1. Introduction

S. Baaj and G. Skandalis [1] introduced the notion of multiplicative unitaries and they studied Hopf $C^*$-algebras associated with them. J. M. Vallin introduced the notion of pseudo-multiplicative unitaries and algebraic structures associated with them ([11], [12]). M. Enock and Vallin [2] studied pseudo-multiplicative unitaries and quantum groupoids associated with inclusions of von Neumann algebras. The author introduced a notion of multiplicative unitary operators (MUO) on Hilbert $C^*$-modules ([8], see also [6] and [7]). It is interesting to study natural algebraic structures associated with MUO's. In this note, we will show the relation between MUO's and coring structures on Hilbert $C^*$-modules. Coring structures were introduced by M. Sweedler [10] in the purely algebraic framework. Y. Watatani [13] showed that inclusions of $C^*$-algebras give natural coring structures in the framework of his index theory. In this note, we introduce notions of coring structures on Hilbert $C^*$-modules and study coring structures associated with MUO's. In Sections 2 and 3, we study coring structures associated with MUO's arising from groupoids and inclusions of $C^*$-algebras of inedex finite type in the sense of Watatani. In the case of groupoids, the base algebras are commutative. In the case of inclusions of

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C*-algebras of index finite type, we do not know any concrete examples of MUO's on infinite-dimensional Hilbert C*-modules. Therefore it is interesting to study concrete examples of MUO’s and the associated coring structures such that the base algebras are not commutative and the Hilbert C*-modules are infinite-dimensional. In the last section, we study an MUO and the associated coring structures on the Hilbert C*-module of compact operators. In this case, the Hilbert C*-module is infinite-dimensional and the base algebra is the C*-algebra of compact operators.

2. Preliminaries

2.1. Multiplicative operators on Hilbert C*-modules. Let $A$ be a C*-algebra, let $E$ be a Hilbert $A$-module and let $\phi$ and $\psi$ be *-homomorphisms of $A$ to $\mathcal{L}_A(E)$. We assume that $\phi$ and $\psi$ commute, that is, $\phi(a)\psi(b) = \psi(b)\phi(a)$ for all $a, b \in A$.

We define a *-homomorphism $\iota \otimes_\phi \psi$ of $A$ to $\mathcal{L}_A(E \otimes_\phi E)$ by $(\iota \otimes_\phi \psi)(a) = I \otimes_\phi \psi(a)$. and define a *-homomorphism $\iota \otimes_\psi \phi$ of $A$ to $\mathcal{L}_A(E \otimes_\psi E)$ by $(\iota \otimes_\psi \phi)(a) = I \otimes_\psi \phi(a)$.

Let $W$ be an operator in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$. We assume that $W$ satisfies the following equations;

\begin{align}
(2.1) \quad & W(\iota \otimes_\phi \psi)(a) = (\phi \otimes_\phi \iota)(a)W, \\
(2.2) \quad & W(\psi \otimes_\psi \iota)(a) = (\iota \otimes_\phi \psi)(a)W, \\
(2.3) \quad & W(\phi \otimes_\psi \iota)(a) = (\psi \otimes_\phi \iota)(a)W
\end{align}

for all $a \in A$. Then we can define following operators;

\begin{align*}
W \otimes_\psi I & \in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_\phi E \otimes_\psi E), \\
I \otimes_\phi \otimes_\psi W & \in \mathcal{L}_A(E \otimes_\phi E \otimes_\psi E, E \otimes_\psi E \otimes_\phi E), \\
W \otimes_\phi I & \in \mathcal{L}_A(E \otimes_\psi E \otimes_\phi E, E \otimes_\phi E \otimes_\phi E), \\
I \otimes_\psi \otimes_\phi W & \in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_\psi E \otimes_\phi E(E \otimes_\phi E)), \\
I \otimes_\iota \otimes_\phi W & \in \mathcal{L}_A(E \otimes_\iota \otimes_\phi (E \otimes_\psi E), E \otimes_\phi E \otimes_\phi E).
\end{align*}
Since $\phi$ and $\psi$ commute, there exists an isomorphism $\Sigma_{12}$ of $E \otimes_{\iota \otimes \psi} (E \otimes_{\phi} E)$ onto $E \otimes_{\iota \otimes \phi} (E \otimes_{\psi} E)$ as Hilbert $A$-modules such that, for $x_i \in E$ ($i = 1, 2, 3$),

$$\Sigma_{12}(x_1 \otimes (x_2 \otimes x_3)) = x_2 \otimes (x_1 \otimes x_3).$$

**Definition 2.1** ([8]). Let $W$ be an element of $\mathcal{L}_A(E \otimes_{\psi} E, E \otimes_{\phi} E)$. Assume that $W$ satisfies the equations (2.1), (2.2) and (2.3). An operator $W$ is said to be multiplicative if it satisfies the pentagonal equation

$$W \otimes_{\phi} (I \otimes_{\phi \otimes \iota} W)(W \otimes_{\psi} I) = (I \otimes_{\iota \otimes \phi} W)\Sigma_{12}(I \otimes_{\psi \otimes \iota} W).$$

**Example 2.2.** Suppose that $A = \mathbb{C}$. Then $E = H$ is a usual Hilbert space and $\mathcal{L}(E) = \mathcal{L}(H)$ is the $C^*$-algebra of bounded linear operators on $H$. Let $\phi = \psi = id$, where $id(\lambda) = \lambda I_H$ for $\lambda \in \mathbb{C}$. Then $E \otimes_{id} E$ is the usual tensor product $H \otimes H$. Let $\Sigma \in \mathcal{L}(H \otimes H)$ be the flip, that is, $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$. Let $W$ be an element of $\mathcal{L}(H \otimes H)$. Then the pentagonal equation (2.4) has the following form:

$$(W \otimes I)(I \otimes W)(W \otimes I) = (I \otimes W)(\Sigma \otimes I)(I \otimes W).$$

Define an operator $\overline{W}$ by $\overline{W} = W \Sigma$. Then $W$ satisfies the pentagonal equation (2.5) if and only if $\overline{W}$ satisfies the usual pentagonal equation;

$$\overline{W}_{12}\overline{W}_{13}\overline{W}_{23} = \overline{W}_{23}\overline{W}_{13}.$$

2.2. **Coproducts on Hilbert $C^*$-modules.** Let $E$ be a Hilbert $A$-module and $\phi$ be a $*$-homomorphism of $A$ to $\mathcal{L}_A(E)$.

**Definition 2.3.** Let $\delta$ be an operator in $\mathcal{L}_A(E, E \otimes_{\phi} E)$. We say that $\delta$ is a coproduct of $(E, \phi)$ if $\delta$ satisfies the following equations;

$$\delta \phi(a) = (\phi \otimes \iota)(a)\delta \quad \text{for all } a \in A,$$

$$(\delta \otimes I_{E})\delta = (I_E \otimes \delta)\delta.$$
Suppose that \( \delta \) is a coproduct for \( E \). For \( \xi, \eta \in E \), we define a product \( \xi \eta \) in \( E \) by \( \xi \eta = \delta^*(\xi \otimes_{\phi} \eta) \). It follows from (2.8) that this product is associative. Then \( E \) is an algebra over \( \mathbb{C} \). Note that we have \( ||\xi \eta|| \leq ||\delta|| ||\xi|| ||\eta|| \).

2.3. Coproducts associated with MUO’s. Let \( E \) be a Hilbert \( A \)-module and let \( \phi \) and \( \psi \) be \(*\)-homomorphisms of \( A \) to \( \mathcal{L}_A(E) \) such that \( \phi \) and \( \psi \) commute. Let \( W \in \mathcal{L}_A(E \otimes_{\psi} E, E \otimes_{\phi} E) \) be a multiplicative unitary operator (MUO).

For an element \( \xi_0 \) of \( E \), we say that \( \xi_0 \) has Property E1 if it satisfies the following conditions;

(i) \( W(\xi_0 \otimes_{\psi} \xi_0) = \xi_0 \otimes_{\phi} \xi_0 \).

(ii) For every \( \xi \in E \), there exists an element \( \pi_{\xi_0}(\xi) \) of \( \mathcal{L}_A(E) \) such that

\[
< \eta, \pi_{\xi_0}(\xi) \zeta > = < W(\xi_0 \otimes_{\psi} \eta), \xi \otimes_{\phi} \zeta > \quad \text{for every } \eta, \zeta \in E .
\]

Fix an element \( \xi_0 \) with Property E1. Define an operator \( \delta = \delta_{\xi_0} \) in \( \mathcal{L}_A(E, E \otimes_{\phi} E) \) by \( \delta(\eta) = W(\xi_0 \otimes_{\psi} \eta) \). Then we have \( \delta^*(\xi \otimes \eta) = \pi_{\xi_0}(\xi)\eta \). Since \( W \) satisfies the pentagonal equation, \( \delta \) is a coproduct of \((E, \phi)\).

For an element \( \xi_0 \) of \( E \), we say that \( \xi_0 \) has Property E2 if it satisfies the following conditions;

(i) \( W(\xi_0 \otimes_{\psi} \xi_0) = \xi_0 \otimes_{\phi} \xi_0 \).

(ii) For every \( \xi \in E \), there exists an element \( \hat{\pi}_{\xi_0}(\xi) \) of \( \mathcal{L}_A(E) \) such that

\[
< \eta, \hat{\pi}_{\xi_0}(\xi) \zeta > = < W^*(\xi_0 \otimes_{\phi} \eta), \xi \otimes_{\psi} \zeta > \quad \text{for every } \eta, \zeta \in E .
\]

Fix an element \( \xi_0 \) with Property E2. Define an operator \( \hat{\delta} = \hat{\delta}_{\xi_0} \) in \( \mathcal{L}_A(E, E \otimes_{\psi} E) \) by \( \hat{\delta}(\eta) = W^*(\xi_0 \otimes_{\phi} \eta) \). Since \( W \) satisfies the pentagonal equation, \( \hat{\delta} \) is a coproduct of \((E, \psi)\).

3. Coring structures on Hilbert \( C^* \)-modules

Let \( E \) be a Hilbert \( A \)-module and let \( \phi \) be a \(*\)-homomorphism of \( A \) to \( \mathcal{L}_A(E) \). Note that \( A \) itself is a Hilbert \( A \)-module with the \( A \)-valued inner product \( < a, b > = a^*b \).
We denote by \( i \) the \(*\)-homomorphism of \( A \) to \( \mathcal{L}_A(A) \) defined by \( i(a)b = ab \). Then there exists a unitary operator \( t \) in \( \mathcal{L}_A(E \otimes \phi A, E) \) defined by \( t(\xi \otimes a) = \xi a \). If \( \phi \) is non-degenerate, then there exists a unitary operator \( t' \) in \( \mathcal{L}_A(A \otimes_{\phi} E, E) \) such that \( t'(a \otimes_{\phi} \xi) = \phi(a)\xi \).

**Definition 3.1.** Suppose that \( \phi \) is non-degenerate. Let \( \delta \) be a coproduct of \((E, \phi)\) and let \( Q \) be an element of \( \mathcal{L}_A(E, A) \), such that \( Q \phi(a) = aQ \) for \( a \in A \).

1. We say that \((E, \phi, \delta, Q)\) is a right counital \( A \)-coring if it satisfies the following equation;

\[
t(I_E \otimes_{\phi} Q)\delta = I_E.
\]

Then \( Q \) is called a right counit.

2. We say that \((E, \phi, \delta, Q)\) is a left counital \( A \)-coring if it satisfies the following equation;

\[
t'(Q \otimes_{\phi} I_E)\delta = I_E.
\]

Then \( Q \) is called a left counit.

3. We say that \((E, \phi, \delta, Q)\) is a counital \( A \)-coring if \( Q \) is a right and left counit. Then \( Q \) is called a counit.

For \( n \geq 2 \), we set

\[
E \otimes_{\phi}^n = E \otimes_{\phi} \cdots \otimes_{\phi} E \quad (n \text{ times}).
\]

Let \((E, \phi, \delta, Q)\) be a left or right counital \( A \)-coring. We define an element \( \omega \) of \( \mathcal{L}_A(E \otimes_{\phi}^4, E \otimes_{\phi}^2) \) by

\[
\omega = \{t(I_E \otimes_{\phi} Q) \otimes_{\phi} I_E\}(I_E \otimes_{\phi} \delta^* \otimes_{\phi} I_E).
\]

Then we have \( \omega(\omega \otimes_{\phi} I) = \omega(I \otimes_{\phi} \omega) \). Therefore we can define a product on \( E \otimes_{\phi} E \) by \( xy = \omega(x \otimes_{\phi} y) \). Then \( E \otimes_{\phi} E \) is an algebra over \( \mathbb{C} \). Note that we have

\[
(\xi_1 \otimes_{\phi} \xi_2)(\eta_1 \otimes_{\phi} \eta_2) = (\xi_1 Q(\xi_2 \eta_1)) \otimes_{\phi} \eta_2.
\]
Definition 3.2. We say that $\delta$ and $Q$ are compatible if the following equation holds:
$$\delta(\xi\eta) = \delta(\xi)\delta(\eta)$$
for every $\xi$, $\eta \in E$.

Example 3.3 ([13]). Let $1 \in A_0 \subset A_1$ be an inclusion of $C^*$-algebras and let $P_1 : A_1 \to A_0$ be a faithful positive conditional expectation of index finite type. Let
$$\{u_i, u_i^*; i = 1, \cdots, N\}$$
be a quasi-basis of $P_1$. Let $E_1 = A_1$ be a Hilbert $A_0$-module with the $A_0$-valued inner product defined by $<a, b> = P_1(a^*b)$. Let $\phi_1 : A_1 \to \mathcal{L}_{A_0}(E_1)$ be a $*$-homomorphism defined by $\phi_1(a)b = ab$. We denote by $\phi_0$ the restriction of $\phi_1$ to $A_0$. Define
$$\delta \in \mathcal{L}_{A_0}(E_1, E_1 \otimes_{\phi_0} E_1)$$
by $\delta(\xi) = \sum_{i=1}^{N}(\xi u_i) \otimes_{\phi_0} u_i^*$. The product on $E_1$ induced by $\delta$ agrees with the product on $A_1$. Then $(E_1, \phi_0, \delta, P_1)$ is a compatible counital $A$-coring.

Example 3.4. Let $G$ be a finite groupoid. Set $A = C(G^{(0)})$ and $E = C(G)$. Then $E$ is a right $A$-module with the right $A$-action defined by $(\xi a)(x) = \xi(x)a(s(x))$ for $\xi \in E$, $a \in A$ and $x \in G$. We define an $A$-valued inner product of $E$ by
$$<\xi, \eta>(u) = \sum_{g \in G_u} \overline{\xi(g)}\eta(g)$$
for $\xi, \eta \in E$ and $u \in G^{(0)}$, where $G_u = s^{-1}(u)$ for $u \in G^{(0)}$. Then $E$ is a Hilbert $A$-module. Define $*$-homomorphisms $\phi$ and $\psi$ of $A$ to $\mathcal{L}_A(E)$ by $(\phi(a)\xi)(x) = a(r(x))\xi(x)$ and $\psi(a) = \xi a$ respectively for $a \in A$, $\xi \in E$ and $x \in G$. Note that we have $E \otimes_{\phi} E = C(G^2(ss))$ and $E \otimes_{\phi} E = C(G^{(2)})$, where $G^2(ss) = \{(g, h) \in G^2; s(g) = s(h)\}$. Let $W \in \mathcal{L}_A(E \otimes_{\phi} E, E \otimes_{\phi} E)$ be the MUO defined by $(W\xi)(g, h) = \xi(h, gh)$. Define an element $a_0 \in A$ by $a_0(u) = |G_u|^{-1/2}$ and define an element $\xi_0 \in E$ by $\xi_0(g) = a_0(s(g))$. Then $\xi_0$ satisfies Properties E1 and E2. Note that we have $||\xi_0|| = 1$. Define an element $\eta_0 \in E$ by $\eta_0 = \chi_{G^{(0)}}a_0^{-1}$. Define operators $Q_{\eta_0}, Q_{\xi_0} : E \to A$ by $Q_{\eta_0}(\xi) = <\eta_0, \xi>$ and $Q_{\xi_0}(\xi) = <\xi_0, \xi>$ respectively. Then $(E, \phi, \delta_{\xi_0}, Q_{\eta_0})$ is a compatible counital $A$-coring. The product on $E$ induced by $\delta_{\xi_0}$ is of the form $\xi\eta = (\xi * \eta)a_0$, where $\xi * \eta$ is the convolution product on $C(G)$. We also have two compatible right counital $A$-corings $(E, \psi, \delta_{\xi_0}, Q_{\xi_0})$. 
and $(E, \psi, \delta_{x_0}, Q_{x_0})$. Two products on $E \otimes_{\psi} E$ associated with above right counital $A$-corings are different.

4. CORING STRUCTURES ASSOCIATED WITH INCLUSIONS OF $C^*$-ALGEBRAS

Let $1 \in A_0 \subset A_1$ be an inclusion of $C^*$-algebras and let $P_1 : A_1 \to A_0$ be a faithful positive conditional expectation of index-finite type with a quasi-basis $\{u_i, u^*_i\}_{i=1}^N$. Let $E_1, \phi_1$ and $\phi_0$ be as in Example 3.3. Set $E_2 = E_1 \otimes_{\phi_0} E_1$ and define a *-homomorphism $\phi_2 : A_1 \to \mathcal{L}_{A_0}(E_2)$ by $\phi_2 = \phi_1 \otimes \iota$. Define a $C^*$-algebra $A$ by $A = \mathcal{L}_{A_0}(E_1, \phi_1)$ and a Hilbert $A$-module $E$ by

$$E = \mathcal{L}_{A_0}((E_1, \phi_1), (E_2, \phi_2)),$$

that is, $E$ is the set of elements $x \in \mathcal{L}_{A_0}(E_1, E_2)$ such that $x\phi_1(a) = \phi_2(a)x$ for all $a \in A$. The $A$-valued inner product on $E$ is defined by $\langle x, y \rangle = x^*y$. We define *-homomorphisms $\phi$ and $\psi$ of $A$ to $\mathcal{L}_A(E)$ by $\phi(a)x = (a \otimes_{\phi_0} I)x$ and $\psi(a)x = (I \otimes_{\phi_0} a)x$ respectively. We suppose that there exists an MUO $W \in \mathcal{L}_A(E \otimes_{\psi} E, E \otimes_{\phi} E)$ such that $V^*\tilde{V} = W \otimes I_{E_1}$, where $V : E \otimes_{\phi} E \otimes I_{E_1} \to E_3$ and $\tilde{V} : E \otimes_{\psi} E \otimes I_{E_1} \to E_3$ are operators defined in [8]. As for sufficient conditions for $W$ to exist, see [7] and [8]. Define an element $x_0 \in E$ by $x_0(\xi) = \xi \otimes_{\phi_0} 1$. Then $x_0$ satisfies Properties E1 and E2. Note that we have $\|x_0\| = 1$. Define an element $\tilde{y}_0 \in E$ by

$$\tilde{y}_0(\xi) = \sum_{i=1}^N (\xi u_i) \otimes_{\phi_0} u^*_i.$$

Note that we have $\tilde{y}_0^*(\xi \otimes_{\phi_0} \eta) = \xi \eta$, where $\xi \eta$ is the product on $A_1$. Define $Q_{x_0}, Q_{\tilde{y}_0} \in \mathcal{L}_A(E, A)$ by $Q_{x_0}(x) = \langle x_0, x \rangle$ and $Q_{\tilde{y}_0}(x) = \langle \tilde{y}_0, x \rangle$ respectively. Then we have the following theorem.

Theorem 4.1. (1) $(E, \phi, \delta_{x_0}, Q_{x_0})$ is a compatible right counital $A$-coring.

(2) Suppose that there exist elements $(u_i, w_i) \in E \times E$ $(i = 1, \cdots K)$ such that

$$\tilde{\delta}_{x_0}(\tilde{y}_0) = \sum_{i=1}^K u_i \otimes_{\psi} w_i.$$
Then $(E, \psi, \tilde{\delta}_{x_0}, Q_{\tilde{y}_0})$ is a compatible counital $A$-coring.

5. CORING STRUCTURES ON THE SET OF COMPACT OPERATORS

Let $H$ be an infinite-dimensional separable Hilbert space. We consider $H$ to be a Hilbert $\mathbb{C}$-module, in particular the inner product is linear in the second variable. We denote by $A$ the $C^*$-algebra $\mathcal{K}(H)$ of compact operators on $H$. Let $E$ be a Hilbert $A$-module $\mathcal{K}(H, H \otimes H)$. The right action of $A$ on $E$ is defined by $(xa)(\xi) = x(a(\xi))$ for $x \in E$, $a \in A$ and $\xi \in H$ and the $A$-valued inner product of $E$ is defined by $\langle x, y \rangle = x^*y$ for $x, y \in E$. Define $*$-homomorphisms $\phi$ and $\psi$ of $A$ to $\mathcal{L}_A(E)$ by $\phi(a)x = (a \otimes I_H)x$ and $\psi(a)x = (I_H \otimes a)x$ for $a \in A$ and $x \in E$ respectively. We denote by $F$ the Hilbert $A$-module $\mathcal{K}(H, H \otimes H \otimes H)$. The right action of $A$ on $F$ and the $A$-valued inner product of $F$ are defined by the same formulas as those in $E$. There exist unitary operators $M \in \mathcal{L}_A(E \otimes_{\phi} E, F)$ and $\overline{M} \in (E \otimes_{\psi} E, F)$ such that

$$M(x \otimes_{\phi} y) = (x \otimes I_H)y,$$
$$\overline{M}(x \otimes_{\psi} y) = (I_H \otimes x)y$$

for $x, y \in E$ respectively. Define $W = M^{-1}\overline{M}$. Then we have the following:

**Theorem 5.1.** The operator $W$ is the unique multiplicative unitary operator in $\mathcal{L}_A(E \otimes_{\psi} E, E \otimes_{\phi} E)$.

Now we introduce a coring structure on $(E, \phi, \psi)$. Recall that an approximate unit $\{u_n\}_{n=1}^{\infty}$ of $A$ is said to be increasing if $u_n \geq 0$ and $u_{n+1} \geq u_n$ for every $n$.

**Definition 5.2.** Let $\delta$ be a coproduct of $(E, \phi)$. For $n = 1, 2, \cdots$, let $Q_n$ be an element of $\mathcal{L}_A(E, A)$ such that $Q_n(\phi(a)x) = aQ_n(x)$ for $a \in A$ and $x \in E$ and let $\{u_n\}_{n=1}^{\infty}$ be an increasing approximate unit of $A$ such that $u_1 \neq 0$ and $u_n \neq u_{n+1}$ for every $n$. Then $(\delta, \{Q_n\}_{n=1}^{\infty}, \{u_n\}_{n=1}^{\infty})$ is called a coring structure on $(E, \phi, \psi)$ if it
satisfies the following equations for every \( n \);
\[
\begin{align*}
t(I_E \otimes \phi Q_n)\delta &= t'(Q_n \otimes \phi I_E)\delta = \psi(u_n), \\
Q_n \psi(u_n) &= Q_n.
\end{align*}
\]

Then \( \{Q_n\} \) is called an approximate counit.

Let \( T \) be an element of \( \mathcal{L}(H, H \otimes H) \). We will say that \( T \) has Property \( D \) if it satisfies the following conditions:

(i) \( (T \otimes I_H)T = (I_H \otimes T)T \).

(ii) There exists a family \( \{K_n\}_{n=1}^{\infty} \) of mutually orthogonal non-trivial finite-dimensional subspaces of \( H \) such that \( H = \bigoplus_{n=1}^{\infty} K_n \) and there exists a complete orthonormal basis \( \{e_{k_{n-1}+1}, \ldots, e_{k_n}\} \) of \( K_n \) for \( n = 1, 2, \ldots \), where \( k_0 = 0 \), such that, if we set \( \lambda_{j,\ell}^i = \langle e_j \otimes e_\ell, Te_i \rangle \), then \( \{\lambda_{j,\ell}^i\} \) satisfies the following conditions;

(a) for \( i = k_{n-1} + 1 \), \( \lambda_{i,i}^i \neq 0 \) and \( \lambda_{j,\ell}^i = \lambda_{\ell,j}^i = 0 \) for every \( j \in \mathbb{N} \) and \( \ell = k_m + 1 \) (\( m = 0, 1, 2, \ldots \)) except for \( j = \ell = i \),

(b) if \( \dim K_n \geq 2 \), for \( i = k_{n-1} + 2, \ldots, k_n \),

\[
\lambda_{i,k_{n-1}+1}^i = \lambda_{k_{n-1}+1,i}^i = \lambda_{k_{n-1}+1,k_{n-1}+1}^i,
\]

and \( \lambda_{j,\ell}^i = \lambda_{i,j}^i = 0 \) for every \( j \in \mathbb{N} \) and \( \ell = k_m + 1 \) (\( m = 0, 1, 2, \ldots \)) except for \( (j, \ell) = (i, k_{n-1} + 1) \).

Then we have the following theorem:

**Theorem 5.3.** There exists a one-to-one correspondence between the set of coring structures \( (\delta, \{Q_n\}, \{u_n\}) \) on \( (E, \phi, \psi) \) and the set of elements \( (T, \{K_n\}, \{e_{k_{n-1}+1}\}) \) which satisfy Property \( D \). The correspondence is given as follows: If \( (T, \{K_n\}, \{e_{k_{n-1}+1}\}) \)
has Property D, set

\[ H_n = \bigoplus_{i=1}^{n} K_i, \]

\[ \xi_n = \sum_{i=1}^{n} \eta_i \in H_n, \quad \text{where} \quad \eta_i = \left( \frac{\lambda_{k_{i-1}+1,k_{i-1}+1}}{k_{i-1}+1} \right)^{-1} e_{k_{i-1}+1} \in K_i, \]

define \( f_n \in H^* \) by \( f_n(\xi) = \langle \xi_n, \xi \rangle \), then \( u_n \in \mathcal{K}(H) \) is the projection onto \( H_n \) and \( \delta \) and \( Q_n \) are given by the following equations;

\[ \delta(x) = M^{-1}(I_H \otimes T)x, \]

\[ Q_n(x) = (I_H \otimes f_n)x. \]

**Question.** Suppose that \( T \) has Property D. Does \( T \) determine \( \{K_n\} \) and \( \{e_{k_{n-1}+1}\} \) uniquely?

The following theorem shows the relation between the coring structures and the multiplicative unitary operator \( W \) defined above:

**Theorem 5.4.** Let \( (\delta, \{Q_n\}, \{u_n\}) \) be a coring structure on \((E, \phi, \psi)\) and let \( T \) be the operator which corresponds to \( (\delta, \{Q_n\}, \{u_n\}) \) by Theorem 5.3. Put \( x_n = T u_n \). Then \( x_n \) is an element of \( E \) and satisfies Property E1. Let \( \delta_n \) be the coproduct of \((E, \phi)\) defined by

\[ \delta_n(x) = W(x_n \otimes_{\psi} x). \]

Then the following equation holds;

\[ \delta = \lim_{n \to \infty} \delta_n \]

with respect to the strict topology on \( \mathcal{L}_A(E, E \otimes_{\phi} E) \).

**References**


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