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Kyoto University
Riesz decomposition and limits at infinity for $p$-precise functions on a half space

1 Introduction

Let $u$ be a nonnegative superharmonic function on $D = \{x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n; x_n > 0\}$, where $n \ge 2$. Then it is known (cf. Lelong-Ferrand [6]) that $u$ is uniquely decomposed as

$$u(x) = ax_n + \int_D G(x, y)d\mu(y) + \int_{\partial D} P(x, y)d\nu(y),$$

where $a$ is a nonnegative number, $\mu$ (resp. $\nu$) is a nonnegative measure on $D$ (resp. $\partial D$), $G$ is the Green function for $D$ and $P$ is the Poisson kernel for $D$. The first author showed in [9] that if $0 \le \beta \le 1$, $1 - n \le \gamma < 1$ and $\int_D y_n^\gamma d\mu(y) + \int_{\partial D} |y|^{\gamma-1}d\nu(y) < \infty$, then

$$\lim_{|x| \to \infty, x \in D - E'} x_n^{-\beta}|x|^{n+\gamma-2+\beta}[u(x) - ax_n] = 0$$

with a suitable exceptional set $E' \subset D$. For related results, we also refer the reader to Essén-Jackson [3, Theorem 4.6], Aikawa [1] and Miyamoto-Yoshida [8].

Our main aim in this paper is to establish the analogue of these results for locally $p$-precise functions $u$ in $D$ satisfying

$$\int_D |\nabla u(x)|^p x_n^\gamma dx < \infty,$$

where $\nabla$ denotes the gradient, $1 < p < \infty$ and $-1 < \gamma < p - 1$ (see Ohtsuka [15] and Ziemer [17] for locally $p$-precise functions).

2 Fine limits at infinity

Denote by $D^{p,\gamma}$ the space of all locally $p$-precise functions on $D$ satisfying (1). Consider the kernel function

$$K_\gamma(x, y) = |x - y|^{1-n}y_n^{-\gamma/p}.$$ 

To evaluate the size of exceptional sets, we use the capacity

$$C_{K_\gamma, p}(E; G) = \inf \int_D g(y)^p dy,$$
where $E$ is a subset of an open set $G$ in $D$ and the infimum is taken over all nonnegative measurable functions $g$ such that $g = 0$ outside $G$ and

$$\int_D K_\gamma(x, y)g(y)dy \geq 1 \quad \text{for all } x \in E.$$ 

We say that $E \subset D$ is $(K_\gamma, p)$-thin at infinity if

$$\sum_{i=1}^{\infty} 2^{-i(n+\gamma-p)}C_{K_\gamma, p}(E_i; D_i) < \infty,$$

(2)

where $E_i = \{x \in E : 2^i \leq |x| < 2^{i+1}\}$ and $D_i = \{x \in D : 2^{i-1} < |x| < 2^{i+2}\}$.

Our first aim in this paper is to establish the following theorem.

**Theorem 1** (cf. [4]). Let $p > 1$, $-1 < \gamma < p - 1$ and $n + \gamma - p \geq 0$. If $u \in D^{p, \gamma}$, then there exist a set $E \subset D$ and a number $A$ such that $E$ is $(K_\gamma, p)$-thin at infinity;

$$\lim_{|x| \to \infty, x \in D - E} |x|^{(n + \gamma - p)/p} [u(x) - A] = 0$$

in case $n + \gamma - p > 0$ and

$$\lim_{|x| \to \infty, x \in D - E} (\log |x|)^{-1/p'} [u(x) - A] = 0$$

in case $n + \gamma - p = 0$, where $p' = p/(p - 1)$.

In fact, if $1 \leq q < p$ and $q < p/(1 + \gamma)$, then Hölder's inequality gives

$$\int_G |\nabla u(x)|^q \, dx \leq \left( \int_G x_n^{-\gamma q/(p-q)} \, dx \right)^{1-q/p} \left( \int_G |\nabla u(x)|^p x_n^\gamma \, dx \right)^{q/p} < \infty$$

for every bounded open set $G \subset D$. Hence we can find a locally $q$-precise extension $\overline{u}$ to $\mathbb{R}^n$ such that $\overline{u}(x', x_n) = u(x', x_n)$ for $x_n > 0$ and $\overline{u}(x', x_n) = u(x', -x_n)$ for $x_n < 0$. We denote by $B(x, r)$ the open ball centered at $x$ with radius $r > 0$. In view of [13], we can find a number $a$ such that

$$\overline{u}(x) = c_n \sum_{i=1}^{n} \int_{B(0,1)} \frac{x_i - y_i}{|x - y|^n} \frac{\partial \overline{u}}{\partial y_i}(y) \, dy + c_n \sum_{i=1}^{n} \int_{\mathbb{R}^n - B(0,1)} \left( \frac{x_i - y_i}{|x - y|^n} - \frac{-y_i}{|y|^n} \right) \frac{\partial \overline{u}}{\partial y_i}(y) \, dy + a$$

for almost every $x \in \mathbb{R}^n$. Here we see that the equality holds for every $x \in D$ except that in a set of $C_{K_\gamma, p}$-capacity zero. Now Theorem 1 is a consequence of [4].

## 3 Riesz decomposition

We denote by $D^{p, \gamma}_0$ the space of all functions $u \in D^{p, \gamma}$ having vertical limit zero at almost every boundary point of $D$, and by $HD^{p, \gamma}$ the space of all harmonic functions on $D$ in $D^{p, \gamma}$. As in Deny-Lions [2], we have the following Riesz decomposition of $u \in D^{p, \gamma}$. 

THEOREM 2. A function \( u \in \mathbb{D}^{p,\gamma} \) is uniquely represented as

\[
    u = u_0 + h, 
\]

where \( u_0 \in \mathbb{D}_0^{p,\gamma} \) and \( h \in \mathbb{H} \mathbb{D}^{p,\gamma} \). More precisely, for fixed \( \xi \in D \),

\[
    u_0(x) = c_n \sum_{i=1}^{n} \int_{D} \left( \frac{x_{i} - y_{i}}{|x - y|^n} - \frac{\bar{x}_{i} - y_{i}}{|ar{x} - y|^n} \right) \frac{\partial u}{\partial y_{i}}(y) dy, 
\]

\[
    h(x) = 2c_n \sum_{i=1}^{n} \int_{D} \left( \frac{\bar{x}_{i} - y_{i}}{|\bar{x} - y|^n} - \frac{\bar{\xi}_{i} - y_{i}}{|ar{\xi} - y|^n} \right) \frac{\partial u}{\partial y_{i}}(y) dy + A, 
\]

where \( \bar{x} = (x_1, \ldots, x_{n-1}, -x_n) \) for \( x = (x_1, \ldots, x_{n-1}, x_n) \), \( c_n = \Gamma(n/2)/(2\pi^{n/2}) \) and \( A \) is a constant depending on \( u \) and \( \xi \).

As applications we are concerned with the limits at infinity of functions in \( \mathbb{D}_0^{p,\gamma} \) and \( \mathbb{H} \mathbb{D}^{p,\gamma} \).

Consider the kernel function

\[
    k_{\beta,\gamma}(x, y) = x_n^{1-\beta}y_n^{-\gamma/p}|x - y|^{1-n}|\bar{x} - y|^{-1} 
\]

for \( x \) and \( y \) in \( D \). To evaluate the size of exceptional sets, we use the capacity

\[
    C_{k_{\beta,\gamma}}(E; G) = \inf \int_{D} g(y)^p dy, 
\]

where \( E \) is a subset of an open set \( G \) in \( D \) and the infimum is taken over all nonnegative measurable functions \( g \) such that \( g = 0 \) outside \( G \) and

\[
    \int_{D} k_{\beta,\gamma}(x, y)g(y)dy \geq 1 \quad \text{for all } x \in E. 
\]

We say that \( E \subset D \) is \((k_{\beta,\gamma}, p)\)-thin at infinity if

\[
    \sum_{i=1}^{\infty} 2^{-i(n+\gamma-(1-\beta)p)}C_{k_{\beta,\gamma}}(E_i; D_i) < \infty. \tag{4}
\]

**THEOREM 3.** Let \( p > 1, -1 < \gamma < p - 1 \) and \( 0 \leq \beta \leq 1 \). If \( u \in \mathbb{D}_0^{p,\gamma} \), then there exists a set \( E \subset D \) such that \( E \) is \((k_{\beta,\gamma}, p)\)-thin at infinity and

\[
    \lim_{|x| \to \infty, x \notin D - E} x_n^{-\beta}|x|^{(n+\gamma-(1-\beta)p)/p}u(x) = 0. 
\]

**THEOREM 4.** Let \( p > 1, -1 < \gamma < p - 1 \) and \( n + \gamma - p \geq 0 \). If \( h \in \mathbb{H} \mathbb{D}^{p,\gamma} \), then there exist a number \( A \) such that

\[
    \lim_{|x| \to \infty, x \in D} x_n^{(n+\gamma-p)/p}[h(x) - A] = 0. 
\]
in case \( n + \gamma - p > 0 \) and
\[
\lim_{|x| \to \infty, x \in D} \left( \max \{ \log(1/x_n), \log |x| \} \right)^{-1/p'} [h(x) - A] = 0
\]
in case \( n + \gamma - p = 0 \).

**Remark 1.** Let \( p > 1 \), \(-1 < \gamma < p - 1 \) and \( n + \gamma - p > 0 \). Then we can find a function \( h \in \text{HD}^{p, \gamma} \) such that
\[
\limsup_{|x| \to \infty, x \in D} |x|^{(n+\gamma-p)/p} h(x) = \infty
\]
and
\[
\lim_{|x| \to \infty, x \in D} x_n^{(n+\gamma-p)/p} h(x) = 0.
\]

For proofs of these theorems, we refer to [14].

## 4 Examples of thin sets at infinity

We are concerned with the measure condition on sets which are thin at infinity.

For a measurable set \( E \subset \mathbb{R}^n \), denote by \( |E| \) the Lebesgue measure of \( E \). Then we can prove that
\[
|E|^{(1-(1-\beta)/n)p} \leq qMC_{k_{\beta, \gamma}, p}(E; D_0)
\]
and
\[
C_{k_{\beta, \gamma}, p}(rE; rD_0) = r^{n+\gamma-(1-\beta)p}C_{k_{\beta, \gamma}, p}(E; D_0)
\]
whenever \( E \subset D \cap B(0, 2) - B(0, 1) \) and \( r > 0 \). Hence we have the following result.

**Proposition 1.** Let \( 0 \leq \beta \leq 1 \) and \(-1 < \gamma < p - 1 \). If (4) holds, then
\[
\sum_{i=1}^{\infty} \left( \frac{|E_i|}{|B_i|} \right)^{(1-(1-\beta)/n)p} < \infty,
\]
where \( E_i = E \cap B_{i+1} - B_i \) with \( B_i = B(0, 2^i) \cap D \).

If \( E \) is well situated, then we have stronger results as in the following.

**Proposition 2.** Let \( 0 \leq \beta \leq 1 \) and \(-1 < \gamma < p - 1 \). Set \( F = \bigcup_{j=1}^{\infty} B_j \), where \( B_j = B(x_j, s_j) \) with \( 2^j \leq |x_j| < 2^{j+1} \) and \( r_j = (x_j)_n > 2s_j \). If \( p < n \) and \( F \) is \((k_{\beta, \gamma}, p)\)-thin at infinity, then
\[
\sum_{j=1}^{\infty} \left( \frac{s_j}{2^j} \right)^{-p} \left( \frac{r_j}{2^j} \right)^{\beta p + \gamma} < \infty; \tag{7}
\]
conversely, if (7) holds, then $F$ is $(k_{\beta,\gamma}, p)$-thin at infinity.

**Proof.** First we show that if $p < n$, then

$$s^{n-p}r^{\beta p+\gamma} \leq MC_{k_{\beta,\gamma}, p}(B; D_0)$$  \hfill (8)

for $B = B(x_0, s)$ with $1 \leq |x_0| < 2$ and $r = (x_0)_n > 2s$. Let $g$ be a nonnegative measurable function such that $g = 0$ outside $D_0$ and

$$\int_{D} k_{\beta,\gamma}(x, y) g(y) \, dy \geq 1$$

for every $x \in B$. Then we have by Fubini’s theorem

$$|B| \leq \int_{B} \left( \int_{D_0} k_{\beta,\gamma}(x, y) g(y) \, dy \right) \, dx$$

$$= \int_{D_0} g(y) y_n^{-\gamma/p} \left( \int_{B} x_n^{1-\beta} |x - y|^{1-n} |\bar{x} - y|^{-1} \, dx \right) \, dy$$

$$\leq Mr^{1-\beta} \int_{D_0} g(y) y_n^{-\gamma/p} \left( \int_{B} |x - y|^{1-n} |\bar{x} - y|^{-1} \, dx \right) \, dy.$$

We set

$$I(y) = \int_{B} |x - y|^{1-n} |\bar{x} - y|^{-1} \, dx$$

and

$$J = \int_{D_0} g(y) y_n^{-\gamma/p} \left( \int_{B} |x - y|^{1-n} |\bar{x} - y|^{-1} \, dx \right) \, dy.$$

If $|y - x_0| < 3s/2$, then

$$I(y) \leq r^{-1} \int_{B} |x - y|^{1-n} \, dx \leq Mr^{-1}s,$$

so that we have by Hölder’s inequality

$$J_1 = \int_{\{y \in D_0: |y - x_0| < 3s/2\}} g(y) y_n^{-\gamma/p} I(y) \, dy$$

$$\leq Mr^{-1}s \int_{\{y \in D_0: |y - x_0| < 3s/2\}} g(y) y_n^{-\gamma/p} \, dy$$

$$\leq Mr^{-1}s \left( \int_{\{y \in D_0: |y - x_0| < 3s/2\}} y_n^{-\gamma p'/p} \, dy \right)^{1/p'} \left( \int_{D_0} g(y)^p \, dy \right)^{1/p}$$

$$\leq Ms^{n-\gamma/p} s^{1-n/p} \left( \int_{D_0} g(y)^p \, dy \right)^{1/p}.$$
If $|y - x_0| \geq 3s/2$ and $y_n \leq x_n/2$, then $|x - y| \geq M(|x' - y| + x_n) \geq M(|x'_0 - y| + r)$, so that

$$I_2(y) = \int_{\{x \in B: y_n \leq x_n/2\}} |x - y|^{1-n} |\bar{x} - y|^{-1} dx$$

$$\leq M(|x'_0 - y| + r)^{-n}s^n.$$

Hence we have by Hölder’s inequality

$$J_2 = \int_{\{y \in D_0: |x_0 - y| \geq 3s/2\}} g(y) y_n^{-\gamma/p} I_2(y) dy$$

$$\leq Ms^n \int_{D_0} g(y) y_n^{-\gamma/p} (|x'_0 - y| + r)^{-n} dy$$

$$\leq Ms^n \left( \int_{D_0} y_n^{-\gamma p'/p} (|x'_0 - y| + r)^{-p'n} dy \right)^{1/p'} \left( \int_{D_0} g(y)^p dy \right)^{1/p}$$

$$\leq Ms^n s^{1-n/p} r^{-1-n/p} \left( \int_{D_0} g(y)^p dy \right)^{1/p},$$

since $p < n$. If $|y - x_0| \geq 3s/2$ and $y_n > x_n/2$, then $|x - y| \geq M(|x_0 - y| + s)$ and $|\bar{x} - y| \geq M(|x_0 - y| + r)$, so that

$$I_3(y) = \int_{\{x \in B: y_n > x_n/2\}} |x - y|^{1-n} |\bar{x} - y|^{-1} dx$$

$$\leq M(|x_0 - y| + s)^{1-n} (|x_0 - y| + r)^{-1}s^n.$$

Consequently, it follows that

$$J_3 = \int_{\{y \in D_0: |x_0 - y| \geq 3s/2, y_n > r/4\}} g(y) y_n^{-\gamma/p} I_3(y) dy$$

$$\leq Ms^n \int_{\{y \in D_0: |y - x_0| \geq 3s/2, y_n > r/4\}} g(y) y_n^{-\gamma/p} (|x_0 - y| + s)^{1-n} (|x_0 - y| + r)^{-1} dy.$$

Setting $t = |x_0 - y|$ and $|(x_0)_n - y_n| = t \cos \theta$, we note that

$$(t + r) \cos \theta \leq |(x_0)_n - y_n| + (x_0)_n \leq 3y_n < 3(r + t)$$

when $y_n > r/4$. Using Hölder’s inequality and applying the polar coordinates about $x_0$, we have

$$J_3 \leq Ms^n \left( \int_{3s/2}^{\infty} (t + s)^{p'(1-n)}(t + r)^{p'(-\gamma/p-1)} t^{n-1} dt \right)^{1/p'} \left( \int_{D_0} g(y)^p dy \right)^{1/p}$$

$$\leq Ms^n s^{1-n/p} r^{-1-n/p} \left( \int_{D_0} g(y)^p dy \right)^{1/p}.$$
since $p < n$. Therefore we obtain

$$|B| \leqq Mr^{-\beta-\gamma/p}s^{(p-n)/p}s^{n}(\int_{D_0} g(y)^p dy)^{1/p}.$$ 

Hence it follows from the definition of $C_{k_{\beta,\gamma},p}$ that

$$r^{\beta p+\gamma s^{n-p}} \leqq MC_{k_{\beta,\gamma},p}(B; D_0),$$

as required.

To obtain the converse inequality, note that for $x \in B$

$$\int_{B}x_{n}^{1-\beta}|x-y|^{1-n}|\overline{x}-y|^{-1}y_{n}^{-\gamma/p}dy \geqq Mr^{-\beta-\gamma/p} \int_{B}|x-y|^{1-n}dy \geqq Mr^{-\beta-\gamma/p}s,$$

so that

$$C_{k_{\beta,\gamma},p}(B; D_0) \leqq Mr^{(\beta+\gamma/p)s^{n-p}} \int_{B}dy = Mr^{\beta p+\gamma s^{n-p}}.$$ 

Thus the proof is completed.

**Proposition 3.** Let $0 \leqq \beta \leqq 1$ and $-1 < \gamma < p-1$. Set $V = \bigcup_{j=1}^{\infty}B(x_{j}, r_{j}) \cap D$ with $x_{j} \in \partial D$, $2^{j} \leqq |x_{j}| < 2^{j+1}$ and $0 < r_{j} \leqq 2^{j+1}$. If $V$ is $(k_{\beta,\gamma},p)$-thin at infinity, then

$$\sum_{j=1}^{\infty} \left( \frac{r_{j}}{2^{j}} \right)^{n+\gamma-(1-\beta)p} < \infty; \quad (9)$$

conversely, if $\gamma > (1-\beta)p$ and (9) holds, then $V$ is $(k_{\beta,\gamma},p)$-thin at infinity.

**Proof.** First we show that if $B_{+} = B(x_{0}, r) \cap D$ with $x_{0} \in \partial D$, $1 \leqq |x_{0}| < 2$ and $0 < r \leqq 2$, then

$$r^{n+\gamma-(1-\beta)p} \leqq MC_{k_{\beta,\gamma},p}(B_{+}; D_0). \quad (10)$$

Let $g$ be a nonnegative measurable function such that $g = 0$ outside $D_0$ and

$$\int_{D} k_{\beta,\gamma}(x, y)g(y) dy \geqq 1$$

for every $x \in B_{+}$. Then we have by Fubini's theorem

$$|B_{+}| \leqq \int_{B_{+}} \left( \int_{D_0} k_{\beta,\gamma}(x, y)g(y)dy \right) dx$$

$$= \int_{D_0} g(y)y_{n}^{-\gamma/p} \left( \int_{B_{+}} x_{n}^{1-\beta}|x-y|^{1-n}|\overline{x}-y|^{-1}dx \right) dy.$$
Here we see that if $|x_0 - y| > 2r$, then
\[ \int_{B_+} x_n^{1-\beta}|x-y|^{1-n}|\overline{x}-y|^{-1}dx \leq M|x_0 - y|^{-n}r^{1-\beta+n} \]
and that if $|x_0 - y| \leq 2r$, then
\[ \int_{B_+} x_n^{1-\beta}|x-y|^{1-n}|\overline{x}-y|^{-1}dx \leq Mr^{1-\beta} \int_{B_+} |x-y|^{1-n}(|x-y| + y_n)^{-1}dx \leq M r^{1-\beta} \log(4r/y_n). \]

Then we have by Hölder's inequality
\[
J_1 = r^{1-\beta} \int_{\{y \in D_0 : |x_0 - y| \leq 2r\}} g(y) y_n^{-\gamma/p} \log(4r/y_n) dy \\
\leq r^{1-\beta} \left( \int_{\{y \in D_0 : |x_0 - y| \leq 2r\}} \{ \log(4r/y_n) \}^{p'} y_n^{-\gamma p'/p} dy \right)^{1/p'} \left( \int_{D_0} g(y)^p dy \right)^{1/p} \\
\leq Mr^{1-\beta - \gamma/p + n/p'} \left( \int_{D_0} g(y)^p dy \right)^{1/p}
\]
and
\[
J_2 = r^{1-\beta + n} \int_{\{y \in D_0 : |x_0 - y| > 2r\}} g(y) y_n^{-\gamma/p} |x_0 - y|^{-n} dy \\
\leq r^{1-\beta + n} \left( \int_{\{y \in D_0 : |x_0 - y| > 2r\}} y_n^{-\gamma p'/p} |x_0 - y|^{-p'n} dy \right)^{1/p'} \left( \int_{D_0} g(y)^p dy \right)^{1/p} \\
\leq Mr^{1-\beta - \gamma/p + n/p'} \left( \int_{D_0} g(y)^p dy \right)^{1/p}.
\]

Therefore we have
\[ |B_+| \leq Mr^{1-\beta - \gamma/p + n/p'} \left( \int_{D_0} g(y)^p dy \right)^{1/p}, \]
so that it follows from the definition of $C_{k_{\beta,\gamma},p}$ that
\[ r^{n+\gamma-(1-\beta)p} \leq M C_{k_{\beta,\gamma},p}(B_+;D_0), \]
as required.

To obtain the converse inequality, note that for $x \in B_+$
\[ \int_{B_+} x_n^{1-\beta}|x-y|^{1-n}|\overline{x}-y|^{-1}y_n^{-\gamma/p} dy \]
\[ \geq \int_{B_+ \cap B(x,x_n/2)} x_n^{1-\beta}|x-y|^{1-n}|\overline{x}-y|^{-1}y_n^{-\gamma/p} dy \]
\[ \geq M x_n^{1-\beta-1-\gamma/p} \int_{B_+ \cap B(x,x_n/2)} |x-y|^{-1} dy \]
\[ \geq M x_n^{1-\beta-\gamma/p} \geq Mr^{1-\beta-\gamma/p}, \]
since $1-\beta < \gamma/p$. Hence it follows from the definition of $C_{k_{\beta,\gamma,p}}$ that

$$C_{k_{\beta,\gamma,p}}(B_+; D_0) \leq M r^{-(1-\beta)p+\gamma} \int_{B_+} dy = M r^{n+\gamma-(1-\beta)p}.$$  

Thus the proof is completed.

For a nondecreasing function $\varphi$ on $\mathbb{R}^1$ such that $0 < \varphi(2t) \leq M \varphi(t)$ for $t > 0$ with a positive constant $M$, we set

$$T_{\varphi} = \{x = (x', x_n); 0 < x_n < \varphi(|x'|)\}.$$  

PROPOSITION 4 (cf. Aikawa [1, Proposition 5.1]). Let $0 < \beta \leq 1$ and $p(1-\beta)-1 < \gamma < p-1$. Assume further that

$$\lim_{r \to \infty} \frac{\varphi(r)}{r} = 0.$$  

Then $T_{\varphi}$ is $(k_{\beta,\gamma,p})$-thin at infinity if and only if

$$\int_{1}^{\infty} \left(\frac{\varphi(t)}{t}\right)^{p(-1+\beta)+\gamma+1} \frac{dt}{t} < \infty.$$  

For example, $\varphi(r) = r[\log(1+r)]^{-\delta}$ satisfies (12), when $\delta\{p(-1+\beta)+\gamma+1\} > 1$.

5 Limits of monotone functions

Finally we consider the limits at infinity for monotone BLD functions. A continuous function $u$ is called monotone on $D$ in the sense of Lebesgue (see [5]) if for every relatively compact open subset $G$ of $D$,

$$\max_{G \cup \partial G} u = \max_{\partial G} u \quad \text{and} \quad \min_{G \cup \partial G} u = \min_{\partial G} u.$$  

For examples and fundamental properties of monotone functions, see [12] and [16]. Among them the following result is only needed for monotone functions.

LEMMA 1. If $u$ is a monotone BLD function on $B(x, 2r)$ and $p > n-1$, then

$$|u(z) - u(x)|^p \leq M r^{p-n} \int_{B(x, 2r)} |\nabla u(y)|^p dy \quad (13)$$

for every $z \in B(x, r)$. 
THEOREM 5. Let \( p > n - 1, -1 < \gamma < p - 1 \) and \( n + \gamma - p \geq 0 \). If \( u \) is a monotone function on \( D \) satisfying (1), then there exist a number \( A \) such that

\[
\lim_{|x| \to \infty, x \in D} x_n^{(n+\gamma-p)/p}[u(x) - A] = 0
\]

in case \( n + \gamma - p > 0 \) and

\[
\lim_{|x| \to \infty, x \in D} (\max \{\log(1/x_n), \log |x|\})^{-1/p'} [u(x) - A] = 0
\]

in case \( n + \gamma - p = 0 \).

PROOF. For \( x \in D \), let \( r = |x|, C(x) = (0, ..., 0, r) \) and \( \rho_D(x) \) denote the distance of \( x \in D \) from the boundary \( \partial D \), that is, \( \rho_D(x) = x_n \). We take a finite covering \( \{B_j\} \), \( B_j = B(X_j, 4^{-1}\rho_D(X_j)) \), such that

(i) \( X_1 = x \) and \( X_{N+1} = C(x) \);

(ii) \( r/2 < |z| < 2r \) for \( z \in A(r) = \bigcup_j 2B_j \), where \( 2B_j = B(X_j, 2^{-1}\rho_D(X_j)) \);

(iii) \( B_j \cap B_{j+1} \neq \emptyset \) for each \( j \);

(iv) \( \sum_j \chi_{2B_j} \) is bounded, where \( \chi_A \) denotes the characteristic function of \( A \).

By the monotonicity of \( u \), we see that

\[
|u(y) - u(X_j)| \leq M \rho_D(X_j)^{(p-n)/p} \int_{2B_j} |\nabla u(z)|^p \rho_D(z)\gamma dz
\]

for \( y \in B_j \). First suppose \( n + \gamma - p > 0 \). Using Theorem 1, we can find a number \( A \) and \( C_1(x) \) such that \( C_1(x) \in B_{N+1} \) and

\[
\lim_{|x| \to \infty} |x|^{(n+\gamma-p)/p}[u(C_1(x)) - A] = 0.
\]

Then we have by Hölder's inequality

\[
|u(x) - A| \leq |u(X_1) - u(X_2)| + |u(X_2) - u(X_3)| + \cdots + |u(X_N) - u(X_{N+1})| + |u(X_{N+1}) - u(C_1(x))| + |u(C_1(x)) - A|
\]

\[
\leq M \sum_j \rho_D(X_j)^{(p-n-\gamma)/p} \left( \int_{2B_j} |\nabla u(z)|^p \rho_D(z)\gamma dz \right)^{1/p} + |u(C_1(x)) - A|
\]

\[
\leq M \left( \sum_j \rho_D(X_j)^{(p-n-\gamma)/p} \right)^{1/p'} \left( \int_{A(r)} |\nabla u(z)|^p \rho_D(z)\gamma dz \right)^{1/p} + |u(C_1(x)) - A|
\]

\[
\leq M x_n^{(p-n-\gamma)/p} \left( \int_{D - B(0, r/2)} |\nabla u(z)|^p \rho_D(z)\gamma dz \right)^{1/p} + |u(C_1(x)) - A|
\]
which proves
\[
\lim_{|x| \to \infty} x_n^{(n+\gamma-p)/p} [u(x) - A] = 0,
\]
as required.

The case \(n + \gamma - p = 0\) can be treated similarly.

**References**


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