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<thead>
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<th>Title</th>
<th>Beurling's Minimum Principle in a Cone (Potential Theory and Related Topics)</th>
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</thead>
<tbody>
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Beurling’s Minimum Principle in a Cone

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Abstract

This paper shows that some characterizations of the harmonic majorization of the Martin function connected with a domain having smooth boundary without a corner e.g a ball and a half-space also hold for a special domain with corners, i.e. a cone.

1. Introduction.

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the set of all real numbers and all positive real numbers, respectively. We denote by $\mathbb{R}^n$ ($n \geq 2$) the n-dimensional Euclidean space. A point in $\mathbb{R}^n$ is denoted by $P = (X, y), X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance of two points $P$ and $Q$ in $\mathbb{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin $O$ of $\mathbb{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set $S$ in $\mathbb{R}^n$ are denoted by $\partial S$ and $\overline{S}$, respectively.

We introduce a system of spherical coordinates $(r, \Theta), \Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbb{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, y)$ by

$$x_1 = r(\Pi_{j=1}^{n-1}\sin \theta_j) \quad (n \geq 2), \quad y = r \cos \theta_1,$$

and if $n \geq 3$, then

$$x_{n+1-k} = r(\Pi_{j=1}^{k-1}\sin \theta_j) \cos \theta_k \quad (2 \leq k \leq n - 1),$$

where $0 \leq r < +\infty$, $-\frac{1}{2}\pi \leq \theta_{n-1} < \frac{3}{2}\pi$, and if $n \geq 3$, then $0 \leq \theta_j \leq \pi \quad (1 \leq j \leq n - 2)$.

The unit sphere and the upper half unit sphere are denoted by $\mathbb{S}^{n-1}$ and $\mathbb{S}_+^{n-1}$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbb{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set $\Omega, \Omega \subset \mathbb{S}^{n-1}$, are often identified with $\Theta$ and $\Omega$, respectively. For two sets $\Lambda \subset \mathbb{R}_+$ and $\Omega \subset \mathbb{S}^{n-1}$, the set

$$\{(r, \Theta) \in \mathbb{R}^n; r \in \Lambda, (1, \Theta) \in \Omega\}$$
in $\mathbb{R}^n$ is simply denoted by $\Lambda \times \Omega$. In particular, the half-space

$$\mathbb{R}_+ \times S^{n-1}_+ = \{(X, y) \in \mathbb{R}^n; y > 0\}$$

will be denoted by $T_n$.

To extend a result of Beurling [7] for $n=2$, Armitage and Kuran [4] said that a sequence $\{P_m\}$ of points $P_m = (X_m, y_m) \in T_n$, $|P_m| \to +\infty$ ($m \to +\infty$) "characterizes the positive harmonic majorization of $y"$, if every positive harmonic function $h$ in $T_n$ which majorizes the function $y$ on the set $\{P_m; m = 1,2,\ldots\}$ majorizes $y$ everywhere in $T_n$, i.e.

$$\inf_{P \in T_n} \frac{h(P)}{y} = \inf_{m} \frac{h(P_m)}{y_m} \quad (P = (X, y) \in T_n).$$

They proved

**THEOREM A** (Beurling [7] for $n = 2$, Armitage and Kuran [4, Theorem 1] for $n \geq 2$). Let $\{P_m\}$ be a sequence of points $\{P_m\}$,

$$P_m = (r_m, \Theta_m) \in T_n; \Theta_m = (\theta_{1,m}, \theta_{2,m}, \ldots, \theta_{(n-1),m})$$

in $T_n$ satisfying

\[(1.1) \quad r_{m+1} \geq a r_m \quad (m = 1, 2, \ldots)\]

for a certain $a > 1$. Then the sequence $\{P_m\}$ characterizes the positive harmonic majorization of $y$ if and only if

\[(1.2) \quad \sum_{m=1}^{\infty} (\cos \theta_{1,m})^n = +\infty.\]

We remark that $y$ is the Martin function at the infinite Martin boundary point $\infty$ of $T_n$, i.e. $y$ is (up to a positive multiplicative constant) the only positive harmonic function in $T_n$ which vanishes on $\partial T_n$. The "if" part of Theorem A is a minimm principle, since if $h$ is a positive harmonic function $h(P)$ of $P = (X, y) \in T_n$, then

$$\lim_{P \in T_n, P \to P'} \inf_{P \in T_n} \{h(P) - y\} \geq 0$$

for every $P'$ on $\partial T_n$ and the majorization of the function $y$ by $h$ on the set of points $P_m$ satisfying (1.1) and (1.2) replaces

$$\lim_{P \in T_n, |P| \to +\infty} \inf_{P \in T_n} \{h(P) - y\} \geq 0.$$
Hence this sort of sequence was said to be “equivalent to ∞” in Beurling [7] and this type of Theorem A was called “Beurling minimum principle” in Ancona [3, p.18] and Maz'ya [15]. This Theorem A was also extended by Maz'ya [15] to positive solutions of a second-order elliptic differential equation in an n-dimensional bounded domain with smooth boundary of class $C^{1,\alpha}$ ($0 < \alpha < 1$).

Let $D$ be a domain in $\mathbb{R}^n$ and $\Delta(D)$ be the Martin boundary of $D$. The Martin function at $Q \in \Delta(D)$ is denoted by $K_Q(P)$ ($P \in D$). Following Armitage and Kuran [4], we say that a subset $E$ of $D$ characterizes the positive harmonic majorization of $K_Q(P)$, if every positive harmonic function $h$ in $D$ which majorizes $K_Q(P)$ on $E$ majorizes $K_Q(P)$ everywhere in $D$, i.e.

$$(1.3) \quad \inf_{P \in D} \frac{h(P)}{K_Q(P)} = \inf_{P \in E} \frac{h(P)}{K_Q(P)}.$$ 

We set

$$B(P, r) = \{ P' \in \mathbb{R}^n; |P' - P| < r \} \quad (r > 0)$$

and

$$d(P) = \inf_{Q \in \partial D} |P - Q|$$

for any $P \in D$. For a subset $E$ of $D$ and a number $\rho$ ($0 < \rho < 1$) we put

$$(1.4) \quad E_\rho = \bigcup_{P \in E} B(P, \rho d(P)).$$

Dahlberg proved

**THEOREM B** (Dahlberg [10, Theorem 1]). Let $D$ be a Liapunov-Dini domain in $\mathbb{R}^n$ and $Q \in \partial D$. If $E \subset D$, then the following conditions on $E$ are equivalent;

(i) $E$ characterizes the positive harmonic majorization of $K_Q(P)$;

(ii) for every $\rho$, $0 < \rho < 1$

$$\int_{E_\rho} \frac{dP}{|P - Q|^n} = +\infty,$$

(iii) for some $\rho$, $0 < \rho < 1$

$$\int_{E_\rho} \frac{dP}{|P - Q|^n} = +\infty.$$ 

Since (1.3) is closely related to the notion of minimally thinness of $E_\rho$ in (1.4) (see Sjögren [17], Ancona [3] and Zhang [19]), which is also seen in Theorem 1 of this paper, Aikawa and Essén [2, Corollary 7.4.7] also proved Theorem B in a different way from Dahlberg's.
By using a suitable Kelvin transformation which maps $T_n$ onto a ball, the following Theorem C follows from Theorem B.

THEOREM C (Dahlberg [10, Theorem 3]). If $E \subset T_n$, then the following conditions on $E$ are equivalent:

(i) $E$ characterizes the positive harmonic majorization of $y$;

(ii) for every $\rho, 0 < \rho < 1$

$$
\int_{E_{\rho}} \frac{dP}{(1 + |P|)^n} = +\infty,
$$

where

$$
E_{\rho} = \bigcup_{P=(X,y) \in E} B(P, \rho y);
$$

(iii) for some $\rho, 0 < \rho < 1$

$$
\int_{E_{\rho}} \frac{dP}{(1 + |P|)^n} = +\infty.
$$

The methods of proving these Theorems A and B were based on the smoothness of the boundary having no wedges e.g. a ball. For a domain having more rough boundary e.g. a Lipschiz domain, Ancona [3, Theorem 7.4] and Zhang [19, Theorem 3] gave more complicated results which generalize Theorem A.

For a Lipshitz domain and an NTA domain $D$, Zhang [19, Corollary 1] and Aikawa [1, Remark and Theorem 1] gave a necessary and sufficient qualitative condition for a subset $E$ of $D$ characterizing the positive harmonic majorization of $K_Q(P)$ by connecting with minimally thinness of $E_{\rho}$ in (1.4), respectively. In his paper Aikawa says that since a general NTA domain may have wedges, Theorem B does not hold for an NTA domain. But when we see our results in this paper, we may ask whether quantitative Theorem B can just be extended to a Lipshitz domain and an NTA domain.

In this paper we shall prove Theorems A and C can be extended to a result at a corner point of a wedge i.e. a result at $\infty$ of a cone. We remark that a half-space is one of cones.

2. Statements of results.

Let $\Omega$ be a domain on $S^{n-1}(n \geq 2)$ with smooth boundary. Consider the Dirichlet problem

$$(\Lambda_n + \tau)f = 0 \quad \text{on } \Omega$$

$$f = 0 \quad \text{on } \partial\Omega,$$

where $\Lambda_n$ is the spherical part of the Laplace operator $\Delta_n$

$$
\Delta_n = \frac{n - 1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2} \Lambda_n
$$
We denote the least positive eigenvalue of this boundary value problem by \( \tau_\Omega \) and the normalized positive eigenfunction corresponding to \( \tau_\Omega \) by \( f_\Omega(\Theta) \):

\[
\int_\Omega f^2_\Omega(\Theta) d\sigma_\Theta = 1,
\]

where \( d\sigma_\Theta \) is the surface element on \( S^{n-1} \). We denote the solutions of the equation

\[
t^2 + (n - 2)t - \tau_\Omega = 0
\]

by \( \alpha_\Omega, -\beta_\Omega \) (\( \alpha_\Omega, \beta_\Omega > 0 \)). If \( \Omega = S_+^{n-1} \), then \( \alpha_\Omega = 1, \beta_\Omega = n - 1 \) and

\[
f_\Omega(\Theta) = (2n s_n^{-1})^{1/2} \cos \theta_1,
\]

where \( s_n \) is the surface area \( 2\pi^{n/2} \Gamma(n/2)^{-1} \) of \( S^{n-1} \).

To make simplify our consideration in the following, we shall assume that if \( n \geq 3 \), then \( \Omega \) is a \( C^{2,\alpha} \)-domain (\( 0 < \alpha < 1 \)) on \( S^{n-1} \) (e.g. see Gilbarg and Trudinger [12, pp.88-89] for the definition of \( C^{2,\alpha} \)-domain). By \( C_n(\Omega) \), we denote the set \( \mathbb{R}_+ \times \Omega \) in \( \mathbb{R}^n \) with the domain \( \Omega \) on \( S^{n-1} (n \geq 2) \). We call it a cone. Then \( T_n \) is a cone obtained by putting \( \Omega = S_+^{n-1} \). It is known that the Martin boundary of \( C_n(\Omega) \) is the set \( \partial C_n(\Omega) \cup \{\infty\} \), each of which is a minimal Martin boundary point. The Martin kernel at \( \infty \) with respect to a reference point chosen suitably is \( K_\infty(P) = r^{\alpha_\Omega} f_\Omega(\Theta) \) (\( P = (r, \Theta) \in C_n(\Omega) \)).

A subset \( E \) of a domain \( D \) in \( \mathbb{R}^n \) is said to be \textit{minimally thin} at \( Q \in \Delta(D) \) (Brelot [8, p.122], Doob [11, p.208]), if there exists a point \( P \in D \) such that

\[
\hat{R}^E_{K,Q}(P) = K_Q(P),
\]

where \( \hat{R}^E_{K,Q}(P) \) is the regularized reduced function of \( K_Q(P) \) relative to \( E \) (Helms [14, p.134]).

The following Theorem 1 which is used to prove Theorem 2 is a specialized version of Aikawa [1, Theorem 1]. Since his proof is so complicated because of an NTA domain we shall give a simple proof based on a function which is a conical version of Dahlberg’s [10, pp.240-241].

**THEOREM 1.** Let \( E \) be a subset of \( C_n(\Omega) \). The following conditions on \( E \) are equivalent:

(i) \( E \) characterizes the positive harmonic majorization of \( K_\infty(P) \);

(ii) for any \( \rho, \ 0 < \rho < 1, \ E_\rho \) is not minimally thin at \( \infty \),

(iii) for some \( \rho, \ 0 < \rho < 1, \ E_\rho \) is not minimally thin at \( \infty \).

The following Theorem 2 extends Theorem C.

**THEOREM 2.** Suppose that \( E \subset C_n(\Omega) \). Then the following conditions on \( E \) are equivalent:
(i) $E$ characterizes the positive harmonic majorization of $K_{\infty}(P);$  
(ii) for every $\rho \ (0 < \rho < 1),$
\[ \int_{E_\rho} \frac{dP}{(1+|P|)^n} = +\infty, \]
(iii) for some $\rho \ (0 < \rho < 1),$
\[ \int_{E_\rho} \frac{dP}{(1+|P|)^n} = +\infty. \]

A sequence $\{P_m\}$ of points $P_m \in D$ is said to be separated, if there exists a positive constant $c$ such that
\[ |P_i - P_j| \geq cd(P_i) \quad (i, j = 1, 2, ..., i = j) \]
(e.g. see Ancona [3, p.18], Aikawa and Essén [2, p.156]).

From Theorem 2 we immediately have the following Corollary which extends Theorem A.

COROLLARY. Let $\{P_m\}, \ P_m \in C_n(\Omega)$ be a separated sequence satisfying
\[ \inf_m |P_m| > 0. \]
The sequence $\{P_m\}$ characterizes the positive harmonic majorization of $K_{\infty}(P)$ if and only if
\[ \sum_{m=1}^{\infty} \left( \frac{d(P_m)}{|P_m|} \right)^n = +\infty. \]

3. Lemmas and proof of Theorem 1.

Let $f$ and $g$ be two positive real valued functions defined on a set $Z$. Then we shall write $f \approx g$, if there exists two constants $A_1, A_2, 0 < A_1 \leq A_2$ such that $A_1 g \leq f \leq A_2 g$ everywhere on $Z$. For a subset $S$ in $\mathbb{R}^n$, the interior of $S$ and the diameter of $S$ are denoted by $\text{int} \ S$ and $\text{diam} \ S$, respectively. For two subsets $S_1$ and $S_2$ in $\mathbb{R}^n$, the distance between $S_1$ and $S_2$ is denoted by $\text{dist}(S_1, S_2)$. A cube of $\mathcal{M}_k$ is of the form
\[ [l_1 2^{-k}, (l_1 + 1) 2^{-k}] \times \cdots \times [l_n 2^{-k}, (l_n + 1) 2^{-k}] \quad (k = 0, \pm 1, \pm 2, \ldots) \]
where $l_1, \ldots, l_n$ are integers. Let $\rho$ be a number satisfying $0 < \rho \leq \frac{1}{2}$. A family of the Whitney cubes of $C_n(\Omega)$ with $\rho$ is the set of cubes having the following properties; 
(i) $\bigcup_i W_i = C_n(\Omega)$,
(ii) \(\text{int } W_i \cap \text{int } W_k = \emptyset \quad (i = k)\),

(iii) \(\left[\frac{8}{3\rho}\right] \text{diam } W_i \leq \text{dist}(W_i, \mathbb{R}^n \setminus C_n(\Omega)) \leq 2 \left(\left[\frac{8}{3\rho}\right] + 1\right) \text{diam } W_i\),

where \([a]\) denotes the integer satisfying \(a \leq a < a + 1\) (Stein [18, p.167, Theorem 1]).

The following Lemma 1 is fundamental in this paper.

**Lemma 1** (I.Miyamoto, M.Yanagishita and H.Yoshida [16, Theorems 2 and 3]). Let a Borel subset \(E\) of \(C_n(\Omega)\) be minimally thin at \(\infty\). Then we have

\[
\int_{E} \frac{dP}{(1 + |P|)^n} < +\infty.
\]

If \(E\) is a union of cubes from a family of the Whitney cubes of \(C_n(\Omega)\) with \(\rho\) \((0 < \rho \leq \frac{1}{2})\), then (3.1) is also sufficient for \(E\) to be minimally thin at \(\infty\).

For a set \(E \subset C_n(\Omega)\) and a number \(\rho\) \((0 < \rho \leq \frac{1}{2})\), define \(E_{\rho}\) and \(E_{\frac{\rho}{4}}\) as in (1.4).

**Lemma 2.** Let \(\{W_i\}_{i \geq 1}\) be a family of the Whitney cubes of \(C_n(\Omega)\) with \(\rho\). Then there exists a subsequence \(\{W_{i_j}\}_{j \geq 1}\) of \(\{W_i\}_{i \geq 1}\) such that

(i) \(\bigcup_j W_{i_j} \subset E_{\rho}\),

(ii) \(W_{i_j} \cap E_{\frac{\rho}{4}} = \emptyset \quad (j = 1, 2, \ldots)\), \(E_{\frac{\rho}{4}} \subset \bigcup_j W_{i_j}\).

**Proof of Lemma 2.** Let \(k\) be an integer. Let \(c = \left(\left[\frac{8}{3\rho}\right] + 1\right)\) and set

\[
I_k = \left\{ P \in C_n(\Omega) \, ; \, c\sqrt{n}2^{-k} < \text{dist}(P, \partial C_n(\Omega)) \leq c\sqrt{n}2^{-k+1}\right\}.
\]

Let \(\{W_{i_j}\}_{j \geq 1}\) be a sequence of all Whitney cubes of \(\{W_i\}_{i \geq 1}\) such that

\[
W_{i_j} \cap E_{\frac{\rho}{4}} = \emptyset \quad (j = 1, 2, \ldots).
\]

Then it is evident that (ii) holds. We shall also show that this \(\{W_{i_j}\}_{j \geq 1}\) satisfies (i). Take any \(W_{i_{j_0}}\) and let \(W_{i_{j_0}}\) be a cube of \(\mathcal{M}_{k_0}\). Since \(W_{i_{j_0}} \cap E_{\frac{\rho}{4}} = \emptyset\), there exists a point \(P_{j_0}\) in \(E\) such that

\[
B(P_{j_0}, \frac{\rho}{4} d(P_{j_0})) \cap W_{i_{j_0}} = \emptyset.
\]

Then \(P_{j_0} \in I_{k_{0}+1} \cup I_{k_0} \cup I_{k_0-1}\). Because, if \(P \in I_k\) and \(W_{i_j}\) is a Whitney cube satisfying \(W_{i_j} \cap B(P, \frac{\rho}{4} d(P)) = \emptyset\), then \(W_{i_j} \in \mathcal{M}_{k+1} \cup \mathcal{M}_k \cup \mathcal{M}_{k-1}\).

If \(P_{j_0} \in I_{k_0+1}\), then

\[
\rho d(P_{j_0}) - \frac{\rho}{4} d(P_{j_0}) = \frac{3}{4} \rho d(P_{j_0}) > \frac{3}{4} \rho \left(\left[\frac{8}{3\rho}\right] + 1\right) \sqrt{n}2^{-(k_{0}+1)} > \sqrt{n}2^{-k_0}.
\]

Since the diameter of \(W_{i_{j_0}}\) is \(\sqrt{n}2^{-k_0}\), we have from (3.2) that \(W_{i_{j_0}} \subset B(P_{j_0}, \rho d(P_{j_0}))\) and hence \(W_{i_{j_0}} \subset E_{\rho}\). Even if \(P_{j_0} \in I_{k_0}\) or \(P_{j_0} \in I_{k_0-1}\), we similarly have \(W_{i_{j_0}} \subset E_{\rho}\).
Thus all cubes of \( \{W_{i_{j}}\}_{j \geq 1} \) are contained in \( E_{\rho} \), which is just \( (i) \).

**Proof of Theorem 1.** Proof of \( (i) \Rightarrow (ii) \). First of all, we shall remark the following fact. Let \( c \) be a positive constant. Since \( E \) characterizes the positive harmonic majorization of \( K_{\infty}(P) \), \( E_{1} = \{ P \in E ; K_{\infty}(P) > c \} \) also characterizes the positive harmonic majorization of \( K_{\infty}(P) \). For otherwise there exists a positive harmonic function \( h(P) \) on \( C_{n}(\Omega) \), satisfying

\[
\inf_{P \in C_{n}(\Omega)} \frac{h(P)}{K_{\infty}(P)} < \inf_{P \in E} \frac{h(P)}{K_{\infty}(P)} = b.
\]

If we put \( h_{1}(P) = h(P) + bc \quad (P \in C_{n}(\Omega)) \), then \( h_{1}(P) \geq bK_{\infty}(P) \) for all \( P \in E \) and hence

\[
\inf_{P \in C_{n}(\Omega)} \frac{h_{1}(P)}{K_{\infty}(P)} = a < b \leq \inf_{P \in E} \frac{h_{1}(P)}{K_{\infty}(P)},
\]

which contradicts the assumption that \( E \) characterizes the positive harmonic majorization of \( K_{\infty}(P) \). If we can show that for any \( \rho \) \((0 < \rho < 1) \ (E_{1})_{\rho} \) is not minimally thin at \( \infty \), then for any \( \rho \) \((0 < \rho < 1) \ E_{\rho} \) is also not minimally thin at \( \infty \). Hence by applying the following argument to \( E_{1} \) if necessary, we may assume that \( K_{\infty}(P) > c \) for every \( P \in E \), without generality.

Suppose that for some number \( \rho \) \((0 < \rho < 1) \ E_{\rho} \) is minimally thin at \( \infty \). Then to obtain a contradiction to \( (i) \) we shall make a positive harmonic function \( h(P) \) on \( C_{n}(\Omega) \) satisfying

\[
\inf_{P \in C_{n}(\Omega)} \frac{h(P)}{K_{\infty}(P)} = a < b \leq \inf_{P \in E} \frac{h(P)}{K_{\infty}(P)}.
\]

If \( E \) is a bounded subset of \( C_{n}(\Omega) \), then let \( h \) be a constant function. When \( E \) is unbounded, we shall follow Dahlberg [10, p.240] to make it.

We can assume \( \rho \leq \frac{1}{2} \). Let \( \{P_{j}\} \) be a sequence of points \( P_{j} \) which are the central points of cubes \( W_{i_{j}} \) in Lemma 2. Then \( \{P_{j}\} \) cannot accumulate to any finite boundary point of \( C_{n}(\Omega) \) and hence \( |P_{j}| \to +\infty \), because \( P_{j} \in E_{\rho} \) from (i) of Lemma 2 and \( K_{\infty}(P) > c \) for any \( P \in E \). Since \( E_{\rho} \) is minimally thin at \( \infty \) and

\[
\int_{W_{i_{j}}} \frac{dP}{(1 + |P|)^{n}} \approx \left( \frac{d(P_{j})}{|P_{j}|} \right)^{n} \quad (j = 1, 2, \ldots),
\]

Lemma 1 and (i) of Lemma 2 give

\[
(3.3) \quad \sum_{j=1}^{\infty} \left( \frac{d(P_{j})}{|P_{j}|} \right)^{n} < +\infty.
\]

Now we shall assume that \( d(P_{j}) \leq \frac{1}{2} |P_{j}| \quad (j = 1, 2, \ldots) \). The general case will be treated at the end of this proof. Take a point \( Q_{j} = (t_{j}, \Phi_{j}) \in \partial C_{\infty}(\Omega) \setminus \{O\} \) satisfying

\[
|P_{j} - Q_{j}| = d(P_{j}) \quad (j = 1, 2, \ldots).
\]
Then we also see $|Q_j| \geq \frac{1}{2}|P_j|$ and hence $|Q_j| \to +\infty$ ($j \to +\infty$). We define a function $h(P)$ by

$$h(P) = \sum_{j=1}^{\infty} P_{Q_j}(P) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha}}$$

where $G(P_1, P_2)$ ($P_1, P_2 \in C_n(\Omega)$) is the Green function of $C_n(\Omega)$ and $\frac{\partial}{\partial n_{Q}}$ denotes the differentiation at $Q \in \partial C_n(\Omega)$ along the inward normal into $C_n(\Omega)$. Then $h$ is well-defined and hence is a positive harmonic function on $C_n(\Omega)$, because at any fixed $P = (r, \Theta) \in C_n(\Omega)$

$$P_{Q_j}(P) \approx r^{\alpha} f_{\Omega}(\Theta) t_j^{-\beta-1} \frac{\partial}{\partial n_{\Phi_j}} f_{\Omega}(\Phi_j)$$

for every $Q_j$ satisfying $t_j \geq 2r$ (see Azarin [6, Lemma 1]).

Now we shall show

$$\inf_{P \in E} \frac{h(P)}{K_\infty(P)} > 0.$$ 

To see first

(3.4)  \[ \frac{h(P_j)}{K_\infty(P_j)} \geq A \quad (j = 1, 2, \ldots) \]

for some positive constant $A$, denote the Poisson kernel of the ball $B_j = B(P_j, d(P_j))$ by $P_j(P, Q)$ ($P \in B_j, Q \in \partial B_j$). Then we see

$$P_{Q_j}(P) \geq P_j(P, Q_j) \quad (P \in B_j; j = 1, 2, \ldots)$$

and hence

$$P_{Q_j}(P_j) \geq P_j(P_j, Q_j) = s_n^{-1}\{d(P_j)\}^{1-n} \quad (j = 1, 2, \ldots).$$

Since

$$f_{\Omega}(\Theta) \approx d(P') \quad (P' = (1, \Theta), \Theta \in \Omega),$$

we obtain

$$h(P_j) \geq P_{Q_j}(P_j) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha}} \geq AK_\infty(P_j) \quad (j = 1, 2, \ldots).$$

Next take any $P \in E$. Then by (ii) of Lemma 2 there exist a point $P_j$ such that

$$|P - P_j| < \frac{1}{2} \text{diam}(W_i) \leq \delta d(P_j),$$

where $\delta = \frac{1}{2} \left[ \frac{8}{3\rho} \right]^{-1}$. Hence we see

$$h(P) \geq \frac{1 - \delta}{(1 + \delta)^{n-1}} h(P_j) \quad \text{and} \quad K_\infty(P) \leq \frac{1 + \delta}{(1 - \delta)^{n-1}} K_\infty(P_j)$$
from the Harnack’s inequalities (see Armitage and Gardiner [5, Theorem 1.4.1]). Thus we have

$$\frac{h(P)}{K_{\infty}(P)} \geq \left(\frac{1 - \delta}{1 + \delta}\right)^n \frac{h(P_j)}{K_{\infty}(P_j)} \geq \left(\frac{1 - \delta}{1 + \delta}\right)^n A$$

from (3.4), which shows

$$\inf_{P \in E} \frac{h(P)}{K_{\infty}(P)} > 0.$$

To show $$\inf_{P \in C_n(\Omega)} \frac{h(P)}{K_{\infty}(P)} = 0$$, fix a ray $$L$$ which is inside $$C_n(\Omega)$$ and starts from $$O$$. We shall show

(3.5) $$\lim_{|P| \to +\infty, P \in L} \frac{h(P)}{K_{\infty}(P)} = 0.$$ 

Put

$$g_j(P) = \frac{P_{Q_j}(P)}{K_{\infty}(P)} |P_j|^{\beta_{\Omega} + 1} \quad (P \in C_n(\Omega), j = 1, 2, \ldots).$$

Then we have

$$\frac{h(P)}{K_{\infty}(P)} = \sum_{j=1}^{\infty} g_j(P) \left(\frac{d(P_j)}{|P_j|}\right)^n.$$

Since

(3.6) $$P_{Q_j}(P) \approx t_j^{\alpha_{\Omega} - 1} r^{-\beta_{\Omega}} f_\Omega(\Theta) \frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j) \quad (P = (r, \Theta) \in C_n(\Omega), r \geq 2t_j)$$

(see Azarin [6, Lemma 1]), we see

$$\lim_{|P| \to +\infty, P \in L} g_j(P) = 0$$

for any fixed $$j$$. Hence if we can show that

(3.7) $$|g_j(P)| \leq M \quad (P \in L, j = 1, 2, \ldots)$$

for some constant $$M$$, then we shall have (3.5) from (3.3).

Now we shall prove (3.7) by dividing into three cases. If $$r \leq \frac{t_j}{2}$$, then we have

$$P_{Q_j}(P) \approx r^{\alpha_{\Omega} - 1} t_j^{-\beta_{\Omega} - 1} f_\Omega(\Theta) \frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j)$$

and hence we have

$$|g_j(P)| \leq M \quad (P = (r, \Theta) \in C_n(\Omega), j = 1, 2, \ldots).$$

If $$r \geq 2t_j$$, then we have

$$|g_j(P)| \leq M \quad (P = (r, \Theta) \in C_n(\Omega); j = 1, 2, \ldots).$$
from (3.6). Lastly, put \( R_1 = \frac{r}{t_j} \), \( u = t_j \) and \( \Theta_1 = \Theta \) in
\[
u^{n-2}G((uR_1, \Theta_1), (uR_2, \Theta_2)) = G((R_1, \Theta_1), (R_2, \Theta_2)) \quad ((R_1, \Theta_1), (R_2, \Theta_2) \in C_n(\Omega)).
\]
When \((R_2, \Theta_2)\) approaches to \((1, \Phi_j)\) along the inward normal, we obtain
\[
\frac{\partial G(P, Q_j)}{\partial n_{Q_j}} = \frac{1}{t_j^{n-1}} \frac{\partial G}{\partial n_{Q_j}} \left( \left( \frac{r}{t_j}, \Theta \right), (1, \Phi_j) \right).
\]
If \( \frac{1}{2}t_j \leq r \leq 2t_j \), then
\[
t_j^{n-1}P_{Q_j}(P) \leq M' \quad (P = (r, \Theta) \in L; \ j = 1, 2, \ldots)
\]
for some constant \( M' \) and hence
\[
|g_j(P)| \leq M \quad (P \in L; \ j = 1, 2, \ldots).
\]
Finally, even if there is a \( j \) such that \( d(P_j) > \frac{1}{2}|P_j| \), there also exists a \( J \) such that \( d(P_j) \leq \frac{1}{2}|P_j| \) for every \( j \geq J \). Define \( h_2 \) by
\[
h_2(P) = \sum_{j=J}^{\infty} P_{Q_j}(P) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha}} \quad (P \in C_n(\Omega)),
\]
which satisfies
\[
h_2(P_j) \geq AK_\infty(P_j) \quad (j \geq J) \quad \text{and} \quad \inf_{P \in C_n(\Omega)} \frac{h_2(P)}{K_\infty(P)} = 0.
\]
Put \( \gamma = \max_{1 \leq j < J} K_\infty(P_j) \). Then the function \( h(P) = h_2(P) + \gamma \) is a positive harmonic function on \( C_n(\Omega) \) such that
\[
\inf_{P \in C_n(\Omega)} \frac{h(P)}{K_\infty(P)} = 0
\]
and
\[
h(P_j) \geq \min\{A, 1\} K_\infty(P_j) \quad (j = 1, 2, \ldots)
\]
from which it follows in the same way as above that
\[
\inf_{P \in E} \frac{h(P)}{K_\infty(P)} > 0.
\]

Proof of (iii) \( \Rightarrow \) (i).
Suppose that \( E \) does not characterize the positive harmonic majorization of \( K_\infty(P) \). Then there exists a positive harmonic function \( h(P) \) in \( C_n(\Omega) \) such that
\[
a = \inf_{P \in C_n(\Omega)} \frac{h(P)}{K_\infty(P)} < \inf_{P \in E} \frac{h(P)}{K_\infty(P)} = b.
\]
If we put \( v(P) = h(P) - aK_{\infty}(P) \) \((P \in C_n(\Omega))\), then \( v(P) \) is a positive harmonic function on \( C_n(\Omega) \) satisfying \( \inf_{P \in C_n(\Omega)} \frac{v(P)}{K_{\infty}(P)} = 0 \). Let \( \rho \) be any positive number satisfying \( 0 < \rho < 1 \). For any \( P \in E_{\rho} \), there exists a point \( P' \in E \) such that \( |P - P'| < \rho d(P') \) and hence

\[
\left( \frac{1 - \rho}{1 + \rho} \right)^n \frac{v(P')}{K_{\infty}(P')} \leq \frac{v(P)}{K_{\infty}(P)}
\]

by Harnack's inequality. (e.g. Armitage and Gardiner [5, Theorem 1.4.1]). Hence we have

\[
\inf_{P \in E_{\rho}} \frac{v(P)}{K_{\infty}(P)} \geq \left( \frac{1 - \rho}{1 + \rho} \right)^n \inf_{P \in E} \frac{v(P)}{K_{\infty}(P)} = \left( \frac{1 - \rho}{1 + \rho} \right)^n (b - a) > 0.
\]

Therefore we obtain

\[
\inf_{P \in C_n(\Omega)} \frac{v(P)}{K_{\infty}(P)} < \inf_{P \in E_{\rho}} \frac{v(P)}{K_{\infty}(P)}.
\]

Since \( v(P) \) is also a positive superharmonic function, \( E_{\rho} \) is minimally thin at \( \infty \) (e.g. Miyamoto, Yanagishita and Yoshida [16. Theorem 1]). This contradicts (iii).

4. Proofs of Theorem 2 and Corollary

Proof of Theorem 2. Proof of (i) \( \Rightarrow \) (ii). Suppose that

\[
\int_{E_{\rho}} \frac{dP}{(1 + |P|)^n} < +\infty
\]

for some \( \rho \) \((0 < \rho < 1)\). We can assume that this \( \rho \) satisfies \( 0 < \rho \leq \frac{1}{2} \). Let \( \{W_{i_{j}}\}_{j \geq 1} \) be a subsequence of \( \{W_{i}\}_{i \geq 1} \) in Lemma 2. Then from (i) of Lemma 2 we also have

\[
\int_{\bigcup_{j} W_{i_{j}}} \frac{dP}{(1 + |P|)^n} < +\infty.
\]

Since \( \bigcup_{j} W_{i_{j}} \) is a union of cubes from the Whitney cubes of \( C_n(\Omega) \) with \( \rho \), we see from the second part of Lemma 1 that \( \bigcup_{j} W_{i_{j}} \) is minimally thin at \( \infty \), and hence from (ii) of Lemma 2 that \( E_{\frac{\rho}{4}} \) is minimally thin at \( \infty \). Since \( E \) characterizes the positive harmonic majorization of \( K_{\infty}(P) \), it follows from Theorem 1 that \( E_{\frac{\rho}{4}} \) is not minimally thin at \( \infty \), which contradicts the conclusion obtained above.

Proof of (iii) \( \Rightarrow \) (i). Suppose that \( E \) does not characterize the positive harmonic majorization of \( K_{\infty}(P) \). Then we see from Theorem 1 that for any \( \rho \) \((0 < \rho < 1)\) \( E_{\rho} \) is minimally thin at \( \infty \). Lemma 1 gives that for any \( \rho \) \((0 < \rho < 1)\)

\[
\int_{E_{\rho}} \frac{dP}{(1 + |P|)^n} < +\infty.
\]
This contradicts (iii).

Proof of Corollary. It is easy to see that if $\{P_m\}$ is a separated sequence, then

$$B(P_i, \rho d(P_i)) \cap B(P_j, \rho d(P_j)) = \emptyset \quad (i, j = 1, 2, \ldots; i = j)$$

for a sufficiently small $\rho$ ($0 < \rho < 1$) and hence

$$\int_{E_{\rho}} \frac{dP}{(1 + |P|)^n} \approx \sum_{m=1}^{\infty} \left( \frac{d(P_m)}{|P_m|} \right)^n.$$

This corollary immediately follows from (iii) of Theorem 2.

References


