphism C^*-algebra $C_0(X) \rtimes_\sigma \mathbb{Z}$. We have that t^0 is injective ([Ka4, Proposition 3.7]). Let $\mathbb{T}$ be the group of complex numbers $z \in \mathbb{C}$ with $|z| = 1$. By the universality of $\mathcal{O}(E)$, there exists an action $\beta : \mathbb{T} \curvearrowright \mathcal{O}(E)$ defined by $\beta_z(t^0(f)) = t^0(f)$ and $\beta_z(t^1(\xi)) = zt^1(\xi)$ for $f \in C_0(E^0)$, $\xi \in C_d(E^1)$ and $z \in \mathbb{T}$. The action β is called the *gauge action*. The next theorem says that the injectivity of T^0 together with the existence of a gauge action implies the universality of T.

Theorem 1.6 ([Ka4, Theorem 4.5]) *For a continuous graph E and a Cuntz-Krieger E-pair T, the natural surjection $\mathcal{O}(E) \to C^*(T)$ is an isomorphism if and only if T^0 is injective and there exists an automorphism β'_z of $C^*(T)$ such that $\beta'_z(T^0(f)) = T^0(f)$ and $\beta'_z(T^1(\xi)) = zT^1(\xi)$ for every $z \in \mathbb{T}$.*

2 Invariant subsets of continuous graphs

We review definitions and results in [Ka5]. Let $E = (E^0, E^1, d, r)$ be a continuous graph.

Definition 2.1 A subset X^0 of E^0 is said to be *positively invariant* if $d(e) \in X^0$ implies $r(e) \in X^0$ for each $e \in E^1$, and to be *negatively invariant* if for $v \in X^0 \cap E^0_{\mathrm{rg}}$, there exists $e \in E^1$ with $r(e) = v$ and $d(e) \in X^0$. A subset X^0 of E^0 is said to be *invariant* if X^0 is both positively and negatively invariant.

These terminologies coincides with the ordinal ones when continuous graphs are arising from dynamical systems. When E is a discrete graph, X^0 is positively invariant if and only if its complement is hereditary, and X^0 is negatively invariant if and only if its complement is saturated (cf. [BHRS]). For a closed positively invariant subset X^0 of E^0, we set $X^1 = d^{-1}(X^0)$. Then $X = (X^0, X^1, d, r)$ is a continuous graph. A closed positively invariant set X^0 is invariant if and only if $X^0_{\mathrm{sg}} \subset E^0_{\mathrm{sg}} \cap X^0$.

Definition 2.2 A pair $\rho = (X^0, Z)$ of closed subsets of E^0 satisfying the following two conditions is called an *admissible* pair;

(i) X^0 is invariant,

(ii) $X^0_{\mathrm{sg}} \subset Z \subset E^0_{\mathrm{sg}} \cap X^0$.

Definition 2.3 For an admissible pair $\rho = (X^0, Z)$, we define a continuous graph $E_\rho = (E^0_\rho, E^1_\rho, d_\rho, r_\rho)$ as follows. Set $Y_\rho = X^0_{\mathrm{rg}} \cap Z$, $\partial Y_\rho = \overline{Y_\rho} \setminus Y_\rho$, and define

$$E^0_\rho = X^0 \amalg_{\partial Y_\rho} \overline{Y_\rho}\ , \qquad E^1_\rho = X^1 \amalg_{d^{-1}(\partial Y_\rho)} d^{-1}(\overline{Y_\rho}).$$

The domain map $d_\rho : E^1_\rho \to E^0_\rho$ is defined from $d : X^1 \to X^0$ and $d : d^{-1}(\overline{Y_\rho}) \to \overline{Y_\rho}$. The range map $r_\rho : E^1_\rho \to E^0_\rho$ is defined from $r : X^1 \to X^0$ and $r : d^{-1}(\overline{Y_\rho}) \to X^0$.

Note that for an admissible pair $\rho = (X^0, Z)$ with $Z = X^0_{\mathrm{rg}}$, we have $E_\rho = X$. Define a C^*-subalgebra $\mathcal{F}^1 \subset \mathcal{O}(E)$ and a $*$-homomorphism $\pi_0^1 : \mathcal{F}^1 \to C_0(E^0_{\mathrm{sg}})$ by

$$\mathcal{F}^1 = \{t^0(f) + \varphi^1(x) \mid f \in C_0(E^0), x \in \mathcal{K}(C_d(E^1))\},$$

and $\pi_0^1(t^0(f) + \varphi^1(x)) = f|_{E^0_{\mathrm{sg}}}$. For an ideal I of $\mathcal{O}(E)$, we define closed subsets X^0_I and Z_I of E^0 by

$$X^0_I = \{v \in E^0 \mid f(v) = 0 \text{ for all } f \in C_0(E^0) \text{ with } t^0(f) \in I\},$$
$$Z_I = \{v \in E^0_{\mathrm{sg}} \mid f(v) = 0 \text{ for all } f \in \pi_0^1(I \cap \mathcal{F}^1)\}.$$

Proposition 2.4 *For an ideal I of $\mathcal{O}(E)$, the pair $\rho_I = (X^0_I, Z_I)$ is an admissible pair.*

By using Theorem 1.6, we can show the following.

Proposition 2.5 *For a gauge-invariant ideal I of $\mathcal{O}(E)$, there exists a natural isomorphism $\mathcal{O}(E)/I \cong \mathcal{O}(E_{\rho_I})$.*

From this proposition and some computation, we get the next theorem.

Theorem 2.6 *The map $I \mapsto \rho_I$ gives us an inclusion reversing one-to-one correspondence between the set of all gauge-invariant ideals and the set of all admissible pairs.*

This theorem is a continuous counterpart of [BHRS, Theorem 3.6]. It is known that gauge-invariant ideals of a homeomorphism C^*-algebra correspond bijectively to closed invariant subsets [T2, Theorem 2]. The next proposition is a generalization of this fact.

Proposition 2.7 *When a continuous graph E satisfies that $E^0_{\mathrm{rg}} = E^0$, the map $I \mapsto X^0_I$ gives an inclusion reversing one-to-one correspondence between the set of all gauge-invariant ideals and the set of closed invariant sets.*

Proof. For a closed invariant set X^0, we have $X^0_{\mathrm{sg}} = E^0_{\mathrm{sg}} \cap X^0 = \emptyset$. Hence admissible pairs correspond bijectively to closed invariant subsets. Now the assertion follows from Theorem 2.6. ∎

3 Free and topologically free continuous graphs

For $n = 2, 3, \ldots$, we define a space E^n of *paths* with length n by

$$E^n = \{(e_n, \ldots, e_2, e_1) \in E^1 \times \cdots \times E^1 \times E^1 \mid d(e_{k+1}) = r(e_k) \ (1 \leq k \leq n-1)\}.$$

We define domain and range maps $d, r : E^n \to E^0$ by $d(e) = d(e_1)$ and $r(e) = r(e_n)$ for $e = (e_n, \ldots, e_1) \in E^n$. A path $e = (e_n, \ldots, e_1) \in E^n$ $(n \geq 1)$ is called a *loop* if $r(e) = d(e)$, and the vertex $r(e) = d(e)$ is called the *base point* of the loop e. A loop $e = (e_n, \ldots, e_1)$ is said to be *without entrances* if $r^{-1}(r(e_k)) = \{e_k\}$ for $k = 1, \ldots, n$.

Definition 3.1 A continuous graph E is said to be *topologically free* if the set of base points of loops without entrances has an empty interior.

This generalizes topological freeness of ordinary dynamical systems and Condition L of graph algebras (see, for example, [T1] and [KPR]).

Theorem 3.2 ([Ka4, Theorem 5.12]) *If a continuous graph $E = (E^0, E^1, d, r)$ is topologically free, then the natural surjection $\mathcal{O}(E) \to C^*(T)$ is an isomorphism for all Cuntz-Krieger E-pair $T = (T^0, T^1)$ such that T^0 is injective.*

By the above theorem, we have the following (cf. Proposition 2.5).

Proposition 3.3 ([Ka5]) *Let I be an ideal of $\mathcal{O}(E)$. If a continuous graph E_{ρ_I} is topologically free, then I is gauge-invariant.*

We define a *positive orbit space* $\mathrm{Orb}^+(v) \subset E^0$ of $v \in E^0$ by

$$\mathrm{Orb}^+(v) = \{v\} \cup \{r(e) \in E^0 \mid e \in E^n \text{ with } d(e) = v \ (n \geq 1)\}.$$

It is easy to see that a subset X^0 of E^0 is positively invariant if and only if $\mathrm{Orb}^+(v) \subset X^0$ for all $v \in X^0$. For $v \in E^0$, we define $L(v) \subset E^0$ by

$$L(v) = \{v' \in \mathrm{Orb}^+(v) \mid v \in \mathrm{Orb}^+(v')\}.$$

Definition 3.4 For a positive integer n, we denote by $\mathrm{Per}_n(E)$ the set of vertices v_1 satisfying the following three conditions;

(i) $L(v_1)$ is a finite set $\{v_1, v_2, \ldots, v_n\}$,

(ii) $\{e \in E^1 \mid d(e), r(e) \in L(v_1)\} = \{e_1, e_2, \ldots, e_n\}$ with $d(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, 2, \ldots, n$ where $v_{n+1} = v_1$,

(iii) v_1 is isolated in $\mathrm{Orb}^+(v_1)$.

We set $\mathrm{Per}(E) = \bigcup_{n=1}^{\infty} \mathrm{Per}_n(E)$ and $\mathrm{Aper}(E) = E^0 \setminus \mathrm{Per}(E)$.

An element in $\mathrm{Per}(E)$ is called a *periodic point* while an element in $\mathrm{Aper}(E)$ is called an *aperiodic point.*

Definition 3.5 A continuous graph E is said to be *free* if $\mathrm{Aper}(E) = E^0$.

This is a generalization of freeness of ordinary dynamical systems and Condition K of graph algebras (see, for example, [KPRR]).

Proposition 3.6 ([Ka5]) *A continuous graph E is free if and only if E_ρ is topologically free for every admissible pair ρ.*

In particular, free continuous graphs are topologically free.

Theorem 3.7 ([Ka5]) *If a continuous graph E is free, then every ideal is gauge-invariant. Hence the set of all ideals corresponds bijectively to the set of all admissible pairs by the map $I \mapsto \rho_I$.*

Proof. Clear from Proposition 3.6, Proposition 3.3 and Theorem 2.6. ∎

4 Crossed products of Cuntz algebras

For $n = 2, 3, \ldots, \infty$, the Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by n isometries $S_1, S_2, \ldots, S_n$ (we also use this notation for $n = \infty$), satisfying

$$\sum_{i=1}^{n} S_i S_i^* = 1 \qquad\qquad \text{if } n < \infty,$$
$$S_i^* S_j = 0 \quad (\text{for any } i, j \text{ with } i \neq j) \qquad \text{if } n = \infty.$$

We fix a locally compact abelian group G whose dual group is denoted by Γ. We always use $+$ for multiplicative operations of abelian groups except for $\mathbb{T}$. The pairing of $t \in G$ and $\gamma \in \Gamma$ is denoted by $\langle t \,|\, \gamma \rangle \in \mathbb{T}$.

Definition 4.1 Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Gamma^n$ be given. We define the action $\alpha^\omega : G \curvearrowright \mathcal{O}_n$ by

$$\alpha_t^\omega(S_i) = \langle t \,|\, \omega_i \rangle S_i \quad (i = 1, 2, \ldots, n,\ t \in G).$$

We recall some elementary facts on the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ by the action α^ω, which was stated in [Ka1]. The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ has a C^*-subalgebra $\mathbb{C}1 \rtimes_{\alpha^\omega} G$, which is isomorphic to $C_0(\Gamma)$ via the Fourier transform. We denote by T^0 the isomorphism

$$T^0 : C_0(\Gamma) \to \mathbb{C}1 \rtimes_{\alpha^\omega} G \subset \mathcal{O}_n \rtimes_{\alpha^\omega} G.$$

The Cuntz algebra $\mathcal{O}_n$ is naturally embedded into the multiplier algebra $M(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is generated as a C^*-algebra by

$$\{S_i T^0(f) \mid i \in \{1, \ldots, n\}, f \in C_0(\Gamma)\}.$$

For $\gamma_0 \in \Gamma$, we define a (reverse) shift automorphism $\sigma_{\gamma_0} : C_0(\Gamma) \to C_0(\Gamma)$ by $(\sigma_{\gamma_0} f)(\gamma) = f(\gamma + \gamma_0)$. Then we have $T^0(f) S_i = S_i T^0(\sigma_{\omega_i} f)$ for all $f \in C_0(\Gamma)$ and $i \in \{1, \ldots, n\}$. From the gauge action of $\mathcal{O}_n$, we can define an action $\beta : \mathbb{T} \curvearrowright \mathcal{O}_n \rtimes_{\alpha^\omega} G$ which is also called a gauge action. We have $\beta_z(T^0(f)) = T^0(f)$ and $\beta_z(S_i T^0(f)) = z S_i T^0(f)$ for $f \in C_0(\Gamma)$, $i \in \{1, \ldots, n\}$, and $z \in \mathbb{T}$.

Definition 4.2 Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Gamma^n$ be given. We define a continuous graph $E_\omega = (E_\omega^0, E_\omega^1, d_\omega, r_\omega)$ as follows. We set $E_\omega^0 = \Gamma$ and $E_\omega^1 = \coprod_{i=1}^n \Gamma_i$ where $\Gamma_i = \Gamma$ for $i = 1, 2, \ldots, n$. The map $d_\omega : E_\omega^1 \to E_\omega^0$ is defined by identity maps on each Γ_i, and the map $r_\omega : E_\omega^1 \to E_\omega^0$ is defined by $r_\omega|_{\Gamma_i}(\gamma) = \gamma + \omega_i$ for $i = 1, 2, \ldots, n$.

Each $v \in E_\omega^0$ receives and emits n-edges. It is easy to see that $E_\omega^0 = (E_\omega^0)_{\mathrm{rg}}$ if $n < \infty$, and $E_\omega^0 = (E_\omega^0)_{\mathrm{inf}}$ if $n = \infty$. Since d_ω is defined by identity maps, we have

$$C_{d_\omega}(E_\omega) = \bigoplus_{i=1}^{n} C_0(\Gamma_i),$$

where $C_0(\Gamma_i) = C_0(\Gamma)$ has natural Hilbert $C_0(\Gamma)$-module structure. The left action $\pi_{r_\omega} : C_0(\Gamma) \to \mathcal{L}(C_{d_\omega}(E_\omega))$ satisfies

$$\pi_{r_\omega}(f)(\xi_1, \xi_2, \ldots, \xi_n) = (\sigma_{\omega_1}(f)\xi_1, \sigma_{\omega_2}(f)\xi_2, \ldots, \sigma_{\omega_n}(f)\xi_n) \in \bigoplus_{i=1}^{n} C_0(\Gamma_i),$$

for $f \in C_0(\Gamma)$ and $(\xi_1, \xi_2, \ldots, \xi_n) \in \bigoplus_{i=1}^n C_0(\Gamma_i)$.

We have a $*$-homomorphism $T^0 : C_0(\Gamma) \to \mathcal{O}_n \rtimes_{\alpha^\omega} G$. We define a linear map $T^1 : \bigoplus_{i=1}^n C_0(\Gamma_i) \to \mathcal{O}_n \rtimes_{\alpha^\omega} G$ by

$$T^1(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{i=1}^n S_i T^0(\xi_i) \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$$

for $(\xi_1, \xi_2, \ldots, \xi_n) \in \bigoplus_{i=1}^n C_0(\Gamma_i)$.

Proposition 4.3 *The pair $T = (T^0, T^1)$ is a Cuntz-Krieger E_ω-pair, and this induces an isomorphism $\mathcal{O}(E_\omega) \cong \mathcal{O}_n \rtimes_{\alpha^\omega} G$.*

Proof. It is not difficult to see that T is a Toeplitz E_ω-pair. When $n = \infty$, T is a Cuntz-Krieger E_ω-pair because $C_0((E_\omega^0)_{\mathrm{rg}}) = 0$. When $n < \infty$, we have $C_0((E_\omega^0)_{\mathrm{rg}}) = C_0(\Gamma)$. For $f \in C_0(\Gamma)$, we see that

$$\pi_{r_\omega}(f) = \sum_{i=1}^n \theta_{\xi_i, \eta_i}$$

where $\xi_i, \eta_i \in C_0(\Gamma_i)$ satisfies that $\xi_i \overline{\eta_i} = \sigma_{\omega_i}(f)$ for $i = 1, 2, \ldots, n$. We have

$$\begin{aligned} \Phi^1(\pi_{r_\omega}(f)) &= \sum_{i=1}^n T^1(\xi_i) T^1(\eta_i)^* = \sum_{i=1}^n S_i T^0(\xi_i) T^0(\eta_i)^* S_i^* \\ &= \sum_{i=1}^n S_i T^0(\sigma_{\omega_i}(f)) S_i^* = \sum_{i=1}^n T^0(f) S_i S_i^* = T^0(f). \end{aligned}$$

Hence T is a Cuntz-Krieger E_ω-pair. By definition, T^0 is injective, and the gauge action on $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ satisfies the condition of Theorem 1.6. Hence the natural surjection $\mathcal{O}(E_\omega) \to \mathcal{O}_n \rtimes_{\alpha^\omega} G$ is an isomorphism. ∎

5 Ideal structures of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ $(n < \infty)$

In this section, we discuss the ideal structure of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ in the case that $n < \infty$. Let n be an integer grater than 1, and take $\omega \in \Gamma^n$. In [Ka1], we introduced the following notion.

Definition 5.1 ([Ka1, Definition 3.2]) A subset X^0 of Γ is called *ω-invariant* if X^0 is a closed set satisfying the following two conditions:

(i) For any $\gamma \in X^0$ and any $i \in \{1, 2, \ldots, n\}$, we have $\gamma + \omega_i \in X^0$.

(ii) For any $\gamma \in X^0$, there exists $i \in \{1, 2, \ldots, n\}$ such that $\gamma - \omega_i \in X^0$.

The condition (i) above corresponds to positive invariance of $X^0 \subset \Gamma = E^0$, and the condition (ii) corresponds to negative invariance of X^0. Hence X^0 is an ω-invariant set if and only if X^0 is a closed invariant set of the continuous graph E_ω. For an ideal I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, we define $X_I^0 \subset \Gamma$ by

$$X_I^0 = \{\gamma \in \Gamma \mid f(\gamma) = 0 \text{ for all } f \in C_0(\Gamma) \text{ with } T^0(f) \in I\}.$$

Then X_I^0 is an ω-invariant subset of Γ ([Kal, Proposition 3.3]). The following is the one of main results in [Kal].

Theorem 5.2 ([Kal, Theorem 3.14]) *The correspondence $I \mapsto X_I^0$ gives an inclusion reversing bijection between the set of gauge-invariant ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and the set of ω-invariant subsets of Γ.*

Proof. This follows from Theorem 2.6 and Proposition 2.7. ∎

Definition 5.3 ([Kal, Definition 4.2]) An ω-invariant subset X of Γ is said to be *bad* if there exists $\gamma_0 \in X$ such that there is only one element $i_0 \in \{1, 2, \ldots, n\}$ with $\gamma_0 - \omega_{i_0} \in X$, and this element i_0 satisfies that $m\omega_{i_0} = 0$ for some positive integer m. An ω-invariant subset X of Γ is said to be *good* if X is not bad.

Lemma 5.4 *An ω-invariant subset X^0 is good if and only if the continuous graph $X = (X^0, X^1, d, r)$ is topologically free.*

Proof. If an ω-invariant subset X^0 is bad, then there exists $\gamma_0 \in X^0$ satisfying that there is only one element $i_0 \in \{1, 2, \ldots, n\}$ with $\gamma_0 - \omega_{i_0} \in X^0$ and $m\omega_{i_0} = 0$ for some positive integer m. Let $V = X^0 \setminus \bigcup_{i \neq i_0} X^0 + \omega_i$. The set V is an open subset of X^0 and it is not empty because $\gamma_0 \in V$. All $\gamma \in V$ is a base point of a loop

$$\overset{\gamma}{\cdot} \longrightarrow \overset{\gamma+\omega_{i_0}}{\cdot} \longrightarrow \cdots\cdots \longrightarrow \overset{\gamma+m\omega_{i_0}=\gamma}{\cdot}$$

which has no entrances in the continuous graph X. Hence the continuous graph X is not topologically free. Conversely if the continuous graph X is not topologically free, then a base point γ of a loop without entrances satisfies that there is only one element $i_0 \in \{1, 2, \ldots, n\}$ with $\gamma_0 - \omega_{i_0} \in X^0$, and for some positive integer m we have $m\omega_{i_0} = 0$. Hence X^0 is bad. ∎

Proposition 5.5 ([Kal, Theorem 4.5]) *Let I be an ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ such that X_I^0 is good. Then I is gauge-invariant.*

Proof. Combine Proposition 3.3 and Lemma 5.4. ∎

An element $\omega \in \Gamma^n$ is said to satisfy *Condition 5.1* if for each $i \in \{1, 2, \ldots, n\}$, one of the following two conditions is satisfied ([Kal]):

(i) For any positive integer k, $k\omega_i \neq 0$.

(ii) There exists $j \neq i$ such that $-\omega_j$ is in the closed semigroup generated by $\omega_1, \ldots, \omega_n$ and $-\omega_i$.

It is not difficult to see that Condition 5.1 is exactly same as the condition that a continuous graph E_ω is free. Hence from Theorem 3.7, we get the following.

Proposition 5.6 ([Kal, Theorem 5.2]) *When ω satisfies Condition 5.1, all ideals are gauge-invariant and there is a one-to-one correspondence between the set of ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and the set of ω-invariant subsets of Γ.*

6 Ideal structures of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$

In [Ka3], we discussed, among others, the ideal structure of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. The argument there was analogous to the case that $n < \infty$ done in [Ka1]. However we need to change some details, for example, the definition of ω-invariant sets. Take $\omega = (\omega_1, \omega_2, \ldots) \in \Gamma^\infty$ and fix it.

Definition 6.1 ([Ka3, Definition 3.3]) A subset X^0 of Γ is called *ω-invariant* if X^0 is a closed set with $X^0 + \omega_i \subset X^0$ for any positive integer i.

An ω-invariant set is same as a closed positively invariant set in the continuous graph E_ω. However, note that every positively invariant subsets of E_ω are invariant because $(E_\omega^0)_{\mathrm{rg}} = \emptyset$. Hence we see that ω-invariant sets are same as closed invariant sets. For an ω-invariant set X^0, we define a closed set H_{X^0} by

$$H_{X^0} = \overline{X^0 \setminus \bigcup_{i=1}^{\infty}(X^0 + \omega_i)} \cup \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty}(X^0 + \omega_i)} \subset X^0.$$

Definition 6.2 ([Ka3, Definition 3.4]) A pair $\widetilde{X} = (X^0, X^\infty)$ of subsets of Γ is called *ω-invariant* if X^0 is an ω-invariant set, and X^∞ is a closed set satisfying $H_{X^0} \subset X^\infty \subset X^0$.

It is not difficult to see that

$$X^0_{\mathrm{sce}} = X^0 \setminus \overline{\bigcup_{i=1}^{\infty}(X^0 + \omega_i)}\,, \qquad X^0_{\mathrm{inf}} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty}(X^0 + \omega_i)},$$

and $H_{X^0} = \overline{X^0_{\mathrm{sce}}} \cup X^0_{\mathrm{inf}} = X^0_{\mathrm{sg}}$. From this fact, we see that the definition of ω-invariant pairs is same as the one of admissible pairs. For an ideal I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and $n \in \mathbb{N}$, we define the closed subset X_I^n of Γ by

$$X_I^n = \{\gamma \in \Gamma \mid f(\gamma) = 0 \text{ for all } f \in C_0(\Gamma) \text{ with } P_n T^0(f) \in I\},$$

where $P_0 = 1$ and $P_n = 1 - \sum_{i=1}^{n} S_i S_i^* \in \mathcal{O}_\infty$. Clearly, the definition of $X_I^0 \subset \Gamma$ is same as in Section 2. Set $X_I^\infty = \bigcap_{n=0}^{\infty} X_I^n$. The pair $\widetilde{X}_I = (X_I^0, X_I^\infty)$ is ω-invariant ([Ka3, Proposition 3.5]). We can see that $X_I^\infty = Z_I$. Hence Theorem 2.6 gives the following.

Theorem 6.3 ([Ka3, Theorem 3.16]) *The correspondence $I \mapsto \widetilde{X}_I$ gives a bijection between the set of gauge-invariant ideals of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and the set of ω-invariant pairs.*

An element $\omega \in \Gamma^\infty$ is said to satisfy Condition 5.1 if for each $i \in \mathbb{Z}_+$, one of the following two conditions is satisfied:

(i) For any positive integer k, $k\omega_i \neq 0$.

(ii) For $k = 1, 2, \ldots$, there exist positive integers $i_{1,k}, \ldots, i_{n_k,k}$ $(n_k \geq 1)$ with $i_{1,k} \neq i$ and $\lim_{k\to\infty} \sum_{j=1}^{n_k} \omega_{i_{j,k}} = 0$.

Similarly as in the case of $n < \infty$, we see that Condition 5.1 is exactly same as the condition that a continuous graph E_ω is free. Hence from Theorem 3.7, we get the following.

Theorem 6.4 ([Ka3, Theorem 5.3]) *Suppose that ω satisfies Condition 5.1. Then all ideal of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is gauge-invariant. Hence there exists a one-to-one correspondence between the set of ideals of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and the set of ω-invariant pairs of subsets of Γ.*

7 Primitive ideal spaces

In [Ka1] and [Ka3], we studied the ideal structures of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ by using primitive ideal spaces when ω does not satisfy Condition 5.1. These works can be considered as continuous counterparts of [HS]. So far, the author has not succeeded in generalizing these results to more general continuous graphs which are not free. Note that a continuous graph E_ω defined here is a special kind of continuous graph which satisfies that every vertices receive and emit same number of edges in the same way.

References

[BHRS] Bates, T.; Hong, J.; Raeburn, I.; Szymański, W. *The ideal structure of the C^*-algebras of infinite graphs.* Preprint.

[FLR] Fowler, N. J.; Laca, M.; Raeburn, I. *The C^*-algebras of infinite graphs.* Proc. Amer. Math. Soc. **128** (2000), no. 8, 2319–2327.

[HS] Hong, J. H.; Szymanski, W. *The primitive ideal space of the C^*-algebras of infinite graphs.* Preprint.

[Ka1] Katsura, T. *The ideal structures of crossed products of Cuntz algebras by quasi-free actions of abelian groups.* To appear in Canad. J. Math.

[Ka2] Katsura, T. *AF-embeddability of crossed products of Cuntz algebras.* To appear in J. Funct. Anal.

[Ka3] Katsura, T. *On crossed products of the Cuntz algebra $\mathcal{O}_\infty$ by quasi-free actions of abelian groups.* To appear in the proceedings of OAMP Conference (Constanta 2001).

[Ka4] Katsura, T. *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras I, fundamental results.* Preprint.

[Ka5] Katsura, T. *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras III, ideal structures.* Preprint.

[KPR] Kumjian, A.; Pask, D.; Raeburn, I. *Cuntz-Krieger algebras of directed graphs.* Pacific J. Math. **184** (1998), no. 1, 161–174.

[KPRR] Kumjian, A.; Pask, D.; Raeburn, I.; Renault, J. *Graphs, groupoids, and Cuntz-Krieger algebras.* J. Funct. Anal. **144** (1997), no. 2, 505–541.

[T1] Tomiyama, J. *The interplay between topological dynamics and theory of C^*-algebras.* Lecture Notes Series, 2. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1992.

[T2] Tomiyama, J. *Structure of ideals and isomorphisms of C^*-crossed products by single homeomorphism.* Tokyo J. Math. **23** (2000), no. 1, 1–13.

[T3] Tomiyama, J. *Hulls and kernels from topological dynamical systems and their applications to homeomorphism C^*-algebras.* Preprint.

[T4] Tomiyama, J. *On the projection theorem for homeomorphism C^*-algebras.* Preprint.