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Continuous graphs and crossed products of Cuntz algebras.

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0 Introduction

In [Ka1, Ka2, Ka3], the author examined the structure of crossed products of Cuntz-algebras by so-called quasi-free actions of abelian groups. Recently, he introduced a new class of $C^*$-algebras which are arising from continuous graphs [Ka4]. These $C^*$-algebras are generalization of graph algebras [KPRR, KPR, FLR] and homeomorphism $C^*$-algebras [T3, T4]. The above crossed products are examples of $C^*$-algebras arising from continuous graphs. From this point of view, some of results in [Ka1] and [Ka3] can be considered as a continuous counterpart of ones in [BHRS] and [HS]. This observation is further studied in [Ka5] for more general settings.

In this short article, we give a definition of continuous graphs and $C^*$-algebras associated with them, and then discuss how the results in [Ka1] and [Ka3] can be interpreted in terms of continuous graphs.

1 $C^*$-algebras arising from continuous graphs

Definition 1.1 Let $E^0$ and $E^1$ be locally compact (Hausdorff) spaces. A map $d : E^1 \to E^0$ is said to be locally homeomorphic if for any $e \in E^1$, there exists a neighborhood $U$ of $e$ such that the restriction of $d$ on $U$ is a homeomorphism onto $d(U)$ and that $d(U)$ is a neighborhood of $d(e)$.

Every local homeomorphisms are continuous and open.

Definition 1.2 ([Ka4, Definition 2.1]) A continuous graph $E = (E^0, E^1, d, r)$ consists of two locally compact spaces $E^0, E^1$, a local homeomorphism $d : E^1 \to E^0$, and a continuous map $r : E^1 \to E^0$.

Note that $d, r : E^1 \to E^0$ are not necessarily surjective nor injective. We think that $E^0$ is a set of vertices and $E^1$ is a set of edges and that an edge $e \in E^1$ is directed from its domain $d(e) \in E^0$ to its range $r(e) \in E^0$. From a homeomorphism $\sigma$ on a locally compact
space $X$, we can define a continuous graph $E = (E^0, E^1, d, r)$ by $E^0 = E^1 = X$, $d = \text{id}$ and $r = \sigma$. In this sense, a continuous graph can be considered as a generalization of dynamical systems.

Let us denote by $C_d(E^1)$ the set of continuous functions $\xi$ of $E^1$ such that $\langle \xi | \xi \rangle(v) = \sum_{e \in d^{-1}(v)} |\xi(e)|^2 < \infty$ for any $v \in E^0$ and $\langle \xi | \xi \rangle \in C_0(E^0)$. For $\xi, \eta \in C_d(E^1)$ and $f \in C_0(E^0)$, we define $\xi f \in C_d(E^1)$ and $\langle \xi | \eta \rangle \in C_0(E^0)$ by

$$
\langle \xi f \rangle(e) = \xi(e) f(d(e)) \quad \text{for } e \in E^1,
$$

$$
\langle \xi | \eta \rangle(v) = \sum_{e \in d^{-1}(v)} \overline{\xi(e)} \eta(e) \quad \text{for } v \in E^0.
$$

With these operations, $C_d(E^1)$ is a (right) Hilbert $C_0(E^0)$-module ([Ka4, Proposition 1.10]). We define a left action $\pi_r$ of $C_0(E^0)$ on $C_d(E^1)$ by $\pi_r(f)\xi(e) = f(r(e))\xi(e)$ for $e \in E^1$, $\xi \in C_d(E^1)$ and $f \in C_0(E^0)$. Thus we get a Hilbert $C_0(E^0)$-bimodule $C_d(E^1)$.

**Definition 1.3** Let $E = (E^0, E^1, d, r)$ be a continuous graph. A Toeplitz $E$-pair on a $C^*$-algebra $A$ is a pair of maps $T = (T^0, T^1)$ where $T^0 : C_0(E^0) \to A$ is a $*$-homomorphism and $T^1 : C_d(E^1) \to A$ is a linear map satisfying that

(i) $T^1(\xi)^* T^1(\eta) = T^0((\xi | \eta))$ for $\xi, \eta \in C_d(E^1)$,

(ii) $T^1(f) T^1(\xi) = T^1(\pi_r(f)\xi)$ for $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$.

For $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$, the equation $T^1(\xi)^* T^0(f) = T^1(\xi f)$ holds automatically from the condition (i). For a Toeplitz $E$-pair $T = (T^0, T^1)$, we write $C^*(T)$ for denoting the $C^*$-algebra generated by the images of the maps $T^0$ and $T^1$. We can define a $*$-homomorphism $\Phi^1 : \mathcal{K}(C_d(E^1)) \to C^*(T)$ by $\Phi^1(\theta_{\xi,\eta}) = T^1(\xi)^* T^1(\eta)$ for $\xi, \eta \in C_d(E^1)$ where $\theta_{\xi,\eta} \in \mathcal{K}(C_d(E^1))$ is defined by $\theta_{\xi,\eta} (\zeta) = \langle \xi | \eta \zeta \rangle$ for $\zeta \in \mathcal{K}(C_d(E^1))$.

**Definition 1.4** Let $E = (E^0, E^1, d, r)$ be a continuous graph. We define three open subsets $E^0_{\text{sc}}$, $E^0_{\text{rg}}$ and $E^0_{\text{inf}}$ of $E^0$ by $E^0_{\text{sc}} = E^0 \setminus r(E^1)$,

$$
E^0_{\text{fin}} = \{ v \in E^0 \mid \text{there exists a neighborhood } V \text{ of } v \text{ such that } r^{-1}(V) \subset E^1 \text{ is compact} \},
$$

and $E^0_{\text{rg}} = E^0_{\text{fin}} \setminus E^0_{\text{sc}}$. We define two closed subsets $E^0_{\text{inf}}$ and $E^0_{\text{sg}}$ of $E^0$ by $E^0_{\text{inf}} = E^0 \setminus E^0_{\text{fin}}$ and $E^0_{\text{sg}} = E^0 \setminus E^0_{\text{rg}}$.

A vertex in $E^0_{\text{sc}}$ is called a source. When $E$ is a discrete graph, $E^0_{\text{fin}}$ is the set of vertices which receive finitely many edges, while $E^0_{\text{inf}}$ is the set of vertices which receive infinitely many edges. A vertex in $E^0_{\text{rg}}$ is said to be regular, and a vertex in $E^0_{\text{sg}}$ is said to be singular. Clearly we have that $E^0_{\text{sc}} \subset E^0_{\text{fin}}$ and $E^0_{\text{sg}} = E^0_{\text{sc}} \cup E^0_{\text{inf}}$. We have that $\ker \pi_r = C_0(E^0_{\text{sc}})$ and $\pi_r^{-1}(\mathcal{K}(C_d(E^1))) = C_0(E^0_{\text{fin}})$ ([Ka4, Proposition 1.24]). Hence the restriction of $\pi_r$ on $C_0(E^0_{\text{rg}})$ is an injection into $\mathcal{K}(C_d(E^1))$.

**Definition 1.5** Let $E = (E^0, E^1, d, r)$ be a continuous graph. A Toeplitz $E$-pair $T = (T^0, T^1)$ is called a Cuntz-Krieger $E$-pair if $T^0(f) = \Phi^1(\pi_r(f))$ for any $f \in C_0(E^0_{\text{rg}})$. We denote by $\mathcal{O}(E)$ the universal $C^*$-algebra generated by a Cuntz-Krieger $E$-pair.
When $E$ is a discrete graph, $\mathcal{O}(E)$ is isomorphic to the graph algebra of the opposite graph of $E$. When a continuous graph $E$ is defined by a homeomorphism $\sigma$ on a locally compact space $X$, $\mathcal{O}(E)$ is isomorphic to the homeomorphism $C^*$-algebra $C_0(X) \rtimes_\sigma \mathbb{Z}$. We have that $t^0$ is injective ([Ka4, Proposition 3.7]). Let $T$ be the group of complex numbers $z \in \mathbb{C}$ with $|z| = 1$. By the universality of $\mathcal{O}(E)$, there exists an action $\beta : T \curvearrowright \mathcal{O}(E)$ defined by $\beta_z(t^0(e)) = t^0(f)$ and $\beta_z(t^1(\xi)) = zt^1(\xi)$ for $f \in C_0(E^0), \xi \in C_a(E^1)$ and $z \in T$. The action $\beta$ is called the gauge action. The next theorem says that the injectivity of $T^0$ together with the existence of a gauge action implies the universality of $T$.

**Theorem 1.6 ([Ka4, Theorem 4.5])** For a continuous graph $E$ and a Cuntz-Krieger $E$-pair $T$, the natural surjection $\mathcal{O}(E) \to C^*(T)$ is an isomorphism if and only if $T^0$ is injective and there exists an automorphism $\beta'_z$ of $C^*(T)$ such that $\beta'_z(T^0(f)) = T^0(f)$ and $\beta'_z(T^1(\xi)) = zT^1(\xi)$ for every $z \in T$.

## 2 Invariant subsets of continuous graphs

We review definitions and results in [Ka5]. Let $E = (E^0, E^1, d, r)$ be a continuous graph.

**Definition 2.1** A subset $X^0$ of $E^0$ is said to be positively invariant if $d(e) \in X^0$ implies $r(e) \in X^0$ for each $e \in E^1$, and to be negatively invariant if for $v \in X^0 \cap E^0_{rg}$, there exists $e \in E^1$ with $r(e) = v$ and $d(e) \in X^0$. A subset $X^0$ of $E^0$ is said to be invariant if $X^0$ is both positively and negatively invariant.

These terminologies coincides with the ordinal ones when continuous graphs are arising from dynamical systems. When $E$ is a discrete graph, $X^0$ is positively invariant if and only if its complement is hereditary, and $X^0$ is negatively invariant if and only if its complement is saturated (cf. [BHRS]). For a closed positively invariant subset $X^0$ of $E^0$, we set $X^1 = d^{-1}(X^0)$. Then $X = (X^0, X^1, d, r)$ is a continuous graph. A closed positively invariant set $X^0$ is invariant if and only if $X^0_{rg} \subset E^0_{rg} \cap X^0$.

**Definition 2.2** A pair $\rho = (X^0, Z)$ of closed subsets of $E^0$ satisfying the following two conditions is called an admissible pair;

(i) $X^0$ is invariant,
(ii) $X^0_{rg} \subset Z \subset E^0_{rg} \cap X^0$.

**Definition 2.3** For an admissible pair $\rho = (X^0, Z)$, we define a continuous graph $E_\rho = (E^0_\rho, E^1_\rho, d_\rho, r_\rho)$ as follows. Set $Y_\rho = X^0_{ri} \cap Z$, $\partial Y_\rho = \overline{Y_\rho} \setminus Y_\rho$, and define

$$E^0_\rho = X^0 \sqcup \partial Y_\rho \ \overline{Y_\rho}, \quad E^1_\rho = X^1 \sqcup d^{-1}(\partial Y_\rho) \ \overline{d^{-1}(Y_\rho)}.$$

The domain map $d_\rho : E^1_\rho \to E^0_\rho$ is defined from $d : X^1 \to X^0$ and $d : d^{-1}(\overline{Y_\rho}) \to \overline{Y_\rho}$. The range map $r_\rho : E^1_\rho \to E^0_\rho$ is defined from $r : X^1 \to X^0$ and $r : d^{-1}(\overline{Y_\rho}) \to X^0$. 
Note that for an admissible pair \( \rho = (X^0, Z) \) with \( Z = X_{\mathrm{rg}}^0 \), we have \( E_\rho = X \). Define a \( C^* \)-subalgebra \( \mathcal{F}^1 \subset \mathcal{O}(E) \) and a \( * \)-homomorphism \( \pi_0^1 : \mathcal{F}^1 \to C_0(E_{\mathrm{rg}}^0) \) by
\[
\mathcal{F}^1 = \{ t^0(f) + \varphi^1(x) \mid f \in C_0(E^0), x \in \mathcal{K}(C_d(E^1)) \},
\]
and \( \pi_0^1(t^0(f) + \varphi^1(x)) = f|_{E_{\mathrm{rg}}^0} \). For an ideal \( I \) of \( \mathcal{O}(E) \), we define closed subsets \( X_I^0 \) and \( Z_I \) of \( E^0 \) by
\[
X_I^0 = \{ v \in E^0 \mid f(v) = 0 \text{ for all } f \in C_0(E^0) \text{ with } t^0(f) \in I \},
\]
\[
Z_I = \{ v \in E_{\mathrm{rg}}^0 \mid f(v) = 0 \text{ for all } f \in \pi_0^1(I \cap \mathcal{F}^1) \}.
\]

**Proposition 2.4** For an ideal \( I \) of \( \mathcal{O}(E) \), the pair \( \rho_I = (X_I^0, Z_I) \) is an admissible pair.

By using Theorem 1.6, we can show the following.

**Proposition 2.5** For a gauge-invariant ideal \( I \) of \( \mathcal{O}(E) \), there exists a natural isomorphism \( \mathcal{O}(E)/I \cong \mathcal{O}(E_{\beta I}) \).

From this proposition and some computation, we get the next theorem.

**Theorem 2.6** The map \( I \mapsto \rho_I \) gives us an inclusion reversing one-to-one correspondence between the set of all gauge-invariant ideals and the set of all admissible pairs.

This theorem is a continuous counterpart of [BHRS, Theorem 3.6]. It is known that gauge-invariant ideals of a homeomorphism \( C^* \)-algebra correspond bijectively to closed invariant subsets [T2, Theorem 2]. The next proposition is a generalization of this fact.

**Proposition 2.7** When a continuous graph \( E \) satisfies that \( E_{\mathrm{rg}}^0 = E^0 \), the map \( I \mapsto X_I^0 \) gives an inclusion reversing one-to-one correspondence between the set of all gauge-invariant ideals and the set of closed invariant sets.

**Proof.** For a closed invariant set \( X^0 \), we have \( X_{\mathrm{rg}}^0 = E_{\mathrm{rg}}^0 \cap X^0 = \emptyset \). Hence admissible pairs correspond bijectively to closed invariant subsets. Now the assertion follows from Theorem 2.6.

## 3 Free and topologically free continuous graphs

For \( n = 2, 3, \ldots \), we define a space \( E^n \) of paths with length \( n \) by
\[
E^n = \{(e_n, \ldots, e_2, e_1) \in E^1 \times \cdots \times E^1 \times E^1 \mid d(e_{k+1}) = r(e_k) (1 \leq k \leq n - 1) \}.
\]

We define domain and range maps \( d, r : E^n \to E^0 \) by \( d(e) = d(e_1) \) and \( r(e) = r(e_n) \) for \( e = (e_n, \ldots, e_1) \in E^n \). A path \( e = (e_n, \ldots, e_1) \in E^n \) \((n \geq 1)\) is called a loop if \( r(e) = d(e) \), and the vertex \( r(e) = d(e) \) is called the base point of the loop \( e \). A loop \( e = (e_n, \ldots, e_1) \) is said to be without entrances if \( r^{-1}(r(e_k)) = \{ e_k \} \) for \( k = 1, \ldots, n \).
Definition 3.1 A continuous graph $E$ is said to be *topologically free* if the set of base points of loops without entrances has an empty interior.

This generalizes topological freeness of ordinary dynamical systems and Condition L of graph algebras (see, for example, [T1] and [KPR]).

Theorem 3.2 ([Ka4, Theorem 5.12]) If a continuous graph $E = (E^0, E^1, d, r)$ is topologically free, then the natural surjection $\mathcal{O}(E) \rightarrow C^*(T)$ is an isomorphism for all Cuntz-Krieger $E$-pair $T = (T^0, T^1)$ such that $T^0$ is injective.

By the above theorem, we have the following (cf. Proposition 2.5).

Proposition 3.3 ([Ka5]) Let $I$ be an ideal of $\mathcal{O}(E)$. If a continuous graph $E_\rho$ is topologically free, then $I$ is gauge-invariant.

We define a *positive orbit space* $\text{Orb}^+(v) \subset E^0$ of $v \in E^0$ by

$$\text{Orb}^+(v) = \{v\} \cup \{r(e) \in E^0 \mid e \in E^n \text{ with } d(e) = v \ (n \geq 1)\}.$$ 

It is easy to see that a subset $X^0$ of $E^0$ is positively invariant if and only if $\text{Orb}^+(v) \subset X^0$ for all $v \in X^0$. For $v \in E^0$, we define $L(v) \subset E^0$ by

$$L(v) = \{v' \in \text{Orb}^+(v) \mid v \in \text{Orb}^+(v')\}.$$ 

Definition 3.4 For a positive integer $n$, we denote by $\text{Per}_n(E)$ the set of vertices $v_i$ satisfying the following three conditions;

(i) $L(v_1)$ is a finite set $\{v_1, v_2, \ldots, v_n\}$,

(ii) $\{e \in E^1 \mid d(e), r(e) \in L(v_1)\} = \{e_1, e_2, \ldots, e_n\}$ with $d(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, 2, \ldots, n$ where $v_{n+1} = v_1$,

(iii) $v_1$ is isolated in $\text{Orb}^+(v_1)$.

We set $\text{Per}(E) = \bigcup_{n=1}^\infty \text{Per}_n(E)$ and $\text{Aper}(E) = E^0 \setminus \text{Per}(E)$.

An element in $\text{Per}(E)$ is called a *periodic point* while an element in $\text{Aper}(E)$ is called an *aperiodic point*.

Definition 3.5 A continuous graph $E$ is said to be *free* if $\text{Aper}(E) = E^0$.

This is a generalization of freeness of ordinary dynamical systems and Condition K of graph algebras (see, for example, [KPRR]).

Proposition 3.6 ([Ka5]) A continuous graph $E$ is free if and only if $E_\rho$ is topologically free for every admissible pair $\rho$.

In particular, free continuous graphs are topologically free.

Theorem 3.7 ([Ka5]) If a continuous graph $E$ is free, then every ideal is gauge-invariant. Hence the set of all ideals corresponds bijectively to the set of all admissible pairs by the map $I \mapsto \rho_I$.

Proof. Clear from Proposition 3.6, Proposition 3.3 and Theorem 2.6. 

}\[\]
4 Crossed products of Cuntz algebras

For \( n = 2, 3, \ldots, \infty \), the Cuntz algebra \( \mathcal{O}_n \) is the universal \( C^* \)-algebra generated by \( n \) isometries \( S_1, S_2, \ldots, S_n \) (we also use this notation for \( n = \infty \)), satisfying

\[
\sum_{i=1}^{n} S_i^* S_i = 1 \quad \text{if} \; n < \infty,
\]

\[
S_i^* S_j = 0 \quad \text{(for any} \; i, j \; \text{with} \; i \neq j) \quad \text{if} \; n = \infty.
\]

We fix a locally compact abelian group \( G \) whose dual group is denoted by \( \Gamma \). We always use + for multiplicative operations of abelian groups except for \( \mathbb{T} \). The pairing of \( t \in G \) and \( \gamma \in \Gamma \) is denoted by \( \langle t \mid \gamma \rangle \in \mathbb{T} \).

**Definition 4.1** Let \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Gamma^n \) be given. We define the action \( \alpha^\omega : G \curvearrowright \mathcal{O}_n \) by

\[
\alpha_t^\omega(S_i) = (\langle t \mid \omega_i \rangle) S_i \quad (i = 1, 2, \ldots, n, \; t \in G).
\]

We recall some elementary facts on the crossed product \( \mathcal{O}_n \times_{\alpha^\omega} G \) by the action \( \alpha^\omega \), which was stated in [Kal1]. The crossed product \( \mathcal{O}_n \times_{\alpha^\omega} G \) has a \( C^* \)-subalgebra \( C \times_{\alpha^\omega} G \), which is isomorphic to \( C_0(\Gamma) \) via the Fourier transform. We denote by \( T^0 \) the isomorphism

\[
T^0 : C_0(\Gamma) \to C_1 \times_{\alpha^\omega} G \subset \mathcal{O}_n \times_{\alpha^\omega} G.
\]

The Cuntz algebra \( \mathcal{O}_n \) is naturally embedded into the multiplier algebra \( M(\mathcal{O}_n \times_{\alpha^\omega} G) \) of \( \mathcal{O}_n \times_{\alpha^\omega} G \). The crossed product \( \mathcal{O}_n \times_{\alpha^\omega} G \) is generated as a \( C^* \)-algebra by

\[
\{S_i T^0(f) \mid i \in \{1, \ldots, n\}, f \in C_0(\Gamma)\}.
\]

For \( \gamma_0 \in \Gamma \), we define a (reverse) shift automorphism \( \sigma_{\gamma_0} : C_0(\Gamma) \to C_0(\Gamma) \) by \( \sigma_{\gamma_0} f(\gamma) = f(\gamma + \gamma_0) \). Then we have \( T^0(f) S_i = S_i T^0(\sigma_0 f) \) for all \( f \in C_0(\Gamma) \) and \( i \in \{1, \ldots, n\} \). From the gauge action of \( \mathcal{O}_n \), we can define an action \( \beta : \mathbb{T} \curvearrowright \mathcal{O}_n \times_{\alpha^\omega} G \) which is also called a gauge action. We have \( \beta_z(T^0(f)) = T^0(f) \) and \( \beta_z(S_i T^0(f)) = z S_i T^0(f) \) for \( f \in C_0(\Gamma) \), \( i \in \{1, \ldots, n\} \), and \( z \in \mathbb{T} \).

**Definition 4.2** Let \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Gamma^n \) be given. We define a continuous graph \( E_\omega = (E_\omega^0, E_\omega^1, d_\omega, r_\omega) \) as follows. We set \( E_\omega^0 = \Gamma \) and \( E_\omega^1 = \prod_{i=1}^{n} \Gamma_i \) where \( \Gamma_i = \Gamma \) for \( i = 1, 2, \ldots, n \). The map \( d_\omega : E_\omega^1 \to E_\omega^0 \) is defined by identity maps on each \( \Gamma_i \), and the map \( r_\omega : E_\omega^0 \to E_\omega^1 \) is defined by \( r_\omega|_{\Gamma_i}(\gamma) = \gamma + \omega_i \) for \( i = 1, 2, \ldots, n \).

Each \( v \in E_\omega^0 \) receives and emits \( n \)-edges. It is easy to see that \( E_\omega^0 = (E_\omega^0)_{rg} \) if \( n < \infty \), and \( E_\omega^0 = (E_\omega^0)_{inf} \) if \( n = \infty \). Since \( d_\omega \) is defined by identity maps, we have

\[
C_{d_\omega}(E_\omega^0) = \bigoplus_{i=1}^{n} C_0(\Gamma_i),
\]

where \( C_0(\Gamma_i) = C_0(\Gamma_i) \) has natural Hilbert \( C_0(\Gamma_i) \)-module structure. The left action \( \pi_{r_\omega} : C_0(\Gamma) \to \mathcal{L}(C_{d_\omega}(E_\omega^0)) \) satisfies

\[
\pi_{r_\omega}(f)(\xi_1, \xi_2, \ldots, \xi_n) = (\sigma_{\omega_1}(f) \xi_1, \sigma_{\omega_2}(f) \xi_2, \ldots, \sigma_{\omega_n}(f) \xi_n) \in \bigoplus_{i=1}^{n} C_0(\Gamma_i),
\]

where \( \sigma_{\omega_i}(f)(\xi) = \xi + \omega_i f(\xi) \) for \( i = 1, 2, \ldots, n \).
for $f \in C_0(\Gamma)$ and $(\xi_1, \xi_2, \ldots, \xi_n) \in \bigoplus_{i=1}^{n} C_0(\Gamma_i)$.

We have a $*$-homomorphism $T^0 : C_0(\Gamma) \to \mathcal{O}_n \rtimes_{\alpha^\omega} G$. We define a linear map $T^1 : \bigoplus_{i=1}^{n} C_0(\Gamma_i) \to \mathcal{O}_n \rtimes_{\alpha^\omega} G$ by

$$T^1(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{i=1}^{n} S_i T^0(\xi_i) \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$$

for $(\xi_1, \xi_2, \ldots, \xi_n) \in \bigoplus_{i=1}^{n} C_0(\Gamma_i)$.

**Proposition 4.3** The pair $T = (T^0, T^1)$ is a Cuntz-Krieger $E_{\omega}$-pair, and this induces an isomorphism $\mathcal{O}(E_\omega) \cong \mathcal{O}_n \rtimes_{\alpha^\omega} G$.

**Proof.** It is not difficult to see that $T$ is a Toeplitz $E_{\omega}$-pair. When $n = \infty$, $T$ is a Cuntz-Krieger $E_{\omega}$-pair because $C_0((E_{\omega}^0)_{\text{reg}}) = 0$. When $n < \infty$, we have $C_0((E_{\omega}^0)_{\text{reg}}) = C_0(\Gamma)$. For $f \in C_0(\Gamma)$, we see that

$$\pi_{\omega}(f) = \sum_{i=1}^{n} \theta_{\xi_i, \eta_i}$$

where $\xi_i, \eta_i \in C_0(\Gamma_i)$ satisfies that $\xi_i \eta_i = \sigma_{\omega} \cdot (f)$ for $i = 1, 2, \ldots, n$. We have

$$\Phi^1(\pi_{\omega}(f)) = \sum_{i=1}^{n} T^1(\xi_i) T^1(\eta_i)^* = \sum_{i=1}^{n} S_i T^0(\xi_i) T^0(\eta_i)^* S_i^*$$

$$= \sum_{i=1}^{n} S_i T^0(\sigma_{\omega} \cdot (f)) S_i^* = \sum_{i=1}^{n} T^0(f) S_i S_i^* = T^0(f).$$

Hence $T$ is a Cuntz-Krieger $E_{\omega}$-pair. By definition, $T^0$ is injective, and the gauge action on $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ satisfies the condition of Theorem 1.6. Hence the natural surjection $\mathcal{O}(E_\omega) \to \mathcal{O}_n \rtimes_{\alpha^\omega} G$ is an isomorphism. 

5 Ideal structures of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ ($n < \infty$)

In this section, we discuss the ideal structure of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ in the case that $n < \infty$. Let $n$ be an integer greater than 1, and take $\omega \in \Gamma^n$. In [Ka1], we introduced the following notion.

**Definition 5.1** ([Ka1, Definition 3.2]) A subset $X^0$ of $\Gamma$ is called $\omega$-invariant if $X^0$ is a closed set satisfying the following two conditions:

(i) For any $\gamma \in X^0$ and any $i \in \{1, 2, \ldots, n\}$, we have $\gamma + \omega_i \in X^0$.

(ii) For any $\gamma \in X^0$, there exists $i \in \{1, 2, \ldots, n\}$ such that $\gamma - \omega_i \in X^0$.

The condition (i) above corresponds to positive invariance of $X^0 \subset \Gamma = E^0$, and the condition (ii) corresponds to negative invariance of $X^0$. Hence $X^0$ is an $\omega$-invariant set.
if and only if $X^0$ is a closed invariant set of the continuous graph $E_\omega$. For an ideal $I$ of $\mathcal{O}_n \times_{\alpha^\omega} G$, we define $X^0_I \subset \Gamma$ by

$$X^0_I = \{ \gamma \in \Gamma \mid f(\gamma) = 0 \text{ for all } f \in C_0(\Gamma) \text{ with } T^0(f) \in I \}.$$ 

Then $X^0_I$ is an $\omega$-invariant subset of $\Gamma$ ([Ka1, Proposition 3.3]). The following is the one of main results in [Ka1].

**Theorem 5.2 ([Ka1, Theorem 3.14])** The correspondence $I \mapsto X^0_I$ gives an inclusion reversing bijection between the set of gauge-invariant ideals of $\mathcal{O}_n \times_{\alpha^\omega} G$ and the set of $\omega$-invariant subsets of $\Gamma$.

**Proof.** This follows from Theorem 2.6 and Proposition 2.7.

**Definition 5.3 ([Ka1, Definition 4.2])** An $\omega$-invariant subset $X$ of $\Gamma$ is said to be bad if there exists $\gamma_0 \in X$ such that there is only one element $i_0 \in \{1, 2, \ldots, n\}$ with $\gamma_0 - \omega_{i_0} \in X$, and this element $i_0$ satisfies that $m\omega_{i_0} = 0$ for some positive integer $m$. An $\omega$-invariant subset $X$ of $\Gamma$ is said to be good if $X$ is not bad.

**Lemma 5.4** An $\omega$-invariant subset $X^0$ is good if and only if the continuous graph $X = (X^0, X^1, d, r)$ is topologically free.

**Proof.** If an $\omega$-invariant subset $X^0$ is bad, then there exists $\gamma_0 \in X^0$ satisfying that there is only one element $i_0 \in \{1, 2, \ldots, n\}$ with $\gamma_0 - \omega_{i_0} \in X^0$ and $m\omega_{i_0} = 0$ for some positive integer $m$. Let $V = X^0 \setminus \bigcup_{i \neq i_0} X^0 + \omega_i$. The set $V$ is an open subset of $X^0$ and it is not empty because $\gamma_0 \in V$. All $\gamma \in V$ is a base point of a loop

$$\gamma \rightarrow \gamma + \omega_{i_0} \rightarrow \ldots \ldots \rightarrow \gamma + m\omega_{i_0} = \gamma$$

which has no entrances in the continuous graph $X$. Hence the continuous graph $X$ is not topologically free. Conversely if the continuous graph $X$ is not topologically free, then a base point $\gamma$ of a loop without entrances satisfies that there is only one element $i_0 \in \{1, 2, \ldots, n\}$ with $\gamma_0 - \omega_{i_0} \in X^0$, and for some positive integer $m$ we have $m\omega_{i_0} = 0$. Hence $X^0$ is bad.

**Proposition 5.5 ([Ka1, Theorem 4.5])** Let $I$ be an ideal of $\mathcal{O}_n \times_{\alpha^\omega} G$ such that $X^0_I$ is good. Then $I$ is gauge-invariant.

**Proof.** Combine Proposition 3.3 and Lemma 5.4.

An element $\omega \in \Gamma^n$ is said to satisfy **Condition 5.1** if for each $i \in \{1, 2, \ldots, n\}$, one of the following two conditions is satisfied ([Ka1]):

(i) For any positive integer $k$, $k\omega_i \neq 0$.

(ii) There exists $j \neq i$ such that $-\omega_j$ is in the closed semigroup generated by $\omega_1, \ldots, \omega_n$ and $-\omega_i$.

It is not difficult to see that Condition 5.1 is exactly same as the condition that a continuous graph $E_\omega$ is free. Hence from Theorem 3.7, we get the following.

**Proposition 5.6 ([Ka1, Theorem 5.2])** When $\omega$ satisfies Condition 5.1, all ideals are gauge-invariant and there is a one-to-one correspondence between the set of ideals of $\mathcal{O}_n \times_{\alpha^\omega} G$ and the set of $\omega$-invariant subsets of $\Gamma$. 


6  Ideal structures of \( \mathcal{O}_\infty \times \alpha^\omega G \)

In [Ka3], we discussed, among others, the ideal structure of \( \mathcal{O}_\infty \times \alpha^\omega G \). The argument there was analogous to the case that \( n < \infty \) done in [Ka1]. However we need to change some details, for example, the definition of \( \omega \)-invariant sets. Take \( \omega = (\omega_1, \omega_2, \ldots) \in \Gamma^\infty \) and fix it.

**Definition 6.1 ([Ka3, Definition 3.3])** A subset \( X^0 \) of \( \Gamma \) is called \( \omega \)-invariant if \( X^0 \) is a closed set with \( X^0 + \omega_i \subset X^0 \) for any positive integer \( i \).

An \( \omega \)-invariant set is same as a closed positively invariant set in the continuous graph \( E_\omega \). However, note that every positively invariant subsets of \( E_\omega \) are invariant because \( (E_\omega^0)^{rg} = \emptyset \). Hence we see that \( \omega \)-invariant sets are same as closed invariant sets. For an \( \omega \)-invariant set \( X^0 \), we define a closed set \( H_{X^0} \) by

\[
H_{X^0} = X^0 \setminus \bigcup_{i=1}^{\infty} (X^0 + \omega_i) \cup \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (X^0 + \omega_i) \subset X^0.
\]

**Definition 6.2 ([Ka3, Definition 3.4])** A pair \( \bar{X} = (X^0, X^\infty) \) of subsets of \( \Gamma \) is called \( \omega \)-invariant if \( X^0 \) is an \( \omega \)-invariant set, and \( X^\infty \) is a closed set satisfying \( H_{X^0} \subset X^\infty \subset X^0 \).

It is not difficult to see that

\[
X^0_{\text{ase}} = X^0 \setminus \bigcup_{i=1}^{\infty} (X^0 + \omega_i), \quad X^0_{\text{inf}} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (X^0 + \omega_i),
\]

and \( H_{X^0} = X^0_{\text{ase}} \cup X^0_{\text{inf}} = X^0_{\text{rg}} \). From this fact, we see that the definition of \( \omega \)-invariant pairs is same as the one of admissible pairs. For an ideal \( I \) of \( \mathcal{O}_\infty \times \alpha^\omega G \) and \( n \in \mathbb{N} \), we define the closed subset \( X^n_I \) of \( \Gamma \) by

\[
X^n_I = \{ \gamma \in \Gamma \mid f(\gamma) = 0 \text{ for all } f \in C_0(\Gamma) \text{ with } P_n T^\gamma (f) \in I \},
\]

where \( P_0 = 1 \) and \( P_n = 1 - \sum_{i=1}^{n} S_i S_i^* \in \mathcal{O}_\infty \). Clearly, the definition of \( X^n_I \subset \Gamma \) is same as in Section 2. Set \( X^\infty_I = \bigcup_{n=0}^{\infty} X^n_I \). The pair \( \bar{X}_I = (X^n_I, X^\infty_I) \) is \( \omega \)-invariant ([Ka3, Proposition 3.5]). We can see that \( X^\infty_I = Z_I \). Hence Theorem 2.6 gives the following.

**Theorem 6.3 ([Ka3, Theorem 3.16])** The correspondence \( I \mapsto \bar{X}_I \) gives a bijection between the set of gauge-invariant ideals of \( \mathcal{O}_\infty \times \alpha^\omega G \) and the set of \( \omega \)-invariant pairs.

An element \( \omega \in \Gamma^\infty \) is said to satisfy Condition 5.1 if for each \( i \in \mathbb{Z}_+ \), one of the following two conditions is satisfied:

(i) For any positive integer \( k \), \( k \omega_i \neq 0 \).

(ii) For \( k = 1, 2, \ldots \), there exist positive integers \( i_1, \ldots, i_{n_k}, k \) (\( n_k \geq 1 \)) with \( i_{1,k} \neq i \) and \( \lim_{k \to \infty} \sum_{j=1}^{n_k} \omega_{i_{j,k}} = 0 \).

Similarly as in the case of \( n < \infty \), we see that Condition 5.1 is exactly same as the condition that a continuous graph \( E_\omega \) is free. Hence from Theorem 3.7, we get the following.

**Theorem 6.4 ([Ka3, Theorem 5.3])** Suppose that \( \omega \) satisfies Condition 5.1. Then all ideal of \( \mathcal{O}_\infty \times \alpha^\omega G \) is gauge-invariant. Hence there exists a one-to-one correspondence between the set of ideals of \( \mathcal{O}_\infty \times \alpha^\omega G \) and the set of \( \omega \)-invariant pairs of subsets of \( \Gamma \).
7 Primitive ideal spaces

In [Kal] and [Ka3], we studied the ideal structures of $\mathcal{O}_n \times \omega G$ by using primitive ideal spaces when $\omega$ does not satisfy Condition 5.1. These works can be considered as continuous counterparts of [HS]. So far, the author has not succeeded in generalizing these results to more general continuous graphs which are not free. Note that a continuous graph $E_\omega$ defined here is a special kind of continuous graph which satisfies that every vertices receive and emit same number of edges in the same way.

References


