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Mean value property for temperatures on an annulus domain

§1. Introduction

Heat balls in $\mathbb{R}^{n+1}$ are characterized by some mean value identity for temperatures (solutions of the heat equation) in [3]. In this paper we give similar theorem for a heat annulus. The corresponding result for harmonic functions is given in [1] (see also [2]).

For a point in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, we write

$$P = (x, t) = (x_1, \ldots, x_n, t).$$

We use $W = W_n$ to denote the Gauss-Weierstrass kernel, defined by

$$W_n(x, t) := \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

where $|x| := (x_1^2 + \cdots + x_n^2)^{1/2}$. The heat ball $\Omega(c)$ centered at the origin and radius $c > 0$ is defined by a level surface of $W_n$, that is,

$$\Omega(c) := \{(x, t) \in \mathbb{R}^{n+1} : W_n(x, -t) > (4\pi c)^{-n/2}\}.$$

Clearly $\Omega(c) \subset \{|x|^2 < 2nc/e, -c < t < 0\}$. We consider the following mean values $M(u, c)$ over the heat sphere $\partial \Omega(c)$ and $V_\alpha(u, c)$ over the heat ball $\Omega(c)$:

$$M(u, c) := \frac{1}{(4\pi c)^{n/2}} \int_{\partial \Omega(c)} Q(x, t)u(x, t)d\sigma(x, t)$$

where $Q(x, t) = |x|^2\left(4|x|^2t^2 + (|x|^2 + 2nt)^2\right)^{-1/2}(t < 0)$, $Q(0,0) = 1$, and

$$V_\alpha(u, c) := \alpha c^{-\alpha} \int_0^c r^{\alpha-1}M(u, r)dr,$$

for $\alpha > 0$. Then,

$$V_\alpha(u, c) = \frac{\alpha}{2^{n+1}\pi n/2c^{n/2}} \int_{\Omega(c)} K_\alpha(x, t)u(x, t)dxdt,$$

where

$$K_\alpha(x, t) := \frac{|x|^2}{(-t)^{(n+4-2\alpha)/2}} \exp\left(\frac{(2\alpha - n)|x|^2}{4n(-t)}\right).$$
For $0 < c_1 < c_2$, we put
\[ A(c_1, c_2) := \Omega(c_2) \setminus \overline{\Omega(c_1)} \]
and call $A(c_1, c_2)$ a heat annulus.

We have the following mean value property for temperatures.

**Theorem 1.** (I) Let $\alpha > 0$ and $c > 0$. If $u$ is a temperature in $\Omega(c)$ and continuous on its closure $\overline{\Omega(c)}$, then $u(0,0) = V_\alpha(u, c)$, that is
\[ u(0,0) = \frac{\alpha}{2^{n+1}n\pi^{n/2}c^\alpha} \int \int_{\Omega(c)} K_\alpha(x,t)u(x,t)dxdt. \]

(II) Let $\alpha > 0$ and $0 < c_1 < c_2$. If $u$ is a temperature in $A(c_1, c_2)$ and continuous on its closure $\overline{A(c_1,c_2)}$, then
\[ M(u, c) = \frac{\alpha}{n2^{n+1}\pi^{n/2}(c_2^\alpha - c_1^\alpha)} \int \int_{A(c_1,c_2)} K_\alpha(x,t)u(x,t)dxdt, \]
where $c$ is a constant defined by
\[ c^{-n/2} := \begin{cases} 
\frac{\alpha(c_2^{\alpha-n/2} - c_1^{\alpha-n/2})}{(\alpha - n/2)(c_2^\alpha - c_1^\alpha)} & \text{if } \alpha \neq n/2 \\
\frac{n\log(c_2/c_1)}{2(c_2^{n/2} - c_1^{n/2})} & \text{if } \alpha = n/2.
\end{cases} \]

The following converse assertions of Theorem 1 are our main results in this paper.

**Theorem 2.** (I) Let $\alpha > 0$, $c > 0$ and let $D$ be a bounded open set in $\mathbb{R}^{n+1}$. If the following conditions are satisfies, then $D = \Omega(c)$:
1. $(\chi_D - \chi_{\Omega(c)})K_\alpha \in L^p(\mathbb{R}^{n+1})$ for some $p > n/2 + 1$.
2. For all $(y, s) \in \mathbb{R}^{n+1} \setminus D$,
\[ \frac{\alpha}{2^{n+1}n\pi^{n/2}c^\alpha} \int \int_D W(x-y,t-s)K_\alpha(x,t)dxdt = W(y,-s). \]

(II) Let $\alpha > 0$, $0 < c_1 < c_2$ and let $D$ be a bounded open set in $\mathbb{R}^{n+1}$. Put $c$ as in (1.4). If the following conditions are satisfies, then $D = A(c_1, c_2)$:
1. $D$ contains $\partial\Omega(c) \setminus \{(0,0)\}$.
2. $(\chi_D - \chi_{A(c_1,c_2)})K_\alpha \in L^p(\mathbb{R}^{n+1})$ for some $p > n/2 + 1$.
3. For all $(y, s) \in \mathbb{R}^{n+1} \setminus D$,
\[ \frac{\alpha}{n2^{n+1}\pi^{n/2}(c_2^\alpha - c_1^\alpha)} \int \int_D W(x-y,t-s)K_\alpha(x,t)dxdt = M(W(\cdot-y,\cdot-s),c). \]
4. $\inf\{s ; (y, s) \in \Omega(c_1) \cap D^c\} = c_1$. 

§2. Proof of Theorems

Proof of Theorem 1. (I) Since $u(0, 0) = M(u, r)$ for $0 < r < c$ (see [4]), we have

$$V_{\alpha}(u, c) = \alpha c^{-\alpha} \int_{0}^{c} r^{\alpha-1} u(0, 0) dr = u(0, 0).$$

To show (II), we first remark that

$$M(u, r) = pr^{-n/2} + q \quad (c_1 < r < c_2)$$

with some constants $p, q$ (see [5]). Hence if $\alpha \neq n/2$, we have

$$\int_{c_1}^{c_2} r^{\alpha-1} M(u, r) dr = p \cdot \frac{\alpha (c_2^{\alpha-n/2} - c_1^{\alpha-n/2})}{(\alpha - n/2) (c_2^\alpha - c_1^\alpha)} + q = pc^{-n/2} + q = M(u, c).$$

On the other hand, by (1.1)

$$\int_{c_1}^{c_2} r^{\alpha-1} M(h, r) dr = \frac{1}{n2^{n+1} \pi^{n/2}} \int \int_{A(c_2, c_1)} K_{\alpha}(x, t) u(x, t) dx dt.$$ 

These equalities give (1.3). The case $\alpha = n/2$ is shown in a similar way.

Proof of Theorem 2. (I) In [3], we gave a proof for the case $\alpha = n/2$. Although its proof is valid for general $\alpha > 0$, we will repeat it for the sake of completeness.

Put $\beta := 2^{n+1} n \pi^{n/2} c^\alpha / \alpha$. By the volume mean value property of temperatures in [4], we have

$$\int \int_{\Omega(c)} K_{\alpha}(x, t) dx dt = \beta$$

and for every $(y, s) \in \mathbb{R}^{n+1} \setminus \Omega(c)$,

$$\int \int_{\Omega(c)} W(x - y, t - s) K_{\alpha}(x, t) dx dt = \beta W(y, -s).$$

Since $D$ is bounded, there is $s < 0$ such that $(y, s) \not\in \Omega(c)$ for all $y \in \mathbb{R}^n$, so that (1.5) gives

$$\beta = \beta \int_{\mathbb{R}^n} W(y, -s) dy = \int_{\mathbb{R}^n} \left( \int_{D} W(x - y, t - s) K_{\alpha}(x, t) dx dt \right) dy = \int \int_{D} K_{\alpha}(x, t) dx dt$$

and hence

$$\int \int_{\Omega(c)} K_{\alpha}(x, t) dx dt = \int \int_{D} K_{\alpha}(x, t) dx dt.$$
Now for every \((y, s) \in \mathbb{R}^{n+1}\), we put
\[
v(y, s) := \int \int_{D} W(x - y, t - s) K_{\alpha}(x, t) dx dt
\]
\[
v_{0}(y, s) := \int \int_{\Omega(c)} W(x - y, t - s) K_{\alpha}(x, t) dx dt
\]
\[
u(y, s) := \beta W(y, -s) - v(y, s)
\]
\[
u_{0}(y, s) := \beta W(y, -s) - v_{0}(y, s)
\]
Then (1.5) implies that
\[
(2.4) \quad u(y, s) = 0, \quad \forall (y, s) \in \mathbb{R}^{n+1} \setminus D
\]
and (2.2) implies
\[
(2.5) \quad u_{0}(y, s) = 0, \quad \forall (y, s) \in \mathbb{R}^{n+1} \setminus \Omega(c)
\]
Further, for \(r > 0\) we see
\[
(2.6) \quad M(W(\cdot - y, \cdot - s), r) = W(y, -s) \wedge (4\pi r)^{-n/2} = \begin{cases} W(y, -s) & \text{if } (y, s) \not\in \Omega(r) \\ (4\pi r)^{-n/2} & \text{if } (y, s) \in \Omega(r) \end{cases}
\]
(see [5]), and hence
\[
(2.7) \quad u_{0}(y, s) > 0, \quad \forall (y, s) \in \Omega(c).
\]
We assert that \(v - v_{0} \in C(\mathbb{R}^{n+1})\). Put \(f := (\chi_{D} - \chi_{\Omega(c)}) K_{\alpha}\). Take \(s = 0\) in (1.5), we see \(\text{supp}(f) \subset \mathbb{R}^{n+1} \times (-\infty, 0]\). For each \(a \leq 0\), let \(f_{a}\) denote the restriction of \(f\) to \(\mathbb{R}^{n} \times (-\infty, a]\), and let \(F_{a} := \text{supp}(f) \cap (\mathbb{R}^{n} \times [a, 0])\). If \(a < 0\) then \(f_{a}\) is bounded, so that the function
\[
(y, s) \mapsto \int \int_{\mathbb{R}^{n+1}} W(x - y, t - s) f_{a}(x, t) dx dt
\]
is continuous. Since
\[
H^{*} \left( \int \int_{F_{a}} W(x - y, t - s) f(x, t) dx dt \right) = 0 \quad \forall (y, s) \in \mathbb{R}^{n+1} \setminus F_{a}
\]
it follows that \(v - v_{0} \in C(\mathbb{R}^{n+1} \setminus F_{a})\). Since \(a\) is arbitrary, \(v - v_{0} \in C(\mathbb{R}^{n+1} \setminus F_{0})\). Finally, if \(q := p/(p-1)\), the exponent conjugate to \(p\), then \(q < (n+2)/n\) and for some constant \(M\) we have
\[
|v - v_{0}|(y, s) \leq M |s|^{(n+2-nq)/2q} \|f\|_{p},
\]
so that condition (1) in (I) implies that \((v - v_{0})(y, s) \to 0\) as \(s \to 0\).
To prove that \(D = \Omega(c)\), it is sufficient to show that \(\chi_{D} = \chi_{\Omega(c)}\) a.e. on \(\mathbb{R}^{n+1}\). For then \(u = u_{0}\), so that (2.4) and (2.7) imply that \(\Omega(c) \subset D\). Therefore \(\Omega(c) = D \setminus F\) for
some relatively closed subset $F$ of $D$ with measure zero. Since $\overline{\Omega(c)}^o = \Omega(c)$, it follows that $D = \Omega(c)$.

Suppose that $\chi_D \neq \chi_{\Omega(c)}$ on a set of positive measure. Since $\chi_{\overline{\Omega(c)}} = \chi_{\Omega(c)}$ a.e., we can choose $P_0 \in D \setminus \overline{\Omega(c)}$, in view of (2.3). If $L$ is any line through $P_0$, we can choose $Q_1, Q_2 \in L \cap \partial D$ such that $P_0$ belongs to the segment $Q_1Q_2$. If $Q_1$ and $Q_2$ both belonged to $\overline{\Omega(c)}$, then by convexity $P_0$ would also belong to $\Omega(c)$, which is false. Therefore $\partial D \setminus \overline{\Omega(c)} \neq \emptyset$. Moreover, $\partial D \setminus \overline{\Omega(c)}$ contains a point $(y_0, s_0)$ with the property that every ball centred there meets $D_+ = D \cap (\mathbb{R}^n \times (s_0, \infty))$. For otherwise $\partial D \setminus \overline{\Omega(c)}$ would be contained in the union of a sequence of parallel hyperplanes, and so $D$ would be unbounded. Choose a ball $B$, centred at $(y_0, s_0)$, such that $B \cap \overline{\Omega(c)} = \emptyset$. The function $u$ is an $H^*$-subtemperature on $B$, is not an $H^*$-temperature on $B \cap D$, and is zero at $(y_0, s_0)$ by (2.4). Since $B \cap D_+ \neq \emptyset$, the maximum principle therefore implies that

$$\sup_B u > 0.$$ 

Put

$$m = \max_{\mathbb{R}^{n+1}} (u - u_0) \quad \text{and} \quad E = (u - u_0)^{-1}(m).$$

Since $B \cap \overline{\Omega(c)} = \emptyset$, we have $u_0 = 0$ on $B$. Therefore $\sup_B (u - u_0) > 0$, and hence $m > 0$. Because $u_0 \geq 0$ by (2.5) and (2.7), we have $u > 0$ on $E$, and hence $E \subseteq D$ by (2.4). On the other hand, for all $(x, t) \in D$ we have

$$H^*(u - u_0)(x, t) = (1 - \chi_{\Omega(c)}(x, t)) \frac{\|x\|^2}{t^2} \geq 0,$$

so that $u - u_0$ is an $H^*$-subtemperature on $D$. The maximum principle now implies that $E \cap \partial D \neq \emptyset$, a contradiction. Hence $D = \Omega(c)$.

To show (II), we first remark that

(2.8) \quad $\chi_{\Omega(r)} K_\alpha \not\in L^{n/2+1}(\mathbb{R}^{n+1})$, \quad ($\forall r > 0$).

Now applying $u \equiv 1$ to (1.3), we have

$$\frac{\alpha}{n2^{n+1} \pi^{n/2} (c_2^\alpha - c_1^\alpha)} \int \int_{A(c_1, c_2)} K_\alpha(x, t) dx dt = 1.$$ 

Furthermore, by the usual limiting argument, (1.3) gives

(2.8) \quad $M(W(-y, -s, c) = \frac{\alpha}{n2^{n+1} \pi^{n/2} (c_2^\alpha - c_1^\alpha)} \int \int_{A(c_1, c_2)} W(x-y, t-s) K_\alpha(x, t) dx dt$

for all $(y, s) \in \mathbb{R}^{n+1} \setminus A(c_1, c_2)$. As in (2.2), we have

(2.9) \quad $\int \int_D K_\alpha(x, t) dx dt = \int \int_{A(c_1, c_2)} K_\alpha(x, t) dx dt$, 


and as in the proof of (I), we put
\[
v(y, s) := \frac{\alpha}{n2^{n+1}\pi^{n/2}(c_{2}^\alpha - c_{1}^\alpha)} \int_{D} W(x - y, t - s) K_{\alpha}(x, t) \, dx \, dt,
\]
\[
v_{0}(y, s) := \frac{\alpha}{n2^{n+1}\pi^{n/2}(c_{2}^\alpha - c_{1}^\alpha)} \int_{A(c_{1}, c_{2})} W(x - y, t - s) K_{\alpha}(x, t) \, dx \, dt,
\]
\[
u(y, s) := M(W(\cdot - y, \cdot - s), c) - v(y, s),
\]
\[
u_{0}(y, s) := M(W(\cdot - y, \cdot - s), c) - v_{0}(y, s).
\]
for every \((y, s) \in \mathbb{R}^{n+1}\). Then \(u - u_{0} \in C(\mathbb{R}^{n+1})\) as in (I). Also, (1.6) implies
\[
(2.10) \quad u(y, s) = 0, \quad \forall (y, s) \in \mathbb{R}^{n+1} \setminus D
\]
and by (2.6) and (2.1) we have
\[
(2.11) \quad \begin{cases} 
u_{0}(y, s) = 0, & \text{if } (y, s) \notin A(c_{1}, c_{2}) \\ 
u_{0}(y, s) > 0, & \text{if } (y, s) \in A(c_{1}, c_{2}). \end{cases}
\]
If we assume \(D \setminus \overline{\Omega(c_{2})} \neq \emptyset\), then we have a contradiction as in the proof of (I). Hence \(D \subset \Omega(c_{2})\).

Next we pay attention to a set \(\partial D \cap \Omega(c_{1})\) and assume that this is empty. Then \(D \subset A(c_{1}, c_{2})\) or \(D \supset \Omega(c_{1})\). In the first case, we have \(\chi_{D} = \chi_{A(c_{1}, c_{2})}\) a.e. by (2.9) and hence \(D = A(c_{1}, c_{2})\) as in (I). The second case does not occur, because of (2.8). The rest of proof is to consider the case \(\partial D \cap \Omega(c_{1}) \neq \emptyset\). Choose a point \((y_{0}, s_{0}) \in \partial D \cap \Omega(c_{1})\) and take \(r > 0\) such that a usual ball \(B := B((y_{0}, s_{0}), r)\) is contained in \(\Omega(c_{1})\). If \(B \cap D \cap \{(y, s); s > s_{0}\} \neq \emptyset\), we have a contradiction by the maximum principle as in (I). On the other hand, if \(B \cap D \cap \{(y, s); s > s_{0}\} = \emptyset\) for all point \((y_{0}, s_{0}) \in \partial D \cap \Omega(c_{1})\), we see that \(D \cap \Omega(c_{1}) = \{(y, s); s < s_{0}\} \cap \Omega(c_{1})\). This contradicts our assumption (4) of (II).

References


