Beurling's Minimum Principle in a Cone

Abstract

This paper shows that some characterizations of the harmonic majorization of the Martin function connected with a domain having smooth boundary without a corner e.g a ball and a half-space also hold for a special domain with corners, i.e. a cone.

1. Introduction.

Let \( \mathbb{R} \) and \( \mathbb{R}_+ \) be the set of all real numbers and all positive real numbers, respectively. We denote by \( \mathbb{R}^n \) (\( n \geq 2 \)) the n-dimensional Euclidean space. A point in \( \mathbb{R}^n \) is denoted by \( P = (X, y), X = (x_1, x_2, \ldots, x_{n-1}) \). The Euclidean distance of two points \( P \) and \( Q \) in \( \mathbb{R}^n \) is denoted by \( |P-Q| \). Also \( |P-O| \) with the origin \( O \) of \( \mathbb{R}^n \) is simply denoted by \( |P| \). The boundary and the closure of a set \( S \) in \( \mathbb{R}^n \) are denoted by \( \partial S \) and \( \overline{S} \), respectively.

We introduce a system of spherical coordinates \( (r, \Theta) = (\theta_1, \theta_2, \ldots, \theta_{n-1}) \), in \( \mathbb{R}^n \) which are related to cartesian coordinates \( (x_1, x_2, \ldots, x_{n-1}, y) \) by

\[
x_1 = r(\Pi_{j=1}^{n-1} \sin \theta_j) \quad (n \geq 2), \quad y = r \cos \theta_1,
\]

and if \( n \geq 3 \), then

\[
x_{n+1-k} = r(\Pi_{j=1}^{k-1} \sin \theta_j) \cos \theta_k \quad (2 \leq k \leq n - 1),
\]

where \( 0 \leq r < +\infty, -\frac{1}{2} \pi \leq \theta_{n-1} < \frac{3}{2} \pi \), and if \( n \geq 3 \), then \( 0 \leq \theta_j \leq \pi \) (\( 1 \leq j \leq n - 2 \)).

The unit sphere and the upper half unit sphere are denoted by \( \mathbb{S}^{n-1} \) and \( \mathbb{S}_{+}^{n-1} \), respectively. For simplicity, a point \( (1, \Theta) \) on \( \mathbb{S}^{n-1} \) and the set \( \{\Theta; (1, \Theta) \in \Omega\} \) for a set \( \Omega \), \( \Omega \subset \mathbb{S}^{n-1} \), are often identified with \( \Theta \) and \( \Omega \), respectively. For two sets \( \Lambda \subset \mathbb{R}_+ \) and \( \Omega \subset \mathbb{S}^{n-1} \), the set

\[
\{(r, \Theta) \in \mathbb{R}^n; \quad r \in \Lambda, \quad (1, \Theta) \in \Omega\}
\]
in $\mathbb{R}^n$ is simply denoted by $\Lambda \times \Omega$. In particular, the half-space

$$\mathbb{R}_+ \times \mathbb{S}^{n-1}_+ = \{(X, y) \in \mathbb{R}^n; y > 0\}$$

will be denoted by $\mathbb{T}_n$.

To extend a result of Beurling [7] for $n=2$, Armitage and Kuran [4] said that a sequence $\{P_m\}$ of points $P_m = (X_m, y_m) \in \mathbb{T}_n$, $|P_m| \to +\infty (m \to +\infty)$ "characterizes the positive harmonic majorization of $y$", if every positive harmonic function $h$ in $\mathbb{T}_n$ which majorizes the function $y$ on the set $\{P_m; m = 1, 2, \ldots\}$ majorizes $y$ everywhere in $\mathbb{T}_n$, i.e.

$$\inf_{P \in \mathbb{T}_n} \frac{h(P)}{y} = \inf_m \frac{h(P_m)}{y_m} \quad (P = (X, y) \in \mathbb{T}_n).$$

They proved

**THEOREM A** (Beurling [7] for $n = 2$, Armitage and Kuran [4, Theorem 1] for $n \geq 2$). Let $\{P_m\}$ be a sequence of points $\{P_m\}$,

$$P_m = (r_m, \Theta_m) \in \mathbb{T}_n, \Theta_m = (\theta_{1,m}, \theta_{2,m}, \ldots, \theta_{(n-1),m})$$

in $\mathbb{T}_n$ satisfying

(1.1) \hspace{1cm} r_{m+1} \geq a r_m \quad (m = 1, 2, \ldots)

for a certain $a > 1$. Then the sequence $\{P_m\}$ characterizes the positive harmonic majorization of $y$ if and only if

(1.2) \hspace{1cm} \sum_{m=1}^{\infty} (\cos \theta_{1,m})^n = +\infty.

We remark that $y$ is the Martin function at the infinite Martin boundary point $\infty$ of $\mathbb{T}_n$, i.e. $y$ is (up to a positive multiplicative constant) the only positive harmonic function in $\mathbb{T}_n$ which vanishes on $\partial \mathbb{T}_n$. The "if" part of Theorem A is a minimm principle, since if $h$ is a positive harmonic function $h(P)$ of $P = (X, y) \in \mathbb{T}_n$, then

$$\liminf_{P \in \mathbb{T}_n, P \to P'} \{h(P) - y\} \geq 0$$

for every $P'$ on $\partial \mathbb{T}_n$ and the majorization of the function $y$ by $h$ on the set of points $P_m$ satisfying (1.1) and (1.2) replaces

$$\liminf_{P \in \mathbb{T}_n, |P| \to +\infty} \{h(P) - y\} \geq 0.$$
Hence this sort of sequence was said to be “equivalent to ∞” in Beurling [7] and this type of Theorem A was called “Beurling minimum principle” in Ancona [3, p.18] and Maz'ya [15]. This Theorem A was also extended by Maz'ya [15] to positive solutions of a second-order elliptic differential equation in an n-dimensional bounded domain with smooth boundary of class $C^{1,\alpha}$ ($0 < \alpha < 1$).

Let $D$ be a domain in $\mathbb{R}^n$ and $\Delta(D)$ be the Martin boundary of $D$. The Martin function at $Q \in \Delta(D)$ is denoted by $K_Q(P)$ ($P \in D$). Following Armitage and Kuran [4], we say that a subset $E$ of $D$ characterizes the positive harmonic majorization of $K_Q(P)$, if every positive harmonic function $h$ in $D$ which majorizes $K_Q(P)$ on $E$ majorizes $K_Q(P)$ everywhere in $D$, i.e.

$$\inf_{P \in D} \frac{h(P)}{K_Q(P)} = \inf_{P \in E} \frac{h(P)}{K_Q(P)}.$$  \hspace{1cm} (1.3)

We set

$$B(P, r) = \{P' \in \mathbb{R}^n; |P' - P| < r\} \ (r > 0)$$

and

$$d(P) = \inf_{Q \in D} |P - Q|$$

for any $P \in D$. For a subset $E$ of $D$ and a number $\rho$ ($0 < \rho < 1$) we put

$$E_\rho = \bigcup_{P \in E} B(P, \rho d(P)).$$  \hspace{1cm} (1.4)

Dahlberg proved

**THEOREM B** (Dahlberg [10, Theorem 1]). *Let $D$ be a Liapunov-Dini domain in $\mathbb{R}^n$ and $Q \in \partial D$. If $E \subset D$, then the following conditions on $E$ are equivalent;\n
(i) $E$ characterizes the positive harmonic majorization of $K_Q(P)$;\n
(ii) for every $\rho, 0 < \rho < 1$

$$\int_{E_\rho} \frac{dP}{|P - Q|^n} = +\infty,$$

(iii) for some $\rho, 0 < \rho < 1$

$$\int_{E_\rho} \frac{dP}{|P - Q|^n} = +\infty.$$ 

Since (1.3) is closely related to the notion of minimally thinness of $E_\rho$ in (1.4) (see Sjö gren [17], Ancona [3] and Zhang [19]), which is also seen in Theorem 1 of this paper, Aikawa and Essén [2, Corollary 7.4.7] also proved Theorem B in a different way from Dahlberg's.
By using a suitable Kelvin transformation which maps $T_n$ onto a ball, the following Theorem C follows from Theorem B.

**Theorem C (Dahlberg [10, Theorem 3]).** If $E \subset T_n$, then the following conditions on $E$ are equivalent:

(i) $E$ characterizes the positive harmonic majorization of $y$;

(ii) for every $\rho, 0 < \rho < 1$

$$
\int_{E_\rho} \frac{dP}{(1 + |P|)^n} = +\infty,
$$

where

$$
E_\rho = \bigcup_{P = (X, y) \in E} B(P, \rho y);
$$

(iii) for some $\rho, 0 < \rho < 1$

$$
\int_{E_\rho} \frac{dP}{(1 + |P|)^n} = +\infty.
$$

The methods of proving these Theorems A and B were based on the smoothness of the boundary having no wedges e.g. a ball. For a domain having more rough boundary e.g. a Lipschiz domain, Ancona [3, Theorem 7.4] and Zhang [19, Theorem 3] gave more complicated results which generalize Theorem A.

For a Lipshitz domain and an NTA domain $D$, Zhang [19, Corollary 1] and Aikawa [1, Remark and Theorem 1] gave a necessary and sufficient qualitative condition for a subset $E$ of $D$ characterizing the positive harmonic majorization of $K_Q(P)$ by connecting with minimally thinness of $E_\rho$ in (1.4), respectively. In his paper Aikawa says that since a general NTA domain may have wedges, Theorem B does not hold for an NTA domain. But when we see our results in this paper, we may ask whether quantitative Theorem B can just be extended to a Lipshitz domain and an NTA domain.

In this paper we shall prove Theorems A and C can be extended to a result at a corner point of a wedge i.e. a result at $\infty$ of a cone. We remark that a half-space is one of cones.

2. Statements of results.

Let $\Omega$ be a domain on $S^{n-1}(n \geq 2)$ with smooth boundary. Consider the Dirichlet problem

$$(\Lambda_n + \tau)f = 0 \quad \text{on } \Omega$$

$$f = 0 \quad \text{on } \partial \Omega,$$

where $\Lambda_n$ is the spherical part of the Laplace operator $\Delta_n$

$$
\Delta_n = \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2} \Lambda_n
$$
We denote the least positive eigenvalue of this boundary value problem by $\tau_\Omega$ and the normalized positive eigenfunction corresponding to $\tau_\Omega$ by $f_\Omega(\Theta)$:

$$\int_\Omega f_\Omega^2(\Theta) d\sigma_\Theta = 1,$$

where $d\sigma_\Theta$ is the surface element on $S^{n-1}$. We denote the solutions of the equation

$$t^2 + (n-2)t - \tau_\Omega = 0$$

by $\alpha_\Omega, -\beta_\Omega$ ($\alpha_\Omega, \beta_\Omega > 0$). If $\Omega = S_+^{n-1}$, then $\alpha_\Omega = 1$, $\beta_\Omega = n - 1$ and

$$f_\Omega(\Theta) = (2n s_n^{-1})^{1/2} \cos \theta_1,$$

where $s_n$ is the surface area $2\pi^{n/2}\Gamma(n/2)^{-1}$ of $S^{n-1}$.

To make simplify our consideration in the following, we shall assume that if $n \geq 3$, then $\Omega$ is a $C^{2,\alpha}$-domain ($0 < \alpha < 1$) on $S^{n-1}$ (e.g. see Gilbarg and Trudinger [12, pp.88-89] for the definition of $C^{2,\alpha}$-domain). By $C_n(\Omega)$, we denote the set $\mathbb{R}_+ \times \Omega$ in $\mathbb{R}^n$ with the domain $\Omega$ on $S^{n-1}$ ($n \geq 2$). We call it a cone. Then $T_n$ is a cone obtained by putting $\Omega = S_+^{n-1}$. It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$, each of which is a minimal Martin boundary point. The Martin kernel at $\infty$ with respect to a reference point chosen suitably is $K_\infty(P) = r^{\alpha_\Omega} f_\Omega(\Theta) \ (P = (r, \Theta) \in C_n(\Omega)).$

A subset $E$ of a domain $D$ in $\mathbb{R}^n$ is said to be minimally thin at $Q \in \Delta(D)$ (Brelot [8, p.122], Doob [11, p.208]), if there exists a point $P \in D$ such that

$$\hat{R}^E_{K_Q(P)} = K_Q(P),$$

where $\hat{R}^E_{K_Q(P)}$ is the regularized reduced function of $K_Q(P)$ relative to $E$ (Helms [14, p.134]).

The following Theorem 1 which is used to prove Theorem 2 is a specialized version of Aikawa [1, Theorem 1]. Since his proof is so complicated because of an NTA domain we shall give a simple proof based on a function which is a conical version of Dahlberg's [10, pp.240-241].

**THEOREM 1.** Let $E$ be a subset of $C_n(\Omega)$. The following conditions on $E$ are equivalent:

(i) $E$ characterizes the positive harmonic majorization of $K_\infty(P)$;

(ii) for any $\rho$, $0 < \rho < 1$, $E_\rho$ is not minimally thin at $\infty$,

(iii) for some $\rho$, $0 < \rho < 1$, $E_\rho$ is not minimally thin at $\infty$.

The following Theorem 2 extends Theorem C.

**THEOREM 2.** Suppose that $E \subset C_n(\Omega)$. Then the following conditions on $E$ are equivalent:
(i) $E$ characterizes the positive harmonic majorization of $K_{\infty}(P)$;
(ii) for every $\rho$ ($0 < \rho < 1$),
\[
\int_{E_{\rho}} \frac{dP}{(1+|P|)^{n}} = +\infty,
\]
(iii) for some $\rho$ ($0 < \rho < 1$),
\[
\int_{E_{\rho}} \frac{dP}{(1+|P|)^{n}} = +\infty.
\]

A sequence $\{P_{m}\}$ of points $P_{m} \in D$ is said to be separated, if there exists a positive constant $c$ such that
\[
|P_{i} - P_{j}| \geq cd(P_{i}) \quad (i, j = 1, 2, ..., i = j)
\]
(e.g. see Ancona [3, p.18], Aikawa and Essén [2, p.156]).

From Theorem 2 we immediately have the following Corollary which extends Theorem A.

**COROLLARY.** Let $\{P_{m}\}$, $P_{m} \in C_{n}(\Omega)$ be a separated sequence satisfying
\[
\inf_{m} |P_{m}| > 0.
\]

The sequence $\{P_{m}\}$ characterizes the positive harmonic majorization of $K_{\infty}(P)$ if and only if
\[
\sum_{m=1}^{\infty} \left( \frac{d(P_{m})}{|P_{m}|} \right)^{n} = +\infty.
\]

3. **Lemmas and proof of Theorem 1.**

Let $f$ and $g$ be two positive real valued functions defined on a set $Z$. Then we shall write $f \equiv g$, if there exists two constants $A_{1}, A_{2}, 0 < A_{1} \leq A_{2}$ such that $A_{1}g \leq f \leq A_{2}g$ everywhere on $Z$. For a subset $S$ in $\mathbb{R}^{n}$, the interior of $S$ and the diameter of $S$ are denoted by int $S$ and diam $S$, respectively. For two subsets $S_{1}$ and $S_{2}$ in $\mathbb{R}^{n}$, the distance between $S_{1}$ and $S_{2}$ is denoted by dist($S_{1}, S_{2}$). A cube of $\mathcal{M}_{k}$ is of the form

\[
[l_{1}2^{-k}, (l_{1}+1)2^{-k}] \times \cdots \times [l_{n}2^{-k}, (l_{n}+1)2^{-k}] \quad (k = 0, \pm 1, \pm 2, \ldots)
\]

where $l_{1}, \ldots, l_{n}$ are integers. Let $\rho$ be a number satisfying $0 < \rho \leq \frac{1}{2}$. A family of the Whitney cubes of $C_{n}(\Omega)$ with $\rho$ is the set of cubes having the following properties;
(i) $\bigcup_{i} W_{i} = C_{n}(\Omega)$,
(ii) $\int W_i \cap W_k = \emptyset \quad (i = k),$

(iii) $\left[ \frac{8}{3\rho} \right] \text{diam} W_i \leq \text{dist}(W_i, \mathbb{R}^n \setminus C_n(\Omega)) \leq 2 \left( \left[ \frac{8}{3\rho} \right] + 1 \right) \text{diam} W_i,$

where $[a]$ denotes the integer satisfying $[a] \leq a < [a] + 1$ (Stein [18, p.167, Theorem 1]).

The following Lemma 1 is fundamental in this paper.

**Lemma 1** (I. Miyamoto, M. Yanagishita and H. Yoshida [16, Theorems 2 and 3]). Let a Borel subset $E$ of $C_n(\Omega)$ be minimally thin at $\infty$. Then we have

\begin{equation}
\int_E \frac{dP}{(1 + |P|)^n} < +\infty.
\end{equation}

If $E$ is a union of cubes from a family of the Whitney cubes of $C_n(\Omega)$ with $\rho$ ($0 < \rho \leq \frac{1}{2}$), then (3.1) is also sufficient for $E$ to be minimally thin at $\infty$.

For a set $E \subset C_n(\Omega)$ and a number $\rho$ ($0 < \rho \leq \frac{1}{2}$), define $E_\rho$ and $E_{\frac{\rho}{4}}$ as in (1.4).

**Lemma 2.** Let $\{W_i\}_{i \geq 1}$ be a family of the Whitney cubes of $C_n(\Omega)$ with $\rho$. Then there exists a subsequence $\{W_{i_j}\}_{j \geq 1}$ of $\{W_i\}_{i \geq 1}$ such that

(i) $\bigcup_{j} W_{i_j} \subset E_\rho$,

(ii) $W_{i_j} \cap E_{\frac{\rho}{4}} = \emptyset \quad (j = 1, 2, \ldots)$

$E_{\frac{\rho}{4}} \subset \bigcup_{j} W_{i_j}$.

**Proof of Lemma 2.** Let $k$ be an integer. Let $c = \left( \left[ \frac{8}{3\rho} \right] + 1 \right)$ and set

$I_k = \{ P \in C_n(\Omega) ; c\sqrt{n}2^{-k} < \text{dist}(P, \partial C_n(\Omega)) \leq c\sqrt{n}2^{-k+1} \}.$

Let $\{W_{i_j}\}_{j \geq 1}$ be a sequence of all Whitney cubes of $\{W_i\}_{i \geq 1}$ such that

$W_{i_j} \cap E_{\frac{\rho}{4}} = \emptyset \quad (j = 1, 2, \ldots).$

Then it is evident that (ii) holds. We shall also show that this $\{W_{i_j}\}_{j \geq 1}$ satisfies (i).

Take any $W_{i_{j_0}}$ and let $W_{i_{j_0}}$ be a cube of $\mathcal{M}_{k_0}$. Since $W_{i_{j_0}} \cap E_{\frac{\rho}{4}} = \emptyset$, there exists a point $P_{j_0}$ in $E$ such that

\begin{equation}
B(P_{j_0}, \frac{\rho}{4}d(P_{j_0})) \cap W_{i_{j_0}} = \emptyset.
\end{equation}

Then $P_{j_0} \in I_{k_0+1} \cup I_{k_0} \cup I_{k_0-1}$. Because, if $P \in I_k$ and $W_{i_j}$ is a Whitney cube satisfying $W_{i_j} \cap B(P, \frac{\rho}{4}d(P)) = \emptyset$, then $W_{i_j} \in \mathcal{M}_{k+1} \cup \mathcal{M}_k \cup \mathcal{M}_{k-1}$.

If $P_{j_0} \in I_{k_0+1}$, then

$\rho d(P_{j_0}) - \frac{\rho}{4} d(P_{j_0}) = \frac{3}{4} \rho d(P_{j_0}) > \frac{3}{4} \rho \left( \left[ \frac{8}{3\rho} \right] + 1 \right) \sqrt{n}2^{-(k_0+1)} > \sqrt{n}2^{-k_0}.$

Since the diameter of $W_{i_{j_0}}$ is $\sqrt{n}2^{-k_0}$, we have from (3.2) that $W_{i_{j_0}} \subset B(P_{j_0}, \rho d(P_{j_0}))$ and hence $W_{i_{j_0}} \subset E_{\rho}$. Even if $P_{j_0} \in I_{k_0}$ or $P_{j_0} \in I_{k_0-1}$, we similarly have $W_{i_{j_0}} \subset E_{\rho}$. 

Thus all cubes of $\{W_{i_{j}}\}_{j \geq 1}$ are contained in $E_{\rho}$, which is just (i).

Proof of Theorem 1. Proof of (i) \(\Rightarrow\) (ii). First of all, we shall remark the following fact. Let $c$ be a positive constant. Since $E$ characterizes the positive harmonic majorization of $K_{\infty}(P)$, $E_{1} = \{P \in E; K_{\infty}(P) > c\}$ also characterizes the positive harmonic majorization of $K_{\infty}(P)$. For otherwise there exists a positive harmonic function $h(P)$ on $C_{n}(\Omega)$, satisfying

$$a = \inf_{P \in C_{n}(\Omega)} \frac{h(P)}{K_{\infty}(P)} < \inf_{P \in E_{1}} \frac{h(P)}{K_{\infty}(P)} = b.$$  

If we put $h_{1}(P) = h(P) + bc \quad (P \in C_{n}(\Omega))$, then $h_{1}(P) \geq bK_{\infty}(P)$ for all $P \in E$ and hence

$$\inf_{P \in C_{n}(\Omega)} \frac{h_{1}(P)}{K_{\infty}(P)} = a < b \leq \inf_{P \in E} \frac{h_{1}(P)}{K_{\infty}(P)},$$

which contradicts the assumption that $E$ characterizes the positive harmonic majorization of $K_{\infty}(P)$. If we can show that for any $\rho \ (0 < \rho < 1)$ $(E_{1})_{\rho}$ is not minimally thin at $\infty$, then for any $\rho \ (0 < \rho < 1)$ $E_{\rho}$ is also not minimally thin at $\infty$. Hence by applying the following argument to $E_{1}$ if necessary, we may assume that $K_{\infty}(P) > c$ for every $P \in E$, without generality.

Suppose that for some number $\rho \ (0 < \rho < 1)$ $E_{\rho}$ is minimally thin at $\infty$. Then to obtain a contradiction to (i) we shall make a positive harmonic function $h(P)$ on $C_{n}(\Omega)$ satisfying

$$\inf_{P \in C_{n}(\Omega)} \frac{h(P)}{K_{\infty}(P)} < \inf_{P \in E} \frac{h(P)}{K_{\infty}(P)}.$$ 

If $E$ is a bounded subset of $C_{n}(\Omega)$, then let $h$ be a constant function. When $E$ is unbounded, we shall follow Dahlberg [10, p.240] to make it.

We can assume $\rho \leq \frac{1}{2}$. Let $\{P_{j}\}$ be a sequence of points $P_{j}$ which are the central points of cubes $W_{i_{j}}$ in Lemma 2. Then $\{P_{j}\}$ can not accumulate to any finite boundary point of $C_{n}(\Omega)$ and hence $|P_{j}| \to +\infty$, because $P_{j} \in E_{\rho}$ from (i) of Lemma 2 and $K_{\infty}(P) > c$ for any $P \in E$. Since $E_{\rho}$ is minimally thin at $\infty$ and

$$\int_{W_{i_{j}}} \frac{dP}{(1 + |P|)^{n}} \approx \left(\frac{d(P_{j})}{|P_{j}|}\right)^{n} (j = 1, 2, \ldots),$$

Lemma 1 and (i) of Lemma 2 give

$$\sum_{j=1}^{\infty} \left(\frac{d(P_{j})}{|P_{j}|}\right)^{n} < +\infty.$$  

Now we shall assume that $d(P_{j}) \leq \frac{1}{2}|P_{j}| \quad (j = 1, 2, \ldots)$. The general case will be treated at the end of this proof. Take a point $Q_{j} = (t_{j}, \Phi_{j}) \in \partial C_{n}(\Omega) \setminus \{O\}$ satisfying

$$|P_{j} - Q_{j}| = d(P_{j}) \quad (j = 1, 2, \ldots).$$
Then we also see $|Q_j| \geq \frac{1}{2}|P_j|$ and hence $|Q_j| \to +\infty (j \to +\infty)$. We define a function $h(P)$ by

$$h(P) = \sum_{j=1}^{\infty} P_{Q_j}(P) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha}}$$

where $G(P_1, P_2)$ ($P_1, P_2 \in C_n(\Omega)$) is the Green function of $C_n(\Omega)$ and $\frac{\partial}{\partial n_Q}$ denotes the differentiation at $Q \in \partial C_n(\Omega)$ along the inward normal into $C_n(\Omega)$. Then $h$ is well-defined and hence is a positive harmonic function on $C_n(\Omega)$, because at any fixed $P = (r, \Theta) \in C_n(\Omega)$

$$P_{Q_j}(P) \approx r^\alpha f(\Theta) t_{j}^{-\beta} \frac{\partial}{\partial n_{\Phi_j}} f(\Phi_j)$$

for every $Q_j$ satisfying $t_j \geq 2r$ (see Azarin [6, Lemma 1]).

Now we shall show

$$\inf_{P \in B} \frac{h(P)}{K_{\infty}(P)} > 0.$$

To see first

$$\frac{h(P_j)}{K_{\infty}(P_j)} \geq A \quad (j = 1, 2, \ldots)$$

for some positive constant $A$, denote the Poisson kernel of the ball $B_j = B(P_j, d(P_j))$ by $P_j(P, Q)$ ($P \in B_j, Q \in \partial B_j$). Then we see

$$P_{Q_j}(P) \geq P_j(P, Q_j) \quad (P \in B_j; j = 1, 2, \ldots)$$

and hence

$$P_{Q_j}(P_j) \geq P_j(P_j, Q_j) = s^{-1}_n \{d(P_j)\}^{1-n} \quad (j = 1, 2, \ldots).$$

Since

$$f(\Theta) \approx d(P') \quad (P' = (1, \Theta), \Theta \in \Omega),$$

we obtain

$$h(P_j) \geq P_{Q_j}(P_j) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha}} \geq AK_{\infty}(P_j) \quad (j = 1, 2, \ldots).$$

Next take any $P \in E$. Then by (ii) of Lemma 2 there exist a point $P_j$ such that

$$|P - P_j| < \frac{1}{2} \text{diam}(W_i) \leq \delta d(P_j),$$

where $\delta = \frac{1}{2} \left[ \frac{8}{3\rho} \right]^{-1}$. Hence we see

$$h(P) \geq \frac{1 - \delta}{(1 + \delta)^{n-1}} h(P_j) \quad \text{and} \quad K_{\infty}(P) \leq \frac{1 + \delta}{(1 - \delta)^{n-1}} K_{\infty}(P_j).$$
from the Harnack's inequalities (see Armitage and Gardiner [5, Theorem 1.4.1]). Thus we have
\[
\frac{h(P)}{K_{\infty}(P)} \geq \left(\frac{1-\delta}{1+\delta}\right)^n \frac{h(P_j)}{K_{\infty}(P_j)} \geq \left(\frac{1-\delta}{1+\delta}\right)^n A
\]
from (3.4), which shows
\[
\inf_{P \in E} \frac{h(P)}{K_{\infty}(P)} > 0.
\]

To show \(\inf_{P \in C_n(\Omega)} \frac{h(P)}{K_{\infty}(P)} = 0\), fix a ray \(L\) which is inside \(C_n(\Omega)\) and starts from \(O\). We shall show

\[
(3.5) \quad \lim_{|P| \to +\infty, P \in L} \frac{h(P)}{K_{\infty}(P)} = 0.
\]

Put
\[
g_j(P) = \frac{P_{Q_j}(P)}{K_{\infty}(P)} |P_j|^{\beta_{\Omega}+1} \quad (P \in C_n(\Omega), j = 1, 2, \ldots).
\]
Then we have
\[
\frac{h(P)}{K_{\infty}(P)} = \sum_{j=1}^{\infty} g_j(P) \left(\frac{d(P_j)}{|P_j|}\right)^n.
\]

Since
\[
P_{Q_j}(P) \approx t_j^{\alpha_{\Omega}-1} r^{-\beta_{\Omega}} f_{\Omega}(\Theta) \frac{\partial}{\partial n_{\Phi_j}} f_{\Omega}(\Phi_j) \quad (P = (r, \Theta) \in C_n(\Omega), r \geq 2t_j)
\]
(see Azarin [6, Lemma 1]), we see
\[
\lim_{|P| \to +\infty, P \in L} g_j(P) = 0
\]
for any fixed \(j\). Hence if we can show that
\[
(3.7) \quad |g_j(P)| \leq M \quad (P \in L, j = 1, 2, \ldots)
\]
for some constant \(M\), then we shall have (3.5) from (3.3).

Now we shall prove (3.7) by dividing into three cases. If \(r \leq \frac{t_j}{2}\), then we have
\[
P_{Q_j}(P) \approx r^{\alpha_{\Omega} t_j^{\beta_{\Omega}-1}} f_{\Omega}(\Theta) \frac{\partial}{\partial n_{\Phi_j}} f_{\Omega}(\Phi_j)
\]
and hence we have
\[
|g_j(P)| \leq M \quad (P = (r, \Theta) \in C_n(\Omega), j = 1, 2, \ldots).
\]
If \(r \geq 2t_j\), then we have
\[
|g_j(P)| \leq M \quad (P = (r, \Theta) \in C_n(\Omega); \ j = 1, 2, \ldots)
\]
from (3.6). Lastly, put \( R_1 = \frac{r}{t_j} \), \( u = t_j \) and \( \Theta_1 = \Theta \) in

\[
u^{n-2}G((uR_1, \Theta_1), (uR_2, \Theta_2)) = G((R_1, \Theta_1), (R_2, \Theta_2)), \quad ((R_1, \Theta_1), (R_2, \Theta_2) \in C_n(\Omega)).
\]

When \( (R_2, \Theta_2) \) approaches to \( (1, \Phi_j) \) along the inward normal, we obtain

\[
(3.8) \quad \frac{\partial G(P, Q_j)}{\partial n_{Q_j}} = \frac{1}{t_j^{n-1}} \frac{\partial G}{\partial n_{Q_j}} \left( \left( \frac{r}{t_j}, \Theta \right), (1, \Phi_j) \right).
\]

If \( \frac{1}{2}t_j \leq r \leq 2t_j \), then

\[
t_j^{n-1}P_{Q_j}(P) \leq M' \quad (P = (r, \Theta) \in L; \ j = 1, 2, \ldots)
\]

for some constant \( M' \) and hence

\[
|g_j(P)| \leq M \quad (P \in L; \ j = 1, 2, \ldots).
\]

Finally, even if there is a \( j \) such that \( d(P_j) > \frac{1}{2}|P_j| \), there also exists a \( J \) such that \( d(P_j) \leq \frac{1}{2}|P_j| \) for every \( j \geq J \). Define \( h_2 \) by

\[
h_2(P) = \sum_{j=J}^{\infty} P_{Q_j}(P) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha}} \quad (P \in C_n(\Omega)),
\]

which satisfies

\[
h_2(P_j) \geq AK_\infty(P_j) \quad (j \geq J) \quad \text{and} \quad \inf_{P \in C_n(\Omega)} \frac{h_2(P)}{K_\infty(P)} = 0.
\]

Put \( \gamma = \max_{1 \leq j < J} K_\infty(P_j) \). Then the function \( h(P) = h_2(P) + \gamma \) is a positive harmonic function on \( C_n(\Omega) \) such that

\[
\inf_{P \in C_n(\Omega)} \frac{h(P)}{K_\infty(P)} = 0
\]

and

\[
h(P_j) \geq \min\{A, 1\} K_\infty(P_j) \quad (j = 1, 2, \ldots)
\]

from which it follows in the same way as above that

\[
\inf_{P \in E} \frac{h(P)}{K_\infty(P)} > 0.
\]

Proof of \( (iii) \Rightarrow (i) \).

Suppose that \( E \) does not characterize the positive harmonic majorization of \( K_\infty(P) \). Then there exists a positive harmonic function \( h(P) \) in \( C_n(\Omega) \) such that

\[
a = \inf_{P \in C_n(\Omega)} \frac{h(P)}{K_\infty(P)} < \inf_{P \in E} \frac{h(P)}{K_\infty(P)} = b.
\]
If we put \(v(P) = h(P) - aK_{\infty}(P)\) \((P \in C_{n}(\Omega))\), then \(v(P)\) is a positive harmonic function on \(C_{n}(\Omega)\) satisfying \(\inf_{P \in C_{n}(\Omega)} \frac{v(P)}{K_{\infty}(P)} = 0\). Let \(\rho\) be any positive number satisfying \(0 < \rho < 1\). For any \(P \in E_{\rho}\), there exists a point \(P' \in E\) such that \(|P - P'| < \rho d(P')\) and hence
\[
\left(\frac{1 - \rho}{1 + \rho}\right)^{n} \frac{v(P')}{K_{\infty}(P')} \leq \frac{v(P)}{K_{\infty}(P)}
\]
by Harnack’s inequality. (e.g. Armitage and Gardiner [5, Theorem 1.4.1]). Hence we have
\[
\inf_{P \in E_{\rho}} \frac{v(P)}{K_{\infty}(P)} \geq \left(\frac{1 - \rho}{1 + \rho}\right)^{n} \inf_{P \in E} \frac{v(P)}{K_{\infty}(P)} = \left(\frac{1 - \rho}{1 + \rho}\right)^{n} (b - a) > 0.
\]
Therefore we obtain
\[
\inf_{P \in C_{n}(\Omega)} \frac{v(P)}{K_{\infty}(P)} < \inf_{P \in E_{\rho}} \frac{v(P)}{K_{\infty}(P)}
\]
Since \(v(P)\) is also a positive superharmonic function, \(E_{\rho}\) is minimally thin at \(\infty\) (e.g. Miyamoto, Yanagishita and Yoshida [16, Theorem 1]). This contradicts \((iii)\).

4. Proofs of Theorem 2 and Corollary

**Proof of Theorem 2.** Proof of \((i) \Rightarrow (ii)\). Suppose that
\[
\int_{E_{\rho}} \frac{dP}{(1 + |P|)^{n}} < +\infty
\]
for some \(\rho (0 < \rho < 1)\). We can assume that this \(\rho\) satisfies \(0 < \rho \leq \frac{1}{2}\). Let \(\{W_{i_{j}}\}_{j \geq 1}\) be a subsequence of \(\{W_{i}\}_{i \geq 1}\) in Lemma 2. Then from \((i)\) of Lemma 2 we also have
\[
\int_{\bigcup_{j}W_{i_{j}}} \frac{dP}{(1 + |P|)^{n}} < +\infty.
\]
Since \(\bigcup_{j}W_{i_{j}}\) is a union of cubes from the Whitney cubes of \(C_{n}(\Omega)\) with \(\rho\), we see from the second part of Lemma 1 that \(\bigcup_{j}W_{i_{j}}\) is minimally thin at \(\infty\), and hence from \((ii)\) of Lemma 2 that \(E_{\frac{\rho}{4}}\) is minimally thin at \(\infty\). Since \(E\) characterizes the positive harmonic majorization of \(K_{\infty}(P)\), it follows from Theorem 1 that \(E_{\frac{\rho}{4}}\) is not minimally thin at \(\infty\), which contradicts the conclusion obtained above.

Proof of \((iii) \Rightarrow (i)\). Suppose that \(E\) does not characterize the positive harmonic majorization of \(K_{\infty}(P)\). Then we see from Theorem 1 that for any \(\rho (0 < \rho < 1)\) \(E_{\rho}\) is minimally thin at \(\infty\). Lemma 1 gives that for any \(\rho (0 < \rho < 1)\)
\[
\int_{E_{\rho}} \frac{dP}{(1 + |P|)^{n}} < +\infty.
\]
This contradicts (iii).

**Proof of Corollary.** It is easy to see that if \( \{P_m\} \) is a separated sequence, then

\[
B(P_i, \rho d(P_i)) \cap B(P_j, \rho d(P_j)) = \emptyset \quad (i, j = 1, 2, \ldots; i = j)
\]

for a sufficiently small \( \rho \) \( (0 < \rho < 1) \) and hence

\[
\int_{E_\rho} \frac{dP}{(1 + |P|)^n} \approx \sum_{m=1}^{\infty} \left( \frac{d(P_m)}{|P_m|} \right)^n.
\]

This corollary immediately follows from (iii) of Theorem 2.

**References**


