Lindelöf type theorems for monotone Sobolev functions on half spaces

Abstract

This paper deals with Lindelöf type theorems for monotone functions in weighted Sobolev spaces.

1 Introduction

Let $\mathbb{R}^n$ ($n \geq 2$) denote the $n$-dimensional Euclidean space. We use the notation $D$ to denote the upper half space of $\mathbb{R}^n$, that is,

$$D = \{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0 \}.$$

We denote by $\rho_D(x)$ the distance of $x$ from the boundary $\partial D$, that is, $\rho_D(x) = |x_n|$ for $x = (x', x_n)$. Denote by $B(x, r)$ the open ball centered at $x$ with radius $r$, and set $\sigma B(x, r) = B(x, \sigma r)$ for $\sigma > 0$ and $S(x, r) = \partial B(x, r)$.

A continuous function $u$ on $D$ is called monotone in the sense of Lebesgue (see [6]) if the equalities

$$\max u = \max_{\partial G} u \quad \text{and} \quad \min u = \min_{\partial G} u$$

hold whenever $G$ is a domain with compact closure $\overline{G} \subset D$. If $u$ is a monotone Sobolev function in $D$ and $p > n - 1$, then

$$|u(x) - u(y)| \leq Mr \left( \frac{1}{r^n} \int_{2B} |\nabla u(z)|^p dz \right)^{1/p} \quad (1)$$

for all $x, y \in B$, where $B$ is an arbitrary ball of radius $r$ with $2B \subset D$ (see [7, Theorem 1] and [5, Theorem 2.8]). For further results of monotone functions, we refer to [3], [14] and [16].

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Our aim in the present note is to extend the second author’s result [13, Theorem 2] to weighted case.

Let \( \mu \) be a Borel measure on \( \mathbb{R}^n \) satisfying the doubling condition:

\[
\mu(2B) \leq c_1 \mu(B)
\]

for every ball \( B \subset \mathbb{R}^n \). We further assume that

\[
\frac{\mu(B')}{\mu(B)} \geq c_2 \left( \frac{r'}{r} \right)^s
\]

for all \( B' = B(\xi', r') \) and \( B = B(\xi, r) \) with \( \xi', \xi \in \partial D \) and \( B' \subset B \), where \( s > 1 \).

**THEOREM 1.** Let \( u \) be a Sobolev function on \( D \) satisfying

\[
|u(x) - u(y)| \leq M \rho_D(z) \left( \int_{\sigma B} |\nabla u(z)|^p d\mu \right)^{1/p}
\]

for every \( x, y \in B = B(z, \rho_D(z)/(2\sigma)) \) with \( z \in D \) and

\[
\int_D |\nabla u(z)|^p d\mu(z) < \infty.
\]

Define

\[
E_1 = \left\{ \xi \in \partial D : \int_{B(\xi,1) \cap D} |\xi - y|^{-n} |\nabla u(y)| dy = \infty \right\}
\]

and

\[
E_2 = \left\{ \xi \in \partial D : \limsup_{r \to 0} \frac{1}{r^{-p} \mu(B(\xi, r))} \int_{B(\xi, r) \cap D} |\nabla u(y)|^p d\mu(y) > 0 \right\}.
\]

Then \( u \) has a nontangential limit at every \( \xi \in \partial D \setminus (E_1 \cup E_2) \).

Remark 1. Note here that \( E_1 \cup E_2 \) is of \( C_{1,p,\mu} \)-capacity zero. In Manfredi-Villamor [9], the exceptional sets are characterized by Hausdorff dimension, so that their result follows from this nontangential limit result.

**THEOREM 2.** Let \( u \) be a function on \( D \) for which there exist a nonnegative function \( g \in L_{loc}^p(D; \mu) \), \( M > 0 \) and \( \sigma \geq 1 \) such that

\[
|u(x) - u(y)| \leq M \rho_D(z) \left( \int_{\sigma B} g^p d\mu \right)^{1/p}
\]

for every \( x, y \in B = B(z, \rho_D(z)/(2\sigma)) \) with \( z \in D \) and

\[
\int_D g(z)^p d\mu(z) < \infty.
\]
Suppose $p > s - 1$ and set

$$E = \left\{ \xi \in \partial \Omega : \limsup_{r \to 0} \left( r^{-p} \mu(B(\xi, r)) \right)^{-1} \int_{B(\xi, r) \cap \Omega} g(z)^p d\mu(z) > 0 \right\}.$$ 

If $\xi \in \partial \Omega \setminus E$ and there exists a curve $\gamma$ in $\Omega$ tending to $\xi$ along which $u$ has a finite limit $\beta$, then $u$ has a nontangential limit $\beta$ at $\xi$.

For $\alpha > -1$, we consider

$$d\nu(x) = |x_n|^\alpha dx$$

as a measure, which satisfies

$$\nu(B(\xi, r)) = \nu(B(0, 1)) r^{n+\alpha}$$

for all $\xi \in \partial \Omega$ and $r > 0$.

Then we obtain the following result.

**Corollary 1.** Let $u$ be a monotone Sobolev function on $\Omega$ satisfying

$$\int_{\Omega} |\nabla u(z)|^p z_n^\alpha dz < \infty$$

for $p > n - 1$ and $-1 < \alpha < p - n + 1$. Consider the set

$$E_{\alpha-n-p} = \left\{ \xi \in \partial \Omega : \limsup_{r \to 0} r^{p-\alpha-n} \int_{B(\xi, r) \cap \Omega} |\nabla u(z)|^p z_n^\alpha dz > 0 \right\}.$$ 

If $\xi \in \partial \Omega \setminus E_{\alpha-n-p}$ and there exists a curve $\gamma$ in $\Omega$ tending to $\xi$ along which $u$ has a finite limit $\beta$, then $u$ has a nontangential limit $\beta$ at $\xi$.

**Remark 2.** We know that $\mathcal{H}^{n+\alpha-p}(E_{\alpha-n-p}) = 0$, where $\mathcal{H}^d$ denotes the $d$-dimensional Hausdorff measure, and hence it is of $C_{1-\alpha/p,p}$-capacity zero; for these results, see Meyers [10, 11] and the second author's book [14].

## 2 Proof of Theorem 2

A sequence $\{x_j\}$ is called regular at $\xi \in \partial \Omega$ if $x_j \to \xi$ and

$$|x_{j+1} - \xi| < |x_j - \xi| < c|x_{j+1} - \xi|$$

for some constant $c > 1$.

First we give the following result, which can be proved by (4).

**Lemma 1.** Let $u$ and $g$ be as in Theorem 2. If $\xi \in \partial \Omega \setminus E$ and there exists a regular sequence $\{x_j\} \subset \Omega$ with $x_j = \xi + (0, ..., 0, r_j)$ such that $u(x_j)$ has a finite limit $\beta$, then $u$ has a nontangential limit $\beta$ at $\xi$. 
PROOF OF THEOREM 2: For \( r > 0 \) sufficiently small, take \( C(r) \in \gamma \cap S(\xi, r) \). Letting \( C_1(r) = \xi + (0, \ldots, 0, r) \), take an end point \( C_2(r) \in \partial D \) of a quarter of circle containing \( C_1(r) \) and \( C(r) \).

We take a finite chain of balls \( B_1, B_2, \ldots, B_N \) \((N \text{ may depend on } r)\) with the following properties:

1. \( B_j = B(z_j, \rho_D(z_j) / (2\sigma)) \) with \( z_j \in \bigcirc C(r)C_1(r), z_1 = C(r) \) and \( z_N = C_1(r) \);
2. \( \rho_D(z_j) \leq \rho_D(z_{j+1}) \) and \( z_{j+1} \notin B_j \);
3. \( B_j \cap B_{j+1} \neq \emptyset \) for each \( j \);
4. \( |C_2(r) - z| \leq 3 \rho_D(z) \) for \( z \in A(\xi, r) = \bigcup_{j=1}^{N} \sigma B_j \subset B(\xi, 2r) \cap D \);
5. \( \sum_j \chi_{\sigma B_j} \leq c_3 \), where \( \chi_A \) denotes the characteristic function of \( A \) and \( c_3 \) is a constant depending only on \( c_1 \) and \( \sigma \);

see Heinonen [2] and Hajłasz-Koskela [1].

Pick \( x_j \in B_{j+1} \cap B_j \) for \( 1 \leq j \leq N - 1 \). By (4), we see that

\[
|u(x_j) - u(x_{j-1})| \leq M \rho_D(z_j) \left( \int_{\sigma B_j} g(z)\rho_D(z)^{-\delta} \, d\mu(z) \right)^{1/p}
\]

for \( 1 \leq j \leq N \), where \( x_0 = C(r) \) and \( x_N = C_1(r) \).

Since \( p > s - 1 \) by our assumption, there is \( \delta > 0 \) such that \( s - p < \delta < 1 \). We have by Hölder’s inequality

\[
|u(C_1(r)) - u(C(r))| \leq |u(x_1) - u(x_0)| + |u(x_2) - u(x_1)| + \cdots + |u(x_N) - u(x_{N-1})|
\]

\[
\leq M \sum_{j=1}^{N} \rho_D(z_j)^{1+\delta/p} \mu(\sigma B_j)^{-1/p} \left( \int_{\sigma B_j} g(z)\rho_D(z)^{-\delta} \, d\mu(z) \right)^{1/p}
\]

\[
\leq M \left( \sum_{j=1}^{N} \rho_D(z_j)^{p'(1+\delta/p)} \mu(\sigma B_j)^{-p'/p} \right)^{1/p'} \left( \int_{A(\xi, r)} g(z)\rho_D(z)^{-\delta} \, d\mu(z) \right)^{1/p}
\]

\[
\leq M \left( \sum_{j=1}^{N} \rho_D(z_j)^{p'(1+\delta/p)} \mu(\sigma B_j)^{-p'/p} \right)^{1/p'} \left( \int_{B(\xi, 2r) \cap D} g(z)\rho_D(z)^{-\delta} \, d\mu(z) \right)^{1/p}
\]

where \( 1/p + 1/p' = 1 \). Here note that \( \mu(B(C_2(r), \rho_D(z_j))) \leq c_4 \mu(\sigma B_j) \), where \( c_4 \) is a positive constant depending only on the doubling constant \( c_1 \). Since \( \delta > s - p \), we see from (2) that

\[
\sum_{j=1}^{N} \rho_D(z_j)^{p'(p+\delta)/p} \mu(\sigma B_j)^{-p'/p} \leq M \sum_{j=1}^{N} \rho_D(z_j)^{p'(p+\delta)/p} \mu(B(C_2(r), \rho_D(z_j)))^{-p'/p}
\]
Moreover, since $0 < \delta < 1$, we note that
\[
\int_{2^{-j}}^{2^{-j+1}} |C_2(r) - z|^{-\delta} dr \leq \int_{2^{-j}}^{2^{-j+1}} |r - |z||^{-\delta} dr \leq M 2^{-j(1-\delta)}. \tag{6}
\]
Hence it follows from (6) that
\[
\int_{2^{-j}}^{2^{-j+1}} |u(C_1(r)) - u(C(r))|^p \frac{dr}{r} \leq M \int_{2^{-j}}^{2^{-j+1}} r^\delta (r^{-p}\mu(B(\xi, r)))^{-1} \left( \int_{B(\xi,2r) \cap \mathrm{D}} g(z)^p |C_2(r) - z|^{-\delta} d\mu(z) \right) \frac{dr}{r}.
\]
\[
\leq M 2^{-j(p+\delta-1)} \mu(B(\xi, 2^{-j}))^{-1} \int_{B(\xi,2^{-j+2}) \cap \mathrm{D}} g(z)^p \left( \int_{2^{-j}}^{2^{-j+1}} |C_2(r) - z|^{-\delta} dr \right) d\mu(z).
\]

Since $\xi \in \partial \mathrm{D} \setminus E$, we can find a sequence \( \{r_j\} \) such that $2^{-j} < r_j < 2^{-j+1}$ and
\[
\lim_{j \to \infty} |u(C_1(r_j)) - u(C(r_j))| = 0.
\]
By our assumption we see that $u(C_1(r_j))$ has a finite limit $\beta$ as $j \to \infty$. If we note that \( \{C_1(r_j)\} \) is regular at $\xi$, then Lemma 1 proves the required conclusion of the theorem.

## 3 \( A_q \) weights

Let $w$ be a Muckenhoupt $A_q$ weight, and define
\[
d\nu(y) = w(y) dy.
\]
Let $u$ be a monotone Sobolev function on $\mathrm{D}$ such that
\[
\int_{\mathrm{D}} |\nabla u(x)|^p d\nu(x) < \infty.
\]
Suppose that $1 < q < p/(n-1)$. Since $p_1 = p/q > n-1$, applying inequality (1) we obtain
\[ |u(x) - u(y)| \leq M r \left( \frac{1}{r^n} \int_{2B} |\nabla u(z)|^{p_1} dz \right)^{1/p_1} \]
for all $x, y \in B$, where $B$ is an arbitrary ball of radius $r$ with $2B \subset \mathbb{D}$. As in the proof of Theorem 2, we insist that
\[ \int_{2^{-j}}^{2^{-j+1}} |u(C_1(r)) - u(C(r))|^{p_1} \frac{dr}{r} \leq M 2^{-jp_1} |B(\xi, 2^{-j})|^{-1} \int_{B(\xi, 2^{-j+2}) \cap \mathbb{D}} |\nabla u(z)|^{p_1} dz. \]
Using Hölder inequality and $A_q$-condition of $w$, we have
\begin{align*}
\int_{2^{-j}}^{2^{-j+1}} |u(C_1(r)) - u(C(r))|^{p_1} \frac{dr}{r} & \leq M 2^{-jp_1} |B(\xi, 2^{-j})|^{-1} \left( \int_{B(\xi, 2^{-j+2}) \cap \mathbb{D}} |\nabla u(z)|^{p_1 q} w(z) dz \right)^{1/q} \left( \int_{B(\xi, 2^{-j+2}) \cap \mathbb{D}} w(z)^{-q'/q} dz \right)^{1/q'} \\
& \leq M \left( 2^{jp} \nu(B(\xi, 2^{-j})) \right)^{-1} \int_{B(\xi, 2^{-j+2}) \cap \mathbb{D}} |\nabla u(z)|^{p} \nu(z) dz,
\end{align*}
where $1/q + 1/q' = 1$. Thus we obtain the following result (cf. Manfredi-Villamor [9]), as in the proof of Theorem 2.

**Corollary 2.** Let $1 \leq q < p/(n-1)$. Let $w \in A_q$ and set $d\nu(y) = w(y)dy$. Suppose that $u$ is a monotone Sobolev function on $\mathbb{D}$ satisfying
\[ \int_{\mathbb{D}} |\nabla u(z)|^{p} \nu(z) < \infty. \] (7)
Set
\[ E = \left\{ \xi \in \partial \mathbb{D} : \limsup_{r \to 0} (r^{-p} \nu(B(\xi, r)))^{-1} \int_{B(\xi, r) \cap \mathbb{D}} |\nabla u(z)|^{p} \nu(z) > 0 \right\}. \]
If $\xi \in \partial \mathbb{D} \setminus E$ and there exists a curve $\gamma$ in $\mathbb{D}$ tending to $\xi$ along which $u$ has a finite limit $\beta$, then $u$ has a nontangential limit $\beta$ at $\xi$.

**Remark 3.** Let $1 \leq q < p/(n-1)$. Let $w$ be a Muckenhoupt $A_q$ weight, and define
\[ d\nu(y) = w(y)dy. \]
Suppose that $u$ is a monotone Sobolev function on $\mathbb{D}$ satisfying (7). Applying Hölder's inequality to (1) with $p$ replaced by $p/q$, we see that
\[ |u(x) - u(y)| \leq M r \left( \int_{2B} |\nabla u(z)|^{p} \nu(z) dz \right)^{1/p} \]
for all $x, y \in B$, where $B$ is an arbitrary ball of radius $r$ with $2B \subset \mathbb{D}$ (see also Manfredi-Villamor [9]).

**Remark 4.** Consider $w(y) = |y_n|^\alpha$. Then $w \in A_q$ if and only if $-1 < \alpha < q - 1$. In this case, Corollary 2 does not imply Corollary 1 when $n \geq 3$. 

4 Generalizations of Lindelöf theorems

For an integer $d$, $1 \leq d < n$, let $P_d : \mathbb{R}^n \rightarrow \mathbb{R}^d$ be the projection, that is,

$$P_d(x) = (x_1, \ldots, x_d, 0, \ldots, 0) \quad \text{for} \quad x = (x_1, x_2, \ldots, x_n).$$

We say that $\Gamma \subset D$ is a $(\lambda_1, \lambda_2, d)$-approach set at $\xi$, where $\lambda_1 \geq 1$ and $\lambda_2 > 0$, if there exists a sequence of positive numbers $\{r_j\}$ tending to zero such that $r_{j+1} < r_j < \lambda_1 r_{j+1}$ and

$$\mathcal{H}^d(P_d(\Gamma \cap (B(\xi, r_j) \setminus B(\xi, r_{j+1})))) \geq \lambda_2 r_j^d.$$

**Theorem 3.** Let $u$ be a function on $D$ with $g$ satisfying (4) and

$$\int_D g(z)^p d\mu(z) < \infty.$$

Suppose $p > s - d$, and define

$$E = \left\{ \xi \in \partial D : \limsup_{r \to 0} \left( r^{-p} \mu(B(\xi, r)) \right)^{-1} \int_{B(\xi, r) \cap D} g(z)^p d\mu(z) > 0 \right\}.$$

If $\xi \in \partial D \setminus E$ and there exists a $(\lambda_1, \lambda_2, d)$-approach set $\Gamma \subset D$ at $\xi$ along which $u$ has a finite limit $\beta$ at $\xi$, then $u$ has a nontangential limit $\beta$ at $\xi$.

**Proof.** By our assumption, we can take $\delta > 0$ such that $s - p < \delta < d$. Set

$$G_j = P_d(\Gamma \cap (B(\xi, r_j) \setminus B(\xi, r_{j+1}))).$$

For $X \in G_j$, take $C(X) \in \Gamma \cap (B(\xi, r_j) \setminus B(\xi, r_{j+1}))$, and set $r(X) = r = |\xi - C(X)|$. Let $C_1(X) = \xi + (0, \ldots, 0, r)$ and $D(X) = P_{n-1}(C(X))$.

We take a finite chain of balls $B_1, B_2, \ldots, B_N$ with the following properties:

(i) $B_j = B(z_j, \rho_D(z_j)/(2\sigma))$ with $z_j \in C(X)C_1(X)$, $z_1 = C(X)$ and $z_N = C_1(X)$;

(ii) $\rho_D(z_j) \leq \rho_D(z_{j+1})$ and $z_{j+1} \notin B_j$;

(iii) $B_j \cap B_{j+1} \neq \emptyset$ for each $j$;

(iv) $|D(X) - z| \leq 3\rho_D(z)$ for $z \in A(\xi, r) = \bigcup_{j=1}^{N} \sigma B_j \subset B(\xi, 2r) \cap D$;

(v) $\sum_j \chi_{\sigma B_j} \leq c_3$.

Since $\delta > s - p$, we have as in the proof of Theorem 2

$$|u(C_1(X)) - u(C(X))|^p \leq M r^\delta \left( r^{-p} \mu(B(\xi, r)) \right)^{-1} \int_{B(\xi, 2r) \cap D} g(z)^p |D(X) - z|^{-\delta} d\mu(z).$$
Further, since $P_d$ is 1-Lipschitz and $0 < \delta < d$, we see that
\[
\int_{G_j} |D(X) - z|^{-\delta} d\mathcal{H}^d(X) \leq \int_{G_j} |X - P_d(z)|^{-\delta} d\mathcal{H}^d(X) \\
\leq \int_{P_d(B(\xi, r_j))} |X - P_d(z)|^{-\delta} d\mathcal{H}^d(X) \\
\leq Mr_j^{d-\delta}.
\]
Hence we have
\[
\int_{G_j} |u(C_1(X)) - u(C(X))|^p d\mathcal{H}^d(X) \leq M(r_j^{-p} \mu(B(\xi, r_j))^{-1} \int_{B(\xi, 2r_j) \cap \Gamma} g(z)^p d\mu(z).
\]
Thus we can find a sequence $\{X_j\}$ such that $X_j \in G_j$ and
\[
\lim_{j \to \infty} |u(C_1(X_j)) - u(C(X_j))| = 0.
\]
Thus we see that $u(C_1(X_j))$ has a finite limit $\beta$ as $j \to \infty$. Since $\{C_1(X_j)\}$ is regular at $\xi$, we can show that $u$ has a nontangential limit $\beta$ at $\xi$ by Lemma 1.

**Corollary 3.** Let $u$ be a harmonic function on $D$ satisfying
\[
\int_{D \cap B(0,N)} |\nabla u(z)|^p z_n^{\alpha} dz < \infty
\]
for every $N > 0$, and $-1 < \alpha < p - n + d$. If $\xi \in \partial D \setminus E_{n+\alpha-p}$ and there exists a $(\lambda_1, \lambda_2, d)$-approach set $\Gamma \subset D$ at $\xi$ along which $u$ has a finite limit $\beta$ at $\xi$, then $u$ has a nontangential limit $\beta$ at $\xi$.

**Remark 5.** The conclusion of Corollary 3 is still valid for $A$-harmonic functions and polyharmonic functions.

**References**


