1. INTRODUCTION

S. Baaj and G. Skandalis [1] introduced the notion of multiplicative unitaries and they studied Hopf $C^*$-algebras associated with them. J. M. Vallin introduced the notion of pseudo-multiplicative unitaries and algebraic structures associated with them ([11], [12]). M. Enock and Vallin [2] studied pseudo-multiplicative unitaries and quantum groupoids associated with inclusions of von Neumann algebras. The author introduced a notion of multiplicative unitary operators (MUO) on Hilbert $C^*$-modules ([8], see also [6] and [7]). It is interesting to study natural algebraic structures associated with MUO's. In this note, we will show the relation between MUO's and coring structures on Hilbert $C^*$-modules. Coring structures were introduced by M. Sweedler [10] in the purely algebraic framework. Y. Watatani [13] showed that inclusions of $C^*$-algebras give natural coring structures in the framework of his index theory. In this note, we introduce notions of coring structures on Hilbert $C^*$-modules and study coring structures associated with MUO's. In Sections 2 and 3, we study coring structures associated with MUO's arising from groupoids and inclusions of $C^*$-algebras of inedex finite type in the sense of Watatani. In the case of groupoids, the base algebras are commutative. In the case of inclusions of

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\( C^* \)-algebras of index finite type, we do not know any concrete examples of MUO’s on infinite-dimensional Hilbert \( C^* \)-modules. Therefore it is interesting to study concrete examples of MUO’s and the associated coring structures such that the base algebras are not commutative and the Hilbert \( C^* \)-modules are infinite-dimensional. In the last section, we study an MUO and the associated coring structures on the Hilbert \( C^* \)-module of compact operators. In this case, the Hilbert \( C^* \)-module is infinite-dimensional and the base algebra is the \( C^* \)-algebra of compact operators.

2. Preliminaries

2.1. Multiplicative operators on Hilbert \( C^* \)-modules. Let \( A \) be a \( C^* \)-algebra, let \( E \) be a Hilbert \( A \)-module and let \( \phi \) and \( \psi \) be \(*\)-homomorphisms of \( A \) to \( \mathcal{L}_A(E) \). We assume that \( \phi \) and \( \psi \) commute, that is, \( \phi(a)\psi(b) = \psi(b)\phi(a) \) for all \( a, b \in A \).

We define a \(*\)-homomorphism \( \iota \otimes \phi \psi \) of \( A \) to \( \mathcal{L}_A(E \otimes \phi E) \) by \((\iota \otimes \phi \psi)(a) = I \otimes \phi \psi(a)\) and define a \(*\)-homomorphism \( \iota \otimes \psi \phi \) of \( A \) to \( \mathcal{L}_A(E \otimes \psi E) \) by \((\iota \otimes \psi \phi)(a) = I \otimes \psi \phi(a)\).

Let \( W \) be an operator in \( \mathcal{L}_A(E \otimes \psi E, E \otimes \phi E) \). We assume that \( W \) satisfies the following equations;

\[
\begin{align*}
(2.1) \quad & W(\iota \otimes \phi \psi)(a) = (\phi \otimes \iota \phi)(a)W, \\
(2.2) \quad & W(\psi \otimes \psi \iota)(a) = (\iota \otimes \psi \psi)(a)W, \\
(2.3) \quad & W(\phi \otimes \psi \iota)(a) = (\psi \otimes \iota \psi)(a)W
\end{align*}
\]

for all \( a \in A \). Then we can define following operators;

\[
\begin{align*}
W \otimes \psi I & \in \mathcal{L}_A(E \otimes \psi E \otimes \psi E, E \otimes \phi E \otimes \psi E), \\
I \otimes \phi \otimes \psi W & \in \mathcal{L}_A(E \otimes \phi E \otimes \psi E, E \otimes \psi E \otimes \phi E), \\
W \otimes \phi I & \in \mathcal{L}_A(E \otimes \psi E \otimes \phi E, E \otimes \phi E \otimes \phi E), \\
I \otimes \psi \otimes \phi W & \in \mathcal{L}_A(E \otimes \psi E \otimes \phi E, E \otimes \iota \psi (E \otimes \phi E)), \\
I \otimes \iota \otimes \phi W & \in \mathcal{L}_A(E \otimes \iota \otimes \phi (E \otimes \psi E), E \otimes \phi E \otimes \phi E).
\end{align*}
\]
Since $\phi$ and $\psi$ commute, there exists an isomorphism $\Sigma_{12}$ of $E \otimes_{\iota \otimes \psi} (E \otimes_{\phi} E)$ onto $E \otimes_{\iota \otimes \phi} (E \otimes_{\psi} E)$ as Hilbert $A$-modules such that, for $x_i \in E$ ($i = 1, 2, 3$),

$$\Sigma_{12}(x_1 \otimes (x_2 \otimes x_3)) = x_2 \otimes (x_1 \otimes x_3).$$

**Definition 2.1** ([8]). Let $W$ be an element of $\mathcal{L}_A(E \otimes_{\psi} E, E \otimes_{\phi} E)$. Assume that $W$ satisfies the equations (2.1), (2.2) and (2.3). An operator $W$ is said to be multiplicative if it satisfies the pentagonal equation

$$(W \otimes_{\phi} I)(I \otimes_{\phi \otimes \iota} W)(W \otimes_{\psi} I) = (I \otimes_{\iota \otimes \phi} W)\Sigma_{12}(I \otimes_{\psi \otimes \iota} W).$$

**Example 2.2.** Suppose that $A = \mathbb{C}$. Then $E = H$ is a usual Hilbert space and $\mathcal{L}_C(E) = \mathcal{L}(H)$ is the $C^*$-algebra of bounded linear operators on $H$. Let $\phi = \psi = id$, where $id(\lambda) = \lambda I_H$ for $\lambda \in \mathbb{C}$. Then $E \otimes_{id} E$ is the usual tensor product $H \otimes H$.

Let $\Sigma \in \mathcal{L}(H \otimes H)$ be the flip, that is, $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$. Let $W$ be an element of $\mathcal{L}(H \otimes H)$. Then the pentagonal equation (2.4) has the following form:

$$(W \otimes I)(I \otimes W)(W \otimes I) = (I \otimes W)(\Sigma \otimes I)(I \otimes W).$$

Define an operator $\overline{W}$ by $\overline{W} = W \Sigma$. Then $W$ satisfies the pentagonal equation (2.5) if and only if $\overline{W}$ satisfies the usual pentagonal equation;

$$\overline{W}_{12}\overline{W}_{13}\overline{W}_{23} = \overline{W}_{23}\overline{W}_{13}. $$

**2.2. Coproducts on Hilbert $C^*$-modules.** Let $E$ be a Hilbert $A$-module and $\phi$ be a *-homomorphism of $A$ to $\mathcal{L}_A(E)$.

**Definition 2.3.** Let $\delta$ be an operator in $\mathcal{L}_A(E, E \otimes_{\phi} E)$. We say that $\delta$ is a coproduct of $(E, \phi)$ if $\delta$ satisfies the following equations:

$$(\delta \otimes I_E)\delta = (I_E \otimes \delta)\delta.$$
Suppose that $\delta$ is a coproduct for $E$. For $\xi, \eta \in E$, we define a product $\xi \eta$ in $E$ by $\xi \eta = \delta^*(\xi \otimes_{\phi} \eta)$. It follows from (2.8) that this product is associative. Then $E$ is an algebra over $\mathbb{C}$. Note that we have $||\xi \eta|| \leq ||\delta||||\xi||||\eta||$.

2.3. Coproducts associated with MUO's. Let $E$ be a Hilbert $A$-module and let $\phi$ and $\psi$ be $*$-homomorphisms of $A$ to $\mathcal{L}_A(E)$ such that $\phi$ and $\psi$ commute. Let $W \in \mathcal{L}_A(E \otimes_{\psi} E, E \otimes_{\phi} E)$ be a multiplicative unitary operator (MUO).

For an element $\xi_0$ of $E$, we say that $\xi_0$ has Property E1 if it satisfies the following conditions:

(i) $W(\xi_0 \otimes_{\psi} \xi_0) = \xi_0 \otimes_{\phi} \xi_0$.

(ii) For every $\xi \in E$, there exists an element $\pi_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$<\eta, \pi_{\xi_0}(\xi)\zeta> = <W(\xi_0 \otimes_{\psi} \eta), \xi \otimes_{\phi} \zeta> \quad \text{for every } \eta, \zeta \in E.$$ 

Fix an element $\xi_0$ with Property E1. Define an operator $\delta = \delta_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_{\phi} E)$ by $\delta(\eta) = W(\xi_0 \otimes_{\phi} \eta)$. Then we have $\delta^*(\xi \otimes \eta) = \pi_{\xi_0}(\xi)\eta$. Since $W$ satisfies the pentagonal equation, $\delta$ is a coproduct of $(E, \phi)$.

For an element $\xi_0$ of $E$, we say that $\xi_0$ has Property E2 if it satisfies the following conditions:

(i) $W(\xi_0 \otimes_{\psi} \xi_0) = \xi_0 \otimes_{\phi} \xi_0$.

(ii) For every $\xi \in E$, there exists an element $\hat{\pi}_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$<\eta, \hat{\pi}_{\xi_0}(\xi)\zeta> = <W^*(\xi_0 \otimes_{\phi} \eta), \xi \otimes_{\psi} \zeta> \quad \text{for every } \eta, \zeta \in E.$$ 

Fix an element $\xi_0$ with Property E2. Define an operator $\hat{\delta} = \hat{\delta}_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_{\psi} E)$ by $\hat{\delta}(\eta) = W^*(\xi_0 \otimes_{\phi} \eta)$. Since $W$ satisfies the pentagonal equation, $\hat{\delta}$ is a coproduct of $(E, \psi)$.

3. Coring structures on Hilbert $C^*$-modules

Let $E$ be a Hilbert $A$-module and let $\phi$ be a $*$-homomorphism of $A$ to $\mathcal{L}_A(E)$. Note that $A$ itself is a Hilbert $A$-module with the $A$-valued inner product $<a, b> = a^*b$. 
We denote by $i$ the $*$-homomorphism of $A$ to $\mathcal{L}_A(A)$ defined by $i(a)b = ab$. Then there exists a unitary operator $t$ in $\mathcal{L}_A(E \otimes_A A, E)$ defined by $t(\xi \otimes_A a) = \xi a$. If $\phi$ is non-degenerate, then there exists a unitary operator $t'$ in $\mathcal{L}_A(A \otimes_{\phi} E, E)$ such that $t'(a \otimes_{\phi} \xi) = \phi(a)\xi$.

**Definition 3.1.** Suppose that $\phi$ is non-degenerate. Let $\delta$ be a coproduct of $(E, \phi)$ and let $Q$ be an element of $\mathcal{L}_A(E, A)$, such that $Q\phi(a) = aQ$ for $a \in A$.

(1) We say that $(E, \phi, \delta, Q)$ is a right counital $A$-coring if it satisfies the following equation:

$$t(I_E \otimes_{\phi} Q)\delta = I_E.$$ 

Then $Q$ is called a right counit.

(2) We say that $(E, \phi, \delta, Q)$ is a left counital $A$-coring if it satisfies the following equation:

$$t'(Q \otimes_{\phi} I_E)\delta = I_E.$$ 

Then $Q$ is called a left counit.

(3) We say that $(E, \phi, \delta, Q)$ is a counital $A$-coring if $Q$ is a right and left counit. Then $Q$ is called a counit.

For $n \geq 2$, we set

$$E^{\otimes_{\phi} n} = E \otimes_{\phi} \cdots \otimes_{\phi} E \quad (n \text{ times}).$$

Let $(E, \phi, \delta, Q)$ be a left or right counital $A$-coring. We define an element $\omega$ of $\mathcal{L}_A(E^{\otimes_{\phi} 4}, E^{\otimes_{\phi} 2})$ by

$$\omega = \{t(I_E \otimes_{\phi} Q) \otimes_{\phi} I_E\}(I_E \otimes_{\phi \otimes \iota} \delta' \otimes_{\phi} I_E).$$

Then we have $\omega(\omega \otimes_{\phi \otimes \iota} I) = \omega(I \otimes_{\phi \otimes \iota} \omega)$. Therefore we can define a product on $E \otimes_{\phi} E$ by $xy = \omega(x \otimes_{\phi \otimes \iota} y)$. Then $E \otimes_{\phi} E$ is an algebra over $\mathbb{C}$. Note that we have

$$(\xi_1 \otimes_{\phi} \xi_2)(\eta_1 \otimes_{\phi} \eta_2) = (\xi_1 Q(\xi_2 \eta_1)) \otimes_{\phi} \eta_2.$$
Definition 3.2. We say that $\delta$ and $Q$ are compatible if the following equation holds:
$$\delta(\xi \eta) = \delta(\xi) \delta(\eta)$$
for every $\xi, \eta \in E$.

Example 3.3 ([13]). Let $1 \in A_0 \subset A_1$ be an inclusion of $C^*$-algebras and let $P_1 : A_1 \to A_0$ be a faithful positive conditional expectation of index finite type. Let \{ $u_i, u_i^*; i = 1, \ldots, N$ \} be a quasi-basis of $P_1$. Let $E_1 = A_1$ be a Hilbert $A_0$-module with the $A_0$-valued inner product defined by $< a, b > = P_1(a^* b)$. Let $\phi_1 : A_1 \to \mathcal{L}_{A_0}(E_1)$ be a $*$-homomorphism defined by $\phi_1(a) b = ab$. We denote by $\phi_0$ the restriction of $\phi_1$ to $A_0$. Define $\delta \in \mathcal{L}_{A_0}(E_1, E_1 \otimes_{\phi_0} E_1)$ by $\delta(\xi) = \sum_{i=1}^{N} (\xi u_i) \otimes_{\phi_0} u_i^*$. The product on $E_1$ induced by $\delta$ agrees with the product on $A_1$. Then $(E_1, \phi_0, \delta, P_1)$ is a compatible counital $A$-coring.

Example 3.4. Let $G$ be a finite groupoid. Set $A = C(G^{(0)})$ and $E = C(G)$. Then $E$ is a right $A$-module with the right $A$-action defined by $(\xi a)(x) = \xi(x)a(s(x))$ for $\xi \in E$, $a \in A$ and $x \in G$. We define an $A$-valued inner product of $E$ by
$$< \xi, \eta > (u) = \sum_{g \in G_u} \bar{\xi}(g) \eta(g)$$
for $\xi, \eta \in E$ and $u \in G^{(0)}$, where $G_u = s^{-1}(u)$ for $u \in G^{(0)}$. Then $E$ is a Hilbert $A$-module. Define $*$-homomorphisms $\phi$ and $\psi$ of $A$ to $\mathcal{L}_A(E)$ by $(\phi(a) \xi)(x) = a(r(x)) \xi(x)$ and $\psi(a) = \xi a$ respectively for $a \in A$, $\xi \in E$ and $x \in G$. Note that we have $E \otimes_{\psi} E = C(G^2(ss))$ and $E \otimes_{\phi} E = C(G^{(2)})$, where $G^2(ss) = \{(g, h) \in G^2; s(g) = s(h)\}$. Let $W \in \mathcal{L}_A(E \otimes_{\phi} E, E \otimes_{\phi} E)$ be the MUO defined by $(W \xi)(g, h) = \xi(h, gh)$. Define an element $a_0 \in A$ by $a_0(u) = |G_u|^{-1/2}$ and define an element $\xi_0 \in E$ by $\xi_0(g) = a_0(s(g))$. Then $\xi_0$ satisfies Properties E1 and E2. Note that we have $||\xi_0|| = 1$. Define an element $\eta_0 \in E$ by $\eta_0 = \chi_{G^{(0)}} a_0^{-1}$. Define operators $Q_{\eta_0}, Q_{\xi_0} : E \to A$ by $Q_{\eta_0}(\xi) = < \eta_0, \xi >$ and $Q_{\xi_0}(\xi) = < \xi_0, \xi >$ respectively. Then $(E, \phi, \delta_{\xi_0}, Q_{\eta_0})$ is a compatible counital $A$-coring. The product on $E$ induced by $\delta_{\xi_0}$ is of the form $\xi \eta = (\xi * \eta) a_0$, where $\xi * \eta$ is the convolution product on $C(G)$. We also have two compatible right counital $A$-corings $(E, \psi, \delta_{\xi_0}, Q_{\xi_0})$. 


and \((E, \psi, \delta_{x_0}, Q_{x_0})\). Two products on \(E \otimes_{\psi} E\) associated with above right counital \(A\)-corings are different.

4. CORING STRUCTURES ASSOCIATED WITH INCLUSIONS OF \(C^*\)-ALGEBRAS

Let \(1 \in A_0 \subset A_1\) be an inclusion of \(C^*\)-algebras and let \(P_1 : A_1 \to A_0\) be a faithful positive conditional expectation of index-finite type with a quasi-basis \(\{u_i, u_i^*\}_{i=1}^N\).

Let \(E_1, \phi_1\) and \(\phi_0\) be as in Example 3.3. Set \(E_2 = E_1 \otimes_{\phi_0} E_1\) and define a \(\ast\)-homomorphism \(\phi_2 : A_1 \to \mathcal{L}_{A_0}(E_2)\) by \(\phi_2 = \phi_1 \otimes \iota\). Define a \(C^*\)-algebra \(A\) by \(A = \mathcal{L}_{A_0}(E_1, \phi_1)\) and a Hilbert \(A\)-module \(E\) by \(E = \mathcal{L}_{A_0}((E_1, \phi_1), (E_2, \phi_2))\), that is, \(E\) is the set of elements \(x \in \mathcal{L}_{A_0}(E_1, E_2)\) such that \(x\phi_1(a) = \phi_2(a)x\) for all \(a \in A\). The \(A\)-valued inner product on \(E\) is defined by \(<x, y> = x^*y\). We define \(\ast\)-homomorphisms \(\phi\) and \(\psi\) of \(A\) to \(\mathcal{L}_A(E)\) by \(\phi(a)x = (a \otimes_{\phi_0} I)x\) and \(\psi(a)x = (I \otimes_{\phi_0} a)x\) respectively. We suppose that there exists an MUO \(W \in \mathcal{L}_A(E \otimes_{\phi} E, E \otimes_{\phi} E)\) such that \(V^*\hat{V} = W \otimes I_{E_1}\), where \(V : E \otimes_{\phi} E \otimes I_{E_1} \to E_3\) and \(\hat{V} : E \otimes_{\phi} E \otimes I_{E_1} \to E_3\) are operators defined in [8]. As for sufficient conditions for \(W\) to exist, see [7] and [8]. Define an element \(x_0 \in E\) by \(x_0(\xi) = \xi \otimes_{\phi_0} 1\). Then \(x_0\) satisfies Properties E1 and E2. Note that we have \(\|x_0\| = 1\). Define an element \(\tilde{y}_0 \in E\) by

\[
\tilde{y}_0(\xi) = \sum_{i=1}^N (\xi u_i) \otimes_{\phi_0} u_i^*.
\]

Note that we have \(\tilde{y}_0^* (\xi \otimes_{\phi_0} \eta) = \xi \eta\), where \(\xi \eta\) is the product on \(A_1\). Define \(Q_{x_0}, Q_{\tilde{y}_0} \in \mathcal{L}_A(E, A)\) by \(Q_{x_0}(x) = <x_0, x>\) and \(Q_{\tilde{y}_0}(x) = <\tilde{y}_0, x>\) respectively.

Then we have the following theorem.

**Theorem 4.1.** (1) \((E, \phi, \delta_{x_0}, Q_{x_0})\) is a compatible right counital \(A\)-coring.

(2) Suppose that there exist elements \((u_i, w_i) \in E \times E (i = 1, \cdots K)\) such that

\[
\hat{\delta}_{x_0}(\tilde{y}_0) = \sum_{i=1}^K u_i \otimes_{\psi} w_i.
\]
Then \((E, \psi, \tilde{\delta}_{x_{0}}, Q_{\tilde{y}_{0}})\) is a compatible counital \(A\)-coring.

5. CORING STRUCTURES ON THE SET OF COMPACT OPERATORS

Let \(H\) be an infinite-dimensional separable Hilbert space. We consider \(H\) to be a Hilbert \(\mathbb{C}\)-module, in particular the inner product is linear in the second variable. We denote by \(A\) the \(C^*\)-algebra \(\mathcal{K}(H)\) of compact operators on \(H\). Let \(E\) be a Hilbert \(A\)-module \(\mathcal{K}(H, H \otimes H)\). The right action of \(A\) on \(E\) is defined by \((xa)(\xi) = x(a(\xi))\) for \(x \in E, a \in A\) and \(\xi \in H\) and the \(A\)-valued inner product of \(E\) is defined by \(<x, y> = x^*y\) for \(x, y \in E\). Define \(*\)-homomorphisms \(\phi\) and \(\psi\) of \(A\) to \(\mathcal{L}_{A}(E)\) by \(\phi(a)x = (a \otimes I_{H})x\) and \(\psi(a)x = (I_{H} \otimes a)x\) for \(a \in A\) and \(x \in E\) respectively. We denote by \(F\) the Hilbert \(A\)-module \(\mathcal{K}(H, H \otimes H \otimes H)\). The right action of \(A\) on \(F\) and the \(A\)-valued inner product of \(F\) are defined by the same formulas as those in \(E\). There exist unitary operators \(M \in \mathcal{L}_{A}(E \otimes_{\phi} E, F)\) and \(\overline{M} \in (E \otimes_{\psi} E, F)\) such that

\[
M(x \otimes_{\phi} y) = (x \otimes I_{H})y,
\]
\[
\overline{M}(x \otimes_{\psi} y) = (I_{H} \otimes x)y
\]

for \(x, y \in E\) respectively. Define \(W = M^{-1}\overline{M}\). Then we have the following:

**Theorem 5.1.** The operator \(W\) is the unique multiplicative unitary operator in \(\mathcal{L}_{A}(E \otimes_{\psi} E, E \otimes_{\phi} E)\).

Now we introduce a coring structure on \((E, \phi, \psi)\). Recall that an approximate unit \(\{u_{n}\}_{n=1}^{\infty}\) of \(A\) is said to be increasing if \(u_{n} \geq 0\) and \(u_{n+1} \geq u_{n}\) for every \(n\).

**Definition 5.2.** Let \(\delta\) be a coproduct of \((E, \phi)\). For \(n = 1, 2, \cdots\), let \(Q_{n}\) be an element of \(\mathcal{L}_{A}(E, A)\) such that \(Q_{n}(\phi(a)x) = aQ_{n}(x)\) for \(a \in A\) and \(x \in E\) and let \(\{u_{n}\}_{n=1}^{\infty}\) be an increasing approximate unit of \(A\) such that \(u_{1} \neq 0\) and \(u_{n} \neq u_{n+1}\) for every \(n\). Then \((\delta, \{Q_{n}\}_{n=1}^{\infty}, \{u_{n}\}_{n=1}^{\infty})\) is called a coring structure on \((E, \phi, \psi)\) if it
satisfies the following equations for every \( n \);

\[
t(I_E \otimes \phi Q_n)\delta = t'(Q_n \otimes \phi I_E)\delta = \psi(u_n),
\]

\[
Q_n\psi(u_n) = Q_n.
\]

Then \( \{Q_n\} \) is called an approximate counit.

Let \( T \) be an element of \( \mathcal{L}(H, H \otimes H) \). We will say that \( T \) has Property D if it satisfies the following conditions:

(i) \( (T \otimes I_H)T = (I_H \otimes T)T \).

(ii) There exists a family \( \{K_n\}_{n=1}^{\infty} \) of mutually orthogonal non-trivial finite-dimensional subspaces of \( H \) such that \( H = \bigoplus_{n=1}^{\infty} K_n \) and there exists a complete orthonormal basis \( \{e_{k_{n-1}+1}, \ldots, e_{k_n}\} \) of \( K_n \) for \( n = 1, 2, \ldots \), where \( k_0 = 0 \), such that, if we set \( \lambda^{i}_{j,\ell} = < e_j \otimes e_{\ell}, Te_i > \), then \( \{\lambda^{i}_{j,\ell}\} \) satisfies the following conditions;

(a) for \( i = k_{n-1} + 1 \), \( \lambda^{i}_{i,i} \neq 0 \) and \( \lambda^{i}_{j,\ell} = \lambda^{i}_{\ell,j} = 0 \) for every \( j \in \mathbb{N} \) and \( \ell = k_m + 1 \) (\( m = 0, 1, 2, \ldots \)) except for \( j = \ell = i \),

(b) if \( \dim K_n \geq 2 \), for \( i = k_{n-1} + 2, \ldots, k_n \),

\[
\lambda^{i}_{i,k_{n-1}+1} = \lambda^{i}_{k_{n-1}+1,i} = \lambda^{k_{n-1}+1}_{k_{n-1}+1,k_{n-1}+1},
\]

and \( \lambda^{i}_{j,\ell} = \lambda^{i}_{\ell,j} = 0 \) for every \( j \in \mathbb{N} \) and \( \ell = k_m + 1 \) (\( m = 0, 1, 2, \ldots \)) except for \( (j,\ell) = (i,k_{n-1}+1) \).

Then we have the following theorem:

**Theorem 5.3.** There exists a one-to-one correspondence between the set of coring structures \( (\delta, \{Q_n\}, \{u_n\}) \) on \( (E, \phi, \psi) \) and the set of elements \( (T, \{K_n\}, \{e_{k_{n-1}+1}\}) \) which satisfy Property D. The correspondence is given as follows: If \( (T, \{K_n\}, \{e_{k_{n-1}+1}\}) \)
has Property $D$, set

\[ H_n = \oplus_{i=1}^{n} K_i, \]

\[ \xi_n = \sum_{i=1}^{n} \eta_i \in H_n, \quad \text{where} \quad \eta_i = \left( \lambda_{k_{i-1}+1}^{k_{i-1}+1} \right)^{-1} e_{k_{i-1}+1} \in K_i, \]

define $f_n \in H^\ast \text{ by } f_n(\xi) = \langle \xi_n, \xi \rangle$, then $u_n \in \mathcal{K}(H)$ is the projection onto $H_n$ and $\delta$ and $Q_n$ are given by the following equations;

\[ \delta(x) = M^{-1}(I_H \otimes T)x, \]

\[ Q_n(x) = (I_H \otimes f_n)x. \]

**Question.** Suppose that $T$ has Property $D$. Does $T$ determine $\{K_n\}$ and $\{e_{k_{n-1}+1}\}$ uniquely?

The following theorem shows the relation between the coring structures and the multiplicative unitary operator $W$ defined above:

**Theorem 5.4.** Let $(\delta, \{Q_n\}, \{u_n\})$ be a coring structure on $(E, \phi, \psi)$ and let $T$ be the operator which corresponds to $(\delta, \{Q_n\}, \{u_n\})$ by Theorem 5.3. Put $x_n = Tu_n$. Then $x_n$ is an element of $E$ and satisfies Property E1. Let $\delta_n$ be the coproduct of $(E, \phi)$ defined by

\[ \delta_n(x) = W(x_n \otimes_{\psi} x). \]

Then the following equation holds;

\[ \delta = \lim_{n \to \infty} \delta_n \]

with respect to the strict topology on $\mathcal{L}_A(E, E \otimes_{\phi} E)$.

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