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CORING STRUCTURES AND HILBERT C^*-MODULES

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1. Introduction

S. Baaj and G. Skandalis [1] introduced the notion of multiplicative unitaries and they studied Hopf C^*-algebras associated with them. J. M. Vallin introduced the notion of pseudo-multiplicative unitaries and algebraic structures associated with them ([11], [12]). M. Enock and Vallin [2] studied pseudo-multiplictive unitaries and quantum groupoids associated with inclusions of von Neumann algebras. The author introduced a notion of multiplicative unitary operators (MUO) on Hilbert C^*-modules ([8], see also [6] and [7]). It is interesting to study natural algebraic structures associated with MUO's. In this note, we will show the relation btween MUO's and coring structures on Hilbert C^*-modules. Coring structures were introduced by M. Sweedler [10] in the purely algebraic framework. Y. Watatani [13] showed that incusions of C^*-algebras give natural coring structures in the framework of his index theory. In this note, we introduce notions of coring structures on Hilbert C^*-modules and study coring structures associated with MUO's. In Sections 2 and 3, we study coring structures associated with MUO's arising from groupoids and inclusions of C^*-algebras of inedx finite type in the sense of Watatani. In the case of groupoids, the base algebras are commutative. In the case of inclusions of

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C^*-algebras of index finite type, we do not know any concrete examples of MUO's on infinite-dimensional Hilbert C^*-modules. Therefore it is interesting to study concrete examples of MUO's and the associated coring structures such that the base algebras are not commutative and the Hilbert C^*-modules are infinite-dimensional. In the last section, we study an MUO and the associatd coring structures on the Hilbert C^*-module of compact operators. In this case, the Hilbert C^*-module is infinite-dimensional and the base algebra is the C^*-algebra of compact operators.

2. Preliminaries

2.1. **Multiplicative operators on Hilbert C^*-modules.** Let A be a C^*-algebra, let E be a Hilbert A-module and let ϕ and ψ be $*$-homomorphisms of A to $\mathcal{L}_A(E)$. We assume that ϕ and ψ commute, that is, $\phi(a)\psi(b) = \psi(b)\phi(a)$ for all a, $b \in A$. We define a $*$-homomorphism $\iota \otimes_\phi \psi$ of A to $\mathcal{L}_A(E \otimes_\phi E)$ by $(\iota \otimes_\phi \psi)(a) = I \otimes_\phi \psi(a)$. and define a $*$-homomorphism $\iota \otimes_\psi \phi$ of A to $\mathcal{L}_A(E \otimes_\psi E)$ by $(\iota \otimes_\psi \phi)(a) = I \otimes_\psi \phi(a)$. Let W be an operator in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$. We assume that W satisfies the following equations;

$$W(\iota \otimes_\psi \phi)(a) = (\phi \otimes_\phi \iota)(a)W, \tag{2.1}$$

$$W(\psi \otimes_\psi \iota)(a) = (\iota \otimes_\phi \psi)(a)W, \tag{2.2}$$

$$W(\phi \otimes_\psi \iota)(a) = (\psi \otimes_\phi \iota)(a)W \tag{2.3}$$

for all $a \in A$. Then we can define following operators;

$$W \otimes_\psi I \in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_\phi E \otimes_\psi E),$$

$$I \otimes_{\phi\otimes\iota} W \in \mathcal{L}_A(E \otimes_\phi E \otimes_\psi E, E \otimes_\psi E \otimes_\phi E),$$

$$W \otimes_\phi I \in \mathcal{L}_A(E \otimes_\psi E \otimes_\phi E, E \otimes_\phi E \otimes_\phi E),$$

$$I \otimes_{\psi\otimes\iota} W \in \mathcal{L}_A(E \otimes_\psi E \otimes_\psi E, E \otimes_{\iota\otimes\psi} (E \otimes_\phi E)),$$

$$I \otimes_{\iota\otimes\phi} W \in \mathcal{L}_A(E \otimes_{\iota\otimes\phi} (E \otimes_\psi E), E \otimes_\phi E \otimes_\phi E).$$

Since ϕ and ψ commute, there exists an isomorphism Σ_{12} of $E \otimes_{\iota\otimes\psi} (E \otimes_\phi E)$ onto $E \otimes_{\iota\otimes\phi} (E \otimes_\psi E)$ as Hilbert A-modules such that, for $x_i \in E$ $(i = 1, 2, 3)$,

$$\Sigma_{12}(x_1 \otimes (x_2 \otimes x_3)) = x_2 \otimes (x_1 \otimes x_3).$$

Definition 2.1 ([8]). Let W be an element of $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$. Assume that W satisfies the equations (2.1), (2.2) and (2.3). An operator W is said to be multiplicative if it satisfies the pentagonal equation

$$(W \otimes_\phi I)(I \otimes_{\phi\otimes\iota} W)(W \otimes_\psi I) = (I \otimes_{\iota\otimes\phi} W)\Sigma_{12}(I \otimes_{\psi\otimes\iota} W). \tag{2.4}$$

Example 2.2. Suppose that $A = \mathbb{C}$. Then $E = H$ is a usual Hilbert space and $\mathcal{L}_\mathbb{C}(E) = \mathcal{L}(H)$ is the C^*-albebra of bounded linear operators on H. Let $\phi = \psi = id$, where $id(\lambda) = \lambda I_H$ for $\lambda \in \mathbb{C}$. Then $E \otimes_{id} E$ is the usual tensor product $H \otimes H$. Let $\Sigma \in \mathcal{L}(H \otimes H)$ be the flip, that is, $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$. Let W be an element of $\mathcal{L}(H \otimes H)$. Then the pentagonal equation (2.4) has the following form:

$$(W \otimes I)(I \otimes W)(W \otimes I) = (I \otimes W)(\Sigma \otimes I)(I \otimes W). \tag{2.5}$$

Defin an operator $\widetilde{W}$ by $\widetilde{W} = W\Sigma$. Then W satisfies the pentagonal equation (2.5) if and only if $\widetilde{W}$ satisfies the usual pentagonal equation ;

$$\widetilde{W}_{12}\widetilde{W}_{13}\widetilde{W}_{23} = \widetilde{W}_{23}\widetilde{W}_{13}. \tag{2.6}$$

2.2. **Coproducts on Hilbert C^*-modules.** Let E be a Hilbert A-module and ϕ be a $*$-homomorphism of A to $\mathcal{L}_A(E)$.

Definition 2.3. Let δ be an operator in $\mathcal{L}_A(E, E \otimes_\phi E)$. We say that δ is a coproduct of (E, ϕ) if δ satisfies the following equations;

$$\delta\phi(a) = (\phi \otimes \iota)(a)\delta \qquad \text{for all } a \in A, \tag{2.7}$$

$$(\delta \otimes I_E)\delta = (I_E \otimes \delta)\delta. \tag{2.8}$$

Suppose that δ is a coproduct for E. For ξ, $\eta \in E$, we define a product $\xi\eta$ in E by $\xi\eta = \delta^*(\xi \otimes_\phi \eta)$. It follows from (2.8) that this product is associative. Then E is an algebra over $\mathbb{C}$. Note that we have $||\xi\eta|| \leq ||\delta||||\xi||||\eta||$.

2.3. **Coproducts associated with MUO's.** Let E be a Hilbert A-module and let ϕ and ψ be $*$-homomorphisms of A to $\mathcal{L}_A(E)$ such that ϕ and ψ commute. Let $W \in \mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ be a multiplicative unitary operator (MUO).

For an element ξ_0 of E, we say that ξ_0 has Property E1 if it satisfies the following conditions;

(i) $W(\xi_0 \otimes_\psi \xi_0) = \xi_0 \otimes_\phi \xi_0$.

(ii) For every $\xi \in E$, there exists an element $\pi_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$< \eta, \pi_{\xi_0}(\xi)\zeta > = < W(\xi_0 \otimes_\psi \eta), \xi \otimes_\phi \zeta > \quad \text{for every } \eta,\ \zeta \in E.$$

Fix an element ξ_0 with Property E1. Define an operator $\delta = \delta_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_\phi E)$ by $\delta(\eta) = W(\xi_0 \otimes_\psi \eta)$. Then we have $\delta^*(\xi \otimes \eta) = \pi_{\xi_0}(\xi)\eta$. Since W satisfies the pentagonal equation, δ is a coproduct of (E, ϕ).

For an element ξ_0 of E, we say that ξ_0 has Property E2 if it satisfies the following conditions;

(i) $W(\xi_0 \otimes_\psi \xi_0) = \xi_0 \otimes_\phi \xi_0$.

(ii) For every $\xi \in E$, there exists an element $\widehat{\pi}_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$< \eta, \widehat{\pi}_{\xi_0}(\xi)\zeta > = < W^*(\xi_0 \otimes_\phi \eta), \xi \otimes_\psi \zeta > \quad \text{for every } \eta,\ \zeta \in E.$$

Fix an element ξ_0 with Property E2. Define an operator $\widehat{\delta} = \widehat{\delta}_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_\psi E)$ by $\widehat{\delta}(\eta) = W^*(\xi_0 \otimes_\phi \eta)$. Since W satisfies the pentagonal equation, $\widehat{\delta}$ is a coproduct of (E, ψ).

3. Coring structures on Hilbert C^*-modules

Let E be a Hilbert A-module and let ϕ be a $*$-homomorphism of A to $\mathcal{L}_A(E)$. Note that A itself is a Hilbert A-module with the A-valued inner product $< a, b >= a^*b$.

We denote by i the $*$-homomorphism of A to $\mathcal{L}_A(A)$ defined by $i(a)b = ab$. Then there exists a unitary operator t in $\mathcal{L}_A(E \otimes_i A, E)$ defined by $t(\xi \otimes_i a) = \xi a$. If ϕ is non-degenerate, then there exists a unitary operator t' in $\mathcal{L}_A(A \otimes_\phi E, E)$ such that $t'(a \otimes_\phi \xi) = \phi(a)\xi$.

Definition 3.1. Suppose that ϕ is non-degenerate. Let δ be a coproduct of (E, ϕ) and let Q be an element of $\mathcal{L}_A(E, A)$, such that $Q\phi(a) = aQ$ for $a \in A$.

(1) We say that (E, ϕ, δ, Q) is a right counital A-coring if it satisfies the following equation;

$$t(I_E \otimes_\phi Q)\delta = I_E.$$

Then Q is called a right counit.

(2) We say that (E, ϕ, δ, Q) is a left counital A-coring if it satisfies the following equation;

$$t'(Q \otimes_\phi I_E)\delta = I_E.$$

Then Q is called a left counit.

(3) We say that (E, ϕ, δ, Q) is a counital A-coring if Q is a right and left counit. Then Q is called a counit.

For $n \geq 2$, we set

$$E^{\otimes_\phi n} = E \otimes_\phi \cdots \otimes_\phi E \qquad (n \text{ times }).$$

Let (E, ϕ, δ, Q) be a left or right counital A-coring. We defin an element ω of $\mathcal{L}_A(E^{\otimes_\phi 4}, E^{\otimes_\phi 2})$ by

$$\omega = \{t(I_E \otimes_\phi Q) \otimes_\phi I_E\}(I_E \otimes_{\phi \otimes \iota} \delta^* \otimes_\phi I_E).$$

Then we have $\omega(\omega \otimes_{\phi \otimes \iota} I) = \omega(I \otimes_{\phi \otimes \iota} \omega)$. Therefore we can difine a product on $E \otimes_\phi E$ by $xy = \omega(x \otimes_{\phi \otimes \iota} y)$. Then $E \otimes_\phi E$ is an algebra over $\mathbb{C}$. Note that we have

$$(\xi_1 \otimes_\phi \xi_2)(\eta_1 \otimes_\phi \eta_2) = (\xi_1 Q(\xi_2 \eta_1)) \otimes_\phi \eta_2.$$

Definition 3.2. We say that δ and Q are compatible if the following equation holds; $\delta(\xi\eta) = \delta(\xi)\delta(\eta)$ for every ξ, $\eta \in E$.

Example 3.3 ([13]). Let $1 \in A_0 \subset A_1$ be an inclusion of C^*-algebras and let $P_1 : A_1 \to A_0$ be a faithful positive conditional expectation of index finite type. Let $\{u_i, u_i^*; i = 1, \cdots, N\}$ be a quasi-basis of P_1. Let $E_1 = A_1$ be a Hilbert A_0-module with the A_0-valued inner product defined by $< a, b >= P_1(a^*b)$. Let $\phi_1 : A_1 \to \mathcal{L}_{A_0}(E_1)$ be a $*$-homomorphism defined by $\phi_1(a)b = ab$. We denote by ϕ_0 the restriction of ϕ_1 to A_0. Define $\delta \in \mathcal{L}_{A_0}(E_1, E_1 \otimes_{\phi_0} E_1)$ by $\delta(\xi) = \sum_{i=1}^{N}(\xi u_i) \otimes_{\phi_0} u_i^*$. The product on E_1 induced by δ agrees with the product on A_1. Then $(E_1, \phi_0, \delta, P_1)$ is a compatible counital A-coring.

Example 3.4. Let G be a finite groupoid. Set $A = C(G^{(0)})$ and $E = C(G)$. Then E is a right A-module with the right A-action defined by $(\xi a)(x) = \xi(x)a(s(x))$ for $\xi \in E$, $a \in A$ and $x \in G$. We define an A-valued inner product of E by

$$< \xi, \eta > (u) = \sum_{g \in G_u} \overline{\xi(g)}\eta(g)$$

for ξ, $\eta \in E$ and $u \in G^{(0)}$, where $G_u = s^{-1}(u)$ for $u \in G^{(0)}$. Then E is a Hilbert A-module. Define $*$-homomorphisms ϕ and ψ of A to $\mathcal{L}_A(E)$ by $(\phi(a)\xi)(x) = a(r(x))\xi(x)$ and $\psi(a) = \xi a$ respectively for $a \in A$, $\xi \in E$ and $x \in G$. Note that we have $E \otimes_\psi E = C(G^2(ss))$ and $E \otimes_\phi E = C(G^{(2)})$, where $G^2(ss) = \{(g, h) \in G^2;\ s(g) = s(h)\}$. Let $W \in \mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ be the MUO defined by $(W\xi)(g, h) = \xi(h, gh)$. Define an element $a_0 \in A$ by $a_0(u) = |G_u|^{-1/2}$ and define an element $\xi_0 \in E$ by $\xi_0(g) = a_0(s(g))$. Then ξ_0 satisfies Properties E1 and E2. Note that we have $\|\xi_0\| = 1$. Define an element $\eta_0 \in E$ by $\eta_0 = \chi_{G^{(0)}} a_0^{-1}$. Define operators Q_{η_0}, $Q_{\xi_0} : E \to A$ by $Q_{\eta_0}(\xi) =< \eta_0, \xi >$ and $Q_{\xi_0}(\xi) =< \xi_0, \xi >$ respectively. Then $(E, \phi, \delta_{\xi_0}, Q_{\eta_0})$ is a compatible counital A-coring. The product on E induced by δ_{ξ_0} is of the form $\xi\eta = (\xi * \eta)a_0$, where $\xi * \eta$ is the convolution product on $C(G)$. We also have two compatible right counital A-corings $(E, \psi, \delta_{\xi_0}, Q_{\xi_0})$

and $(E, \psi, \delta_{\xi_0}, Q_{\eta_0})$. Two products on $E \otimes_\psi E$ associated with above right counital A-corings are different.

4. Coring structures associated with inclusions of C^*-algebras

Let $1 \in A_0 \subset A_1$ be an inclusion of C^*-algebras and let $P_1 : A_1 \to A_0$ be a faithful positive conditional expectation of index-finite type with a quasi-basis $\{u_i, u_i^*\}_{i=1}^N$. Let E_1, ϕ_1 and ϕ_0 be as in Example 3.3. Set $E_2 = E_1 \otimes_{\phi_0} E_1$ and define a $*$-homomorphism $\phi_2 : A_1 \to \mathcal{L}_{A_0}(E_2)$ by $\phi_2 = \phi_1 \otimes \iota$. Define a C^*-algebra A by $A = \mathcal{L}_{A_0}(E_1, \phi_1)$ and a Hilbert A-module E by

$$E = \mathcal{L}_{A_0}((E_1, \phi_1), (E_2, \phi_2)),$$

that is, E is the set of elements $x \in \mathcal{L}_{A_0}(E_1, E_2)$ such that $x\phi_1(a) = \phi_2(a)x$ for all $a \in A$. The A-valued inner product on E is defined by $< x, y >= x^*y$. We define $*$-homomorphisms ϕ and ψ of A to $\mathcal{L}_A(E)$ by $\phi(a)x = (a \otimes_{\phi_0} I)x$ and $\psi(a)x = (I \otimes_{\phi_0} a)x$ respectively. We suppose that there exists an MUO $W \in \mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^*\widetilde{V} = W \otimes_i I_{E_1}$, where $V : E \otimes_\phi E \otimes_i E_1 \to E_3$ and $\widetilde{V} : E \otimes_\psi E \otimes_i E_1 \to E_3$ are operators defined in [8]. As for sufficient conditions for W to exist, see [7] and [8]. Define an element $x_0 \in E$ by $x_0(\xi) = \xi \otimes_{\phi_0} 1$. Then x_0 satisfies Properties E1 and E2. Note that we have $\|x_0\| = 1$. Define an element $\widetilde{y_0} \in E$ by

$$\widetilde{y_0}(\xi) = \sum_{i=1}^N (\xi u_i) \otimes_{\phi_0} u_i^*.$$

Note that we have $\widetilde{y_0}^*(\xi \otimes_{\phi_0} \eta) = \xi\eta$, where $\xi\eta$ is the product on A_1. Define $Q_{x_0}, Q_{\widetilde{y_0}} \in \mathcal{L}_A(E, A)$ by $Q_{x_0}(x) =< x_0, x >$ and $Q_{\widetilde{y_0}}(x) =< \widetilde{y_0}, x >$ respecively. Then we have the following theorem.

Theorem 4.1. (1) *$(E, \phi, \delta_{x_0}, Q_{x_0})$ is a compatible right counital A-coring.*

(2) *Suppose that there esist elements $(v_i, w_i) \in E \times E$ $(i = 1, \cdots K)$ such that*

$$\widehat{\delta}_{x_0}(\widetilde{y_0}) = \sum_{i=1}^K v_i \otimes_\psi w_i.$$

Then $(E, \psi, \widehat{\delta}_{x_0}, Q_{\widetilde{y_0}})$ is a compatible counital A-coring.

5. Coring structures on the set of compact operators

Let H be an infinite-dimensional separable Hilbert space. We consider H to be a Hilbert $\mathbb{C}$-module, in particular the inner product is linear in the second variable. We denote by A the C^*-algebra $\mathcal{K}(H)$ of compact operators on H. Let E be a Hilbert A-module $\mathcal{K}(H, H \otimes H)$. The right action of A on E is defined by $(xa)(\xi) = x(a(\xi))$ for $x \in E$, $a \in A$ and $\xi \in H$ and the A-valued inner product of E is defined by $< x, y >= x^*y$ for x, $y \in E$. Define $*$-homomorphisms ϕ and ψ of A to $\mathcal{L}_A(E)$ by $\phi(a)x = (a \otimes I_H)x$ and $\psi(a)x = (I_H \otimes a)x$ for $a \in A$ and $x \in E$ respectively. We denote by F the Hilbert A-module $\mathcal{K}(H, H \otimes H \otimes H)$. The right action of A on F and the A-valued inner product of F are defined by the same formulas as those in E. There exist unitary operators $M \in \mathcal{L}_A(E \otimes_\phi E, F)$ and $\widetilde{M} \in (E \otimes_\psi E, F)$ such that

$$M(x \otimes_\phi y) = (x \otimes I_H)y,$$

$$\widetilde{M}(x \otimes_\psi y) = (I_H \otimes x)y$$

for x, $y \in E$ respectively. Define $W = M^{-1}\widetilde{M}$. Then we have the following:

Theorem 5.1. *The operator W is the unique multiplicative unitary operator in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$.*

Now we introduce a coring structure on (E, ϕ, ψ). Recall that an approximate unit $\{u_n\}_{n=1}^\infty$ of A is said to be increasing if $u_n \geq 0$ and $u_{n+1} \geq u_n$ for every n.

Definition 5.2. Let δ be a coproduct of (E, ϕ). For $n = 1, 2, \cdots$, let Q_n be an element of $\mathcal{L}_A(E, A)$ such that $Q_n(\phi(a)x) = aQ_n(x)$ for $a \in A$ and $x \in E$ and let $\{u_n\}_{n=1}^\infty$ be an increasing approximate unit of A such that $u_1 \neq 0$ and $u_n \neq u_{n+1}$ for every n. Then $(\delta, \{Q_n\}_{n=1}^\infty, \{u_n\}_{n=1}^\infty)$ is called a coring structure on (E, ϕ, ψ) if it satisfies the following equations for every n;

$$t(I_E \otimes_\phi Q_n)\delta = t'(Q_n \otimes_\phi I_E)\delta = \psi(u_n),$$

$$Q_n \psi(u_n) = Q_n.$$

Then $\{Q_n\}$ is called an approximate counit.

Let T be an element of $\mathcal{L}(H, H \otimes H)$. We will say that T has Property D if it satisfies the following conditions:

(i) $(T \otimes I_H)T = (I_H \otimes T)T$.

(ii) There exists a family $\{K_n\}_{n=1}^\infty$ of mutually orthogonal non-trivial finite-dimensional subspaces of H such that $H = \oplus_{n=1}^\infty K_n$ and there exists a complete orthonormal basis $\{e_{k_{n-1}+1}, \cdots, e_{k_n}\}$ of K_n for $n = 1, 2, \cdots$, where $k_0 = 0$, such that, if we set $\lambda^i_{j,\ell} =< e_j \otimes e_\ell, Te_i >$, then $\{\lambda^i_{j.\ell}\}$ satisfies the following conditions;

(a) for $i = k_{n-1} + 1$, $\lambda^i_{i,i} \neq 0$ and $\lambda^i_{j,\ell} = \lambda^i_{\ell,j} = 0$ for every $j \in \mathbb{N}$ and $\ell = k_m + 1$ $(m = 0, 1, 2, \cdots)$ except for $j = \ell = i$,

(b) if dim $K_n \geq 2$, for $i = k_{n-1} + 2, \cdots, k_n$,

$$\lambda^i_{i,k_{n-1}+1} = \lambda^i_{k_{n-1}+1,i} = \lambda^{k_{n-1}+1}_{k_{n-1}+1,k_{n-1}+1},$$

and $\lambda^i_{j,\ell} = \lambda^i_{\ell,j} = 0$ for every $j \in \mathbb{N}$ and $\ell = k_m + 1$ $(m = 0, 1, 2, \cdots)$ except for $(j, \ell) = (i, k_{n-1} + 1)$.

Then we have the following theorem:

Theorem 5.3. *There exists a one-to-one correspondence between the set of coring structures $(\delta, \{Q_n\}, \{u_n\})$ on (E, ϕ, ψ) and the set of elements $(T, \{K_n\}, \{e_{k_{n-1}+1}\})$ which satisfy Property D. The correspondence is given as follows: If $(T, \{K_n\}, \{e_{k_{n-1}+1}\})$*

has Property D, set

$$H_n = \oplus_{i=1}^{n} K_i,$$

$$\xi_n = \sum_{i=1}^{n} \eta_i \in H_n, \quad \textit{where } \eta_i = (\overline{\lambda_{k_{i-1}+1,k_{i-1}+1}^{k_{i-1}+1}})^{-1} e_{k_{i-1}+1} \in K_i,$$

define $f_n \in H^*$ *by* $f_n(\xi) = < \xi_n, \xi >$, *then* $u_n \in \mathcal{K}(H)$ *is the projection onto* H_n *and* δ *and* Q_n *are given by the following equations;*

$$\delta(x) = M^{-1}(I_H \otimes T)x,$$

$$Q_n(x) = (I_H \otimes f_n)x.$$

Question. Suppose that T has Property D. Does T determine $\{K_n\}$ and $\{e_{k_{n-1}+1}\}$ uniquely?

The following theorem shows the relation between the coring structures and the multiplicative unitary operator W defined above:

Theorem 5.4. *Let* $(\delta, \{Q_n\}, \{u_n\})$ *be a coring structure on* (E, ϕ, ψ) *and let* T *be the operator which corresponds to* $(\delta, \{Q_n\}, \{u_n\})$ *by Theorem 5.3. Put* $x_n = Tu_n$. *Then* x_n *is an element of* E *and satisfies Property E1. Let* δ_n *be the coproduct of* (E, ϕ) *defined by*

$$\delta_n(x) = W(x_n \otimes_\psi x).$$

Then the following equation holds;

$$\delta = \lim_{n\to\infty} \delta_n$$

with respect to the strict topology on $\mathcal{L}_A(E, E \otimes_\phi E)$.

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