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ISOTROPY REPRESENTATION AND PROJECTION TO THE PRV-COMPONENT

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1. INTRODUCTION

The notion of *isotropy representation* is attributed to Vogan ([14], [15]), and it gives a refinement of the associated cycle attached to Harish-Chandra modules for real reductive groups. In fact, the multiplicities in the associated cycle are not just positive integers, but they can be interpreted as the dimension of the corresponding isotropy representations. An approach to the isotropy representation has been made in [18], [19] and [20], by means of the invariant differential operators of gradient-type on Riemannian symmetric spaces.

The purpose of this paper is to describe the isotropy representation $\mathcal{W}_\lambda$ attached to every singular unitary highest weight module $L(\lambda)$. To be precise, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a symmetric decomposition of a complex simple Lie algebra $\mathfrak{g}$. We assume that the reductive Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is *not* semisimple, or equivalently, $(\mathfrak{g}, \mathfrak{k})$ is assumed to be the complexification of an irreducible Hermitian symmetric pair $(\mathfrak{g}_0, \mathfrak{k}_0)$ (see [13]). We write $\tilde{K}_{\mathbb{C}}$ for the connected, and simply connected Lie group with Lie algebra $\mathfrak{k}$. Let $\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-$ be the triangular decomposition of $\mathfrak{g}$ which arises from the $\mathfrak{k}_0$-invariant complex structure on $\mathfrak{p}_0$. Then, $L(\lambda)$ in question is the irreducible, unitarizable $(\mathfrak{g}, \tilde{K}_{\mathbb{C}})$-module with highest weight λ, such that its associated variety $\mathcal{V}_\lambda$ is strictly contained in $\mathfrak{p}_+$. As is well-known, $\mathcal{V}_\lambda$ is the closure of a single nilpotent $\tilde{K}_{\mathbb{C}}$-orbit $\mathcal{O}$ (depending on λ) in $\mathfrak{p}_+$, which is an irreducible affine algebraic cone. We take an element X in $\mathcal{O}$. Then $\mathcal{W}_\lambda$ is a finite-dimensional representation of the isotropy subgroup $\tilde{K}_{\mathbb{C}}(X)$ of X in $\tilde{K}_{\mathbb{C}}$ (see (2.2) for the definition), and the associated cycle of $L(\lambda)$ turns to be $\dim \mathcal{W}_\lambda \cdot [\overline{\mathcal{O}}]$.

If $(\mathfrak{g}, \mathfrak{k})$ corresponds to the irreducible Hermitian symmetric space of type AIII or CI (resp. DIII), all (resp. almost all) unitary highest weight modules are obtained by decomposing tensor products of the oscillator (Segal-Shale-Weil) representation. In this oscillator setting, it has been already shown in [17], [18] (see also [12] for the multiplicity) that the assignment $\mathcal{W}_\lambda^* \leftrightarrow L(\lambda)$ essentially coincides with the Howe duality correspondence with respect to a reductive dual pair in the stable range, where the smaller member of the pair is compact. In this paper, we focus our attention on the remaining singular $L(\lambda)$'s which can not be realized by the Howe correspondence (non oscillator setting). By using the projection onto the PRV-component, the isotropy representations are explicitly determined for such highest weight modules. This together with our previous work [18] in the oscillator setting establishes the following theorem, which is the main result of this paper.

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Theorem 1.1. *The representation $\mathcal{W}_\lambda$ of $\tilde{K}_{\mathbb{C}}(X)$ is irreducible for every singular unitary highest weight module $L(\lambda)$, and it is explicitly described in Theorems 3.1 and 4.7 (infinitesimally in the non oscillator setting).*

Our results for DI and EVII give a clearer understanding of some multiplicity formulae obtained by Kato and Ochiai ([9], [8]). It should be mentioned that, for the case EVII, a representation isomorphic to our $\mathcal{W}_\lambda$ of B_4 (the semisimple part of the Lie algebra of $\tilde{K}_{\mathbb{C}}(X)$) in Theorem 4.7, appears in the work [3] of Dvorsky and Sahi on tensor products of singular unitary representations.

We organize this paper as follows. Section 2 shows how to characterize the dual $\mathcal{W}_\lambda^*$ of the isotropy representation for any (not necessarily singular) unitary highest weight module $L(\lambda)$, in terms of the projection to the PRV-component (Proposition 2.1). This PRV-projection gives the principal symbol of an invariant differential operator of gradient-type whose ($\tilde{K}_{\mathbb{C}}$-finite) kernel realizes the dual of $L(\lambda)$. By looking at the weights of the $\tilde{K}_{\mathbb{C}}$-modules in question, we can provide ourselves an effective method for constructing a nonzero $\tilde{K}_{\mathbb{C}}(X)$-submodule of $\mathcal{W}_\lambda^*$ (Proposition 2.4). In Section 3, we specify after [18] the $\tilde{K}_{\mathbb{C}}(X)$-module $\mathcal{W}_\lambda^*$ in the oscillator setting (Theorem 3.1). The last section, Section 4, is the principal part of this paper. In the non oscillator setting, the isotropy representation $\mathcal{W}_\lambda$ is explicitly determined for every singular unitary highest weight module $L(\lambda)$ of non scalar type (Theorem 4.7). Such $L(\lambda)$'s are listed in table (4.1) by virtue of Joseph's result [11, Section 7], and we can examine each $L(\lambda)$ in the list, separately.

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2. Isotropy representation for unitary highest weight module

2.1. **A characterization of $\mathcal{W}_\lambda^*$.** We fix once and for all a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$. The totality of compact (resp. noncompact) roots will be denoted by Δ_c (resp. by Δ_n). We choose a positive system Δ^+ of Δ so that $\mathfrak{p}_+$ equals the sum of root subspaces corresponding to noncompact positive roots. Set $\Delta_c^+ := \Delta_c \cap \Delta^+$ and $\Delta_n^+ := \Delta_n \cap \Delta^+$. Construct a maximal family $\beta_1, \ldots, \beta_r$ of strongly orthogonal noncompact positive roots as in [4, 1.4] (cf. [18, 2.1]), where $\beta := \beta_1$ is the highest root of Δ^+, and r is the rank of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$.

We set $\mu_j := \beta_1 + \cdots + \beta_j$ $(j = 1, \ldots, r)$. Then μ_j is Δ_c^+-dominant and integral, and a result of Schmid says that the symmetric algebras $S(\mathfrak{p}_\pm)$ of $\mathfrak{p}_\pm$ decompose as $\mathfrak{k}$-modules ($\tilde{K}_{\mathbb{C}}$-modules) in the following way:

$$S(\mathfrak{p}_\pm) \simeq \bigoplus_{\ell_1, \ldots, \ell_r \in \mathbb{Z}_{\geq 0}} V_{\pm(\ell_1\mu_1 + \cdots + \ell_r\mu_r)}. \tag{2.1}$$

Here, V_ν stands for the irreducible finite-dimensional $\mathfrak{k}$-module with extreme weight ν. Note that $V_{\pm\mu_j}$ occur in the homogeneous components $S^j(\mathfrak{p}_\pm)$ of degree j, respectively.

Let $L(\lambda)$ be the irreducible unitary $(\mathfrak{g}, \tilde{K}_{\mathbb{C}})$-module with Δ^+-highest weight $\lambda \in \mathfrak{t}^*$. Such $L(\lambda)$'s have been classified first by Enright, Howe and Wallach [5]. As in Section 1, let $\mathcal{V}_\lambda = \overline{\mathcal{O}} \subset \mathfrak{p}_+$ denote the associated variety of $L(\lambda)$. In this section, we do *not* assume that $L(\lambda)$ is singular. Fix an element X in the $\tilde{K}_{\mathbb{C}}$-orbit $\mathcal{O}$, and let $\mathfrak{m}(X)$ be the maximal ideal of $S(\mathfrak{p}_-)$ generated by the elements

$$Y - B(X, Y) \quad (Y \in \mathfrak{p}_-),$$

where B is the Killing form of $\mathfrak{g}$. Under the identification $S(\mathfrak{p}_-) = \mathbb{C}[\mathfrak{p}_+]$ ($\mathbb{C}[\mathfrak{p}_+]$ is the polynomial ring on $\mathfrak{p}_+$) through B, the maximal ideal $\mathfrak{m}(X) \subset \mathbb{C}[\mathfrak{p}_+]$ determines the one point variety $\{X\}$ in $\mathfrak{p}_+$. Then it follows from [18, Cor. 3.9] that the isotropy representation $\mathcal{W}_\lambda$ for $L(\lambda)$ turns to be

$$\mathcal{W}_\lambda = L(\lambda)/\mathfrak{m}(X)L(\lambda), \tag{2.2}$$

where the isotropy subgroup $\tilde{K}_{\mathbb{C}}(X)$ acts on the quotient space $L(\lambda)/\mathfrak{m}(X)L(\lambda)$ in the canonical way. Note that $\mathcal{W}_\lambda$ never vanishes because $\dim \mathcal{W}_\lambda$ is equal to the multiplicity in the associated cycle of $L(\lambda)$. If $(\xi_X, \mathbb{C}_{\xi_X})$ denotes the one-dimensional representation of abelian Lie algebra $\mathfrak{p}_-$ defined by $\xi_X(Y) := B(X, Y)$ $(Y \in \mathfrak{p}_-)$, the dual space $\mathcal{W}_\lambda^*$ of $\mathcal{W}_\lambda$ is naturally isomorphic to the space of $\mathfrak{p}_-$-homomorphisms from $L(\lambda)$ to $\mathbb{C}_{\xi_X}$:

$$\mathcal{W}_\lambda^* \simeq \mathrm{Hom}_{\mathfrak{p}_-}(L(\lambda), \mathbb{C}_{\xi_X}). \tag{2.3}$$

We are going to look at the dual $\tilde{K}_{\mathbb{C}}(X)$-module $\mathcal{W}_\lambda^*$ instead of the original $\mathcal{W}_\lambda$. For this, we need more detailed structure on the $(\mathfrak{g}, \tilde{K}_{\mathbb{C}})$-module $L(\lambda)$. Let $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}_+)} V_\lambda$ be the generalized Verma module induced from irreducible $\tilde{K}_{\mathbb{C}}$-module V_λ, where $U(\mathfrak{x})$ denotes the universal enveloping algebra of a Lie algebra $\mathfrak{x}$, and the $\mathfrak{p}_+$-action on V_λ is defined to be null. As a $\tilde{K}_{\mathbb{C}}$-module, $M(\lambda)$ is decomposed into a direct sum of homogeneous components as

$$M(\lambda) \simeq S(\mathfrak{p}_-) \otimes V_\lambda = \bigoplus_{j=0}^{\infty} (S^j(\mathfrak{p}_-) \otimes V_\lambda).$$

Further, $M(\lambda)$ has a unique maximal $(\mathfrak{g}, \tilde{K}_{\mathbb{C}})$-submodule $N(\lambda)$, and $L(\lambda)$ is realized as the irreducible quotient of $M(\lambda)$ by $N(\lambda)$:

$$L(\lambda) = M(\lambda)/N(\lambda).$$

Let Λ_{red} denote the *set of reduction points* consisting of highest weights λ of $L(\lambda)$ with reducible $M(\lambda)$: $\Lambda_{red} := \{\lambda \mid N(\lambda) \neq \{0\}\}$. For $\lambda \in \Lambda_{red}$, let i be the smallest positive integer such that

$$N(\lambda) \cap (S^i(\mathfrak{p}_-) \otimes V_\lambda) \neq \{0\}.$$

This integer i is called the *level of reduction* of $L(\lambda)$. By Enright-Joseph [4] one gets $1 \leq i \leq r$, and the maximal submodule $N(\lambda)$ is generated over $S(\mathfrak{p}_-)$ $(= U(\mathfrak{p}_-))$ by the *PRV-component* $V_{\lambda-\mu_i}$ of the tensor product $V_{-\mu_i} \otimes V_\lambda$:

$$N(\lambda) = S(\mathfrak{p}_-)V_{\lambda-\mu_i} \quad \text{with} \quad V_{\lambda-\mu_i} \subset S^i(\mathfrak{p}_-) \otimes V_\lambda. \tag{2.4}$$

Note that $V_{-\mu_i}$ is viewed as a $\tilde{K}_{\mathbb{C}}$-submodule of $S^i(\mathfrak{p}_-)$ through the isomorphism (2.1). The above inclusion $V_{\lambda-\mu_i}$ into $S^i(\mathfrak{p}_-) \otimes V_\lambda$ gives rise to a surjective $\tilde{K}_{\mathbb{C}}$-homomorphism

$$P : S^i(\mathfrak{p}_+) \otimes V_\lambda^* \longrightarrow V_{\lambda-\mu_i}^*,$$

by taking the dual, where $S^i(\mathfrak{p}_+) \otimes V_\lambda^* = (S^i(\mathfrak{p}_-) \otimes V_\lambda)^*$ through the Killing form B restricted to $\mathfrak{p}_+ \times \mathfrak{p}_-$. As shown in [18], P gives rise to the principal symbol of a differential operator of gradient-type (on Hermitian symmetric space) whose kernel realizes the maximal globalization of the dual $L(\lambda)^*$ of $L(\lambda)$.

The following proposition allows us to characterize $\mathcal{W}_\lambda^*$ by means of the projection P.

Proposition 2.1 (cf. [18, Section 3.3]). *The natural map*

$$V_\lambda \to M(\lambda) \to L(\lambda) \to \mathcal{W}_\lambda = L(\lambda)/\mathfrak{m}(X)L(\lambda)$$

from V_λ to $\mathcal{W}_\lambda$ is surjective, and it induces a $\tilde{K}_{\mathbb{C}}(X)$-isomorphism

$$\mathcal{W}_\lambda^* \simeq \begin{cases} \{v^* \in V_\lambda^* \mid P(X^i \otimes v^*) = 0\} & (\lambda \in \Lambda_{red}), \\ V_\lambda^* & (\lambda \notin \Lambda_{red}). \end{cases} \tag{2.5}$$

So, we get $\dim \mathcal{W}_\lambda \leq \dim V_\lambda$ for any $L(\lambda)$. Moreover, the associated variety $\mathcal{V}_\lambda$ of $L(\lambda)$ is characterized as

$$\mathcal{V}_\lambda = \{Z \in \mathfrak{p}_+ \mid \operatorname{Ker} P(Z^i \otimes \ \cdot\ ) \neq \{0\}\} \tag{2.6}$$

Remark 2.2. (1) If $\lambda \notin \Lambda_{red}$, then $L(\lambda)$ is not singular: $\mathcal{V}_\lambda = \mathfrak{p}_+$, and the above proposition says that $\mathcal{W}_\lambda^*$ is just V_λ^* viewed as a $\tilde{K}_{\mathbb{C}}(X)$-module by restriction.

(2) $L(\lambda)$ is called *of scalar type* of $\dim V_\lambda = 1$. In this case also, $\mathcal{W}_\lambda^*$ is the one-dimensional character of $\tilde{K}_{\mathbb{C}}(X)$ acting on V_λ^*.

Remark 2.3. It has been shown in [18] that $\mathcal{W}^*$ is isomorphic to the space of generalized Whittaker vectors for $L(\lambda)$ with respect to the (C^∞-induced) generalized Gelfand-Graev representation $\Gamma_{\mathcal{O}_{\mathbb{R}}}$ attached to the Cayley transform $\mathcal{O}_{\mathbb{R}}$ of $\mathcal{O}$:

$$\operatorname{Hom}_{\mathfrak{g},\tilde{K}_{\mathbb{C}}}(L(\lambda), \Gamma_{\mathcal{O}_{\mathbb{R}}}) \simeq \mathcal{W}_\lambda^*.$$

Here, $\mathcal{O}_{\mathbb{R}}$ is the nilpotent orbit in $\mathfrak{g}_0$ corresponding to $\mathcal{O}$ through the Kostant-Sekiguchi correspondence (see also [7]).

2.2. **Observations on tensor product $U \otimes V_\lambda^*$.** In order to find a nonzero $\tilde{K}_{\mathbb{C}}(X)$-submodule of $\mathcal{W}_\lambda^*$, let us now make simple observations on tensor products of finite dimensional $\tilde{K}_{\mathbb{C}}$-modules. Let U be any finite-dimensional $\tilde{K}_{\mathbb{C}}$-module with a weight μ. Suppose that there exists a nonzero $\tilde{K}_{\mathbb{C}}$-homomorphism $P : U \otimes V_\lambda^* \to V_{\lambda-\mu}^*$. We take an element s_0 in the Weyl group W_c for $(\mathfrak{k}, \mathfrak{t})$ so that $s_0(\lambda - \mu)$ is Δ_c^+-dominant. Put

$$U_{<s_0\mu} := \bigoplus_{\delta < s_0\mu} U(\delta),$$

where $U(\delta)$ denotes the weight space for δ, and $<$ is a lexicographic order on $\mathfrak{t}^*$ such that $\gamma > 0$ for all $\gamma \in \Delta_c^+$.

For an $x \in U_{<s_0\mu}$, we define a subset $\mathcal{N}(x)$ of $\tilde{K}_{\mathbb{C}}$ as follows:

$$\mathcal{N}(x) := \{k \in \tilde{K}_{\mathbb{C}} \mid k \cdot x \in U_{<s_0\mu}\}. \tag{2.7}$$

Let $\tilde{K}_{\mathbb{C}}(x)$ be the isotropy subgroup of x in $\tilde{K}_{\mathbb{C}}$. We denote by $v^*_{s_0\lambda}$ a nonzero weight vector in V^*_λ with extreme weight $-s_0\lambda$. Clearly, $\mathcal{N}(x)$ is right $\tilde{K}_{\mathbb{C}}(x)$-stable, and so,

$$\mathcal{U}^* := \langle \mathcal{N}(x)^{-1} \cdot v^*_{s_0\lambda} \rangle_{\mathbb{C}} \quad \text{(the complex linear span of } \mathcal{N}(x)^{-1} \cdot v^*_{s_0\lambda})$$

forms a $\tilde{K}_{\mathbb{C}}(x)$-submodule of V^*_λ.

Then we readily obtain

Proposition 2.4. *One gets* $P(x \otimes v^*) = 0$ *for every* $v^* \in \mathcal{U}^*$. *Namely,* $\mathcal{U}^*$ *is a* $\tilde{K}_{\mathbb{C}}(x)$-*submodule of* $\operatorname{Ker} P(x \otimes \ \cdot\ )$.

Proof. Let $y = \sum_{\delta < s_0\mu} y_\delta$ ($y_\delta \in U(\delta)$) be an element of $U_{<s_0\mu}$. Note that $y_\delta \otimes v^*_{s_0\lambda}$ has weight $\delta - s_0\lambda$, which is smaller than the lowest weight $-s_0(\lambda - \mu)$ of $V^*_{\lambda-\mu}$. Then, one sees that $P(y_\delta \otimes v^*_{s_0\lambda})$ can not be a nonzero weight vector of $V^*_{\lambda-\mu}$. This implies that

$$P(y \otimes v^*_{s_0\lambda}) = 0 \quad \text{for } y \in U_{<s_0\delta}. \tag{2.8}$$

For each $k \in \mathcal{N}(x)$. We deduce from the above equality that

$$P(x \otimes k^{-1} \cdot v^*_{s_0\lambda}) = k^{-1} \cdot P(k \cdot x \otimes v^*_{s_0\lambda}) = 0,$$

since $k \cdot x \in U_{<s_0\lambda}$. This proves the lemma. □

In Section 4, we will apply this proposition for $U = \mathfrak{p}_+$ and $\mu = \beta$, and give a nonzero submodule of $\mathcal{W}^*_\lambda$ attached to each singular unitary highest weight module of non oscillator and non scalar type (see Proposition 4.4).

Remark 2.5. A similar argument can be used to construct a nonzero quotient of the isotropy representation for (not necessarily holomorphic) discrete series representation (see [16] and [20]).

3. Oscillator setting [18]

This section gives a brief summary of our previous work [18, Section 5] concerning the isotropy representations $\mathcal{W}_\lambda$ for $L(\lambda)$ in the oscillator setting.

Let (G, G'_k) be one of the reductive dual pairs

$$(SU(p,q), U(k)),\ (Sp(n,\mathbb{R}), O(k)) \text{ and } (SO^*(2n), Sp(k)) \quad \text{with} \quad k = 1, 2, \ldots .$$

Then, the corresponding (complexified) symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ are of the form

$$(\mathfrak{sl}(n,\mathbb{C}),\ \mathfrak{sl}(p,\mathbb{C}) \oplus \mathfrak{sl}(q,\mathbb{C}) \oplus \mathbb{C}),\ \ (\mathfrak{sp}(n,\mathbb{C}),\ \mathfrak{gl}(n,\mathbb{C})),\ \text{ and } (\mathfrak{so}(2n,\mathbb{C}),\ \mathfrak{gl}(n,\mathbb{C})),$$

respectively. They have rank $r = \min(p,q)$, n, $[n/2]$. We set $\epsilon = 1$ for $G = SU(p,q)$ and $Sp(n,\mathbb{C})$, and $\epsilon = 2$ for $G = SO^*(2n)$. Let us consider the oscillator representation $(\omega_k, \mathbb{C}[M_{n,\epsilon k}])$ (Fock model) of $\mathfrak{g} \times G'_k$, acting on the polynomial ring $\mathbb{C}[M_{n,\epsilon k}]$ over the space $M_{n,\epsilon k}$ of complex matrices of size $n \times \epsilon k$ (see [18, 5.1] for the precise definition). Note that the elements of $\mathfrak{p}_-$ act on $\mathbb{C}[M_{n,\epsilon k}]$ by multiplication of homogeneous polynomials of degree 2.

The oscillator representation ω_k of $\mathfrak{g} \times G'_k$ decomposes into irreducibles without multiplicity as follows.

$$\mathbb{C}[M_{n,\epsilon k}] \simeq \bigoplus_{\sigma \in \Sigma(k)} L(\lambda_\sigma) \otimes \sigma. \tag{3.1}$$

Here, $\Sigma(k)$ is a subset of $\hat{G}'_k$ (the set of equivalence classes of irreducible finite-dimensional representations of the compact group G'_k), and $L(\lambda_\sigma)$ is the unitary $(\mathfrak{g}, \tilde{K}_\mathbb{C})$-module with highest weight λ_σ. The assignment $\sigma \mapsto L(\lambda_\sigma)$ is one to one, and $L(\lambda_\sigma)$ is called the theta lift of σ.

The associated variety $\mathcal{V}_{\lambda_\sigma} = \overline{\mathcal{O}}$ does not depend on $\sigma \in \Sigma(k)$ so far as we fix k. The highest weight module $L(\lambda_\sigma)$ is singular if and only if $k < r$. Note that $\Sigma(k)$ equals the whole $\hat{G}'_k$ if the pair (G, G'_k) is in the stable range: $k \leq r$.

Let us now assume that $k \leq r$, since we are interested in singular unitary representations in this paper. Then, it is shown that the reductive part of the isotropy subgroup $\tilde{K}_\mathbb{C}(X)$ $(X \in \mathcal{O})$ contains a factor, which is a covering group of the complexification $(G'_k)_\mathbb{C}$ of G'_k. This yields a natural group homomorphism

$$\pi : \tilde{K}_\mathbb{C}(X) \to (G'_k)_\mathbb{C}.$$

Theorem 3.1 (cf. [18, Th. 5.14]). *The isotropy representation $\mathcal{W}_{\lambda_\sigma}$ for $L(\lambda_\sigma)$ $(\sigma \in \Sigma(k) = \hat{G}'_k$ with $k \leq r)$ is described as*

$$\mathcal{W}_{\lambda_\sigma} \simeq \delta_k \otimes (\sigma^* \circ \pi), \tag{3.2}$$

where δ_k is a one-dimensional character of $\tilde{K}_\mathbb{C}$ (of determinant type). In particular, $\mathcal{W}_{\lambda_\sigma}$ is an irreducible $\tilde{K}_\mathbb{C}(X)$-module.

This theorem says that $\mathcal{W}^*_{\lambda_\sigma} \leftrightarrow L(\lambda_\sigma)$ sets up the Howe duality correspondence, up to a central character of $\tilde{K}_\mathbb{C}$. Note that $\mathcal{W}_{\lambda_\sigma}$ can be described also for the case $k > r$. We refer to Theorems 5.14 and 5.15 in [18] for the precise statement.

4. Non oscillator setting

In this section, we determine the isotropy representation $\mathcal{W}_\lambda$ (infinitesimally) for every singular unitary highest weight module $L(\lambda)$ which can not be realized through the oscillator representation.

4.1. **A list by Joseph.** In view of Remark 2.2 (2), it is enough to consider such $L(\lambda)$'s of non scalar type. By virtue of [11, Section 7], they are enumerated as follows:

(4.1)

$(\mathfrak{g}, \mathfrak{k})$	r	$(\mathfrak{g}, \alpha)$	λ $(k = 1, 2, \ldots)$	$\dim \mathcal{O}$
BI	2	(B_n, α_1)	$\varpi_n + ((1/2) - n)\varpi_1$	$2n - 2$
DI	2	(D_n, α_1)	$k\varpi_n + (2 - k - n)\varpi_1$	$2n - 3$
			$k\varpi_{n-1} + (2 - k - n)\varpi_1$	$2n - 3$
DIII (tube)	n	(D_{2n}, α_{2n})	$k\varpi_1 + (2 - k - 2n)\varpi_{2n}$	$2n^2 - n - 1$
EVII	3	(E_7, α_7)	$k\varpi_6 + (-2k - 8)\varpi_7$	26

Here we use the standard notation of Bourbaki [1, Planches II, IV, VI] on irreducible root systems (e.g., ϖ_i is the fundamental weight associated to a simple root α_i), and α denotes the unique noncompact simple root.

Remark 4.1. The above representation $L(\varpi_n+((1/2)-n)\varpi_1)$ for BI is somewhat special in the classification theory of unitary highest weight modules. Namely, after the terminology of [5], this is in "Case III" , where two associated root systems $Q(\lambda_0)$ and $R(\lambda_0)$ with $\lambda_0 = \varpi_n + (1-2n)\varpi_1$ do not coincide with each other.

Four cases in (4.1) have some properties in common. First, the level of reduction i of $L(\lambda)$ is equal to 1 for each case. This means that λ is at the last unitarizable place. (λ is also at the first reduction point except for EVII.) Second, every $(\mathfrak{g},\mathfrak{k})$ in (4.1) is of tube type, and the dense $\tilde{K}_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathcal{V}_\lambda$ has codimension 1 in $\mathfrak{p}_+$. Third, the highest weight λ is of the form

(4.2) $$\lambda = k\varpi_a + u(k)\zeta \quad \text{with} \quad 2u(k) \in \mathbb{Z},$$

where ϖ_a (resp. ζ) is the fundamental weight attached to some compact simple root α_a (resp. the noncompact simple root α). The irreducible $\mathfrak{k}$-module V_{ϖ_a} with highest weight ϖ_a is a basic representation identified (up to a central character of $\mathfrak{k}$) as follows.

(4.3)

$(\mathfrak{g},\mathfrak{k})$	$\mathfrak{k}$	V_{ϖ_a}
BI	$\mathfrak{o}(2n-1,\mathbb{C})\oplus\mathbb{C}$	spin representation
DI	$\mathfrak{o}(2n-2,\mathbb{C})\oplus\mathbb{C}$	two half-spin representations
DIII (tube)	$\mathfrak{gl}(2n,\mathbb{C})$	natural representation
EVII	$E_6\oplus\mathbb{C}$	27 dimensional representation on $\mathfrak{p}_-$

4.2. **Explicit formula for $\mathcal{W}_\lambda$.** To specify the representations $\mathcal{W}_\lambda$ for λ's in (4.1), we begin with the following lemma, which is shown by case-by-case analysis.

Lemma 4.2. *There exists an $s_0 \in W_c$ with the following properties* (1)–(3):
(1) $s_0\lambda = \lambda$, (2) $s_0(\lambda-\beta)$ *is Δ_c^+-dominant,* (3) $\mathcal{O}\cap(\mathfrak{p}_+)_{<s_0\beta} \neq \emptyset$.

Remark 4.3. Set $\Delta_c(\lambda) := \{\gamma \in \Delta_c \mid (\lambda,\gamma) = 0\}$. Decomposing $\Delta_c(\lambda)$ into a disjoint union of simple root systems, let $Q'(\lambda)$ be the simple component of $\Delta_c(\lambda)$, connected to $-\beta$ in the extended Dynkin diagram. Except for BI, the longest element s_0 of the Weyl group of $Q'(\lambda)$ has the properties (1)–(3) (see [2, Prop. 6.8]).

By virtue of the property (3), we can take an element $X \in \mathcal{O}\cap(\mathfrak{p}_+)_{<s_0\beta}$. Then, Proposition 2.4 can be applied to get a nonzero $\tilde{K}_{\mathbb{C}}(X)$-submodule of $\mathcal{W}_\lambda^*$, as follows.

Proposition 4.4. *Let $\mathcal{U}_\lambda^*$ be the $\tilde{K}_{\mathbb{C}}(X)$-submodule of V_λ^* generated by the lowest weight vector $v_\lambda^* \in V_\lambda^*$. Then one gets $\mathcal{U}_\lambda^* \subset \mathcal{W}_\lambda^*$.*

Now one has $\tilde{K}_{\mathbb{C}}$-isomorphisms

$$V_\lambda^* \simeq V_{k\varpi_a}^* \otimes \mathbb{C}_{-u(k)\zeta} \quad \text{and} \quad V_{\lambda-\beta}^* \simeq V_{k\varpi_a - s_0\beta}^* \otimes \mathbb{C}_{-u(k)\zeta},$$

where $\mathbb{C}_{b\zeta}$ denotes the central character of $\tilde{K}_{\mathbb{C}}$ defined by $b\zeta$ ($b \in \mathbb{R}$). Note that $V_{k\varpi_a}^*$ is the k-fold *Young product* $(V_{\varpi_a}^*)^k$ of $V_{\varpi_a}^*$, which is realized on the irreducible $\tilde{K}_{\mathbb{C}}$-submodule of

$S^k(V^*_{\varpi_a})$ (= the k-fold symmetric tensor of $V^*_{\varpi_a}$) generated by lowest weight vector

$$(v^*_{\varpi_a})^k := \overbrace{v^*_{\varpi_a}\cdots v^*_{\varpi_a}}^{k} \in S^k(V^*_{\varpi_a}).$$

In this way, $\mathcal{W}^*_\lambda$ is looked upon as

$$\mathcal{W}^*_\lambda \subset V^*_\lambda = (V^*_{\varpi_a})^k \otimes \mathbb{C}_{-u(k)\zeta} \subset S^k(V^*_{\varpi_a}) \otimes \mathbb{C}_{-u(k)\zeta}. \tag{4.4}$$

With the help of Proposition 4.4 we can prove the following fact, which reduces our task of describing $\mathcal{W}^*_\lambda$ to the special case $k=1$.

Proposition 4.5. *Let $\mathcal{W}^*_{\varpi_a}$ denote the $\tilde{K}_{\mathbb{C}}(X)$-submodule of $V^*_{\varpi_a}$ defined by $\mathcal{W}^*_{\varpi_a} := \{v^* \in V^*_{\varpi_a} \mid P_a(X \otimes v^*) = 0\}$ through the PRV-projection $P_a : \mathfrak{p}_+ \otimes V^*_{\varpi_a} \to V^*_{\varpi_a - \beta}$. Then we deduce*

$$\mathcal{W}^*_\lambda = V^*_\lambda \cap (S^k(\mathcal{W}^*_{\varpi_a}) \otimes \mathbb{C}_{-u(k)\zeta}) = (\mathcal{W}^*_{\varpi_a})^k \otimes \mathbb{C}_{-u(k)\zeta}. \tag{4.5}$$

Remark 4.6. The first equality in (4.5) follows from a standard argument on tensor (Young) products, while we need a case-by-case study to prove the second equality.

Finally, we arrive at the following theorem, by investigating $\mathcal{W}^*_{\varpi_a}$ in detail through the PRV-projection $P_a : \mathfrak{p}_+ \otimes V^*_{\varpi_a} \to V^*_{\varpi_a - s_0\beta} = V^*_{\varpi_a - \beta}$ given by a kind of contraction ([6]).

Theorem 4.7. (1) *It holds that $\mathcal{U}^*_\lambda = \mathcal{W}^*_\lambda$.*

(2) *Let $\mathfrak{k}(X)_{ss}$ denote the semisimple part of the Lie algebra of $\tilde{K}_{\mathbb{C}}(X)$. Then, $\mathcal{W}_\lambda$ is irreducible as a $\mathfrak{k}(X)_{ss}$-module, and it is described explicitly as*

$$\mathcal{W}_\lambda \simeq \begin{cases} V^{B_{n-2}}_{\varpi_{n-2}} & \text{(Case BI)}, \\ V^{D_{n-2}}_{k\varpi_{n-2}} \ \text{ or } \ V^{D_{n-2}}_{k\varpi_{n-3}} & \text{(Case DI)}, \\ V^{A_1\times C_{n-1}}_{k\varpi_1\times 0} & \text{(Case DIII)}, \\ V^{B_4}_{k\varpi_1} & \text{(Case EVII)}. \end{cases} \tag{4.6}$$

For instance, the above isomorphism for BI *reads that $\mathfrak{k}(X)_{ss}$ is of type B_{n-2}, and that $\mathcal{W}_\lambda$ is an irreducible module over the simple Lie algebra $\mathfrak{o}(2n-3,\mathbb{C})$ of type B_{n-2}, with highest weight ϖ_{n-2} (i.e., the spin representation).*

This together with Theorem 3.1 establishes Theorem 1.1, the main result of this paper.

Remark 4.8. (1) For Cases DI and EVII, the multiplicity $\dim \mathcal{W}_\lambda$ has been computed by Kato and Ochiai ([8], [9]), by a completely different method. They rely on the Poincaré polynomial of $L(\lambda)$ and on a decomposition formula of Kazhdan-Lusztig type. Proposition 4.4 together with their result would give a shorter proof of the isomorphism (4.6) for these two cases. Nevertheless, we can prove (4.6) directly by means of the PRV-projection P_a.

(2) For Case EVII, the direct sum of $V^{B_4}_{k\varpi_1}$ ($k = 0,1,2,\dots$) is isomorphic to the quasi-regular representation of $SO(9)$ on the sphere $S^8 \simeq SO(9)/SO(8)$. This compact symmetric space of rank 1 appears in the work of Dvorsky and Sahi [3, Section 2], where they give an extension of the theta correspondence by using tensor products of singular unitary representations (see also [5, Lemma 13.6]).

We end this paper by illustrating how to describe $\mathcal{W}^*_{\varpi_6} = \mathcal{W}^*_{\varpi_a}$ for the most interesting case.

Example 4.9 (Case EVII). In this case, one gets $\tilde{K}_{\mathbb{C}}$-isomorphisms:

$$V^*_{\varpi_6} \simeq \mathfrak{p}_+ \otimes \mathbb{C}_{-2\zeta}, \quad \text{and} \quad V^*_{\varpi_6-\beta} \simeq \mathfrak{p}_- \quad \text{with} \quad \zeta = \varpi_7. \tag{4.7}$$

So, the PRV-projection $P_a = P_6$ turns to be

$$P_6 : \mathfrak{p}_+ \otimes (\mathfrak{p}_+ \otimes \mathbb{C}_{-2\zeta}) \to \mathfrak{p}_-.$$

We consider the gradation of $\mathfrak{g}$ defined by the adjoint action of $H_\beta \in \mathfrak{t}$ corresponding to the highest root $\beta = \varepsilon_8 - \varepsilon_7$. Then, the Lie algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2),$$

where $\mathfrak{g}(j)$ denotes the j-eigenspace for $\operatorname{ad} H_\beta$. This naturally induces eigenspace decompositions of $\mathfrak{k}$ and $\mathfrak{p}_\pm$ by $\operatorname{ad} H_\beta$, respectively as follows:

$$\mathfrak{k} = \mathfrak{k}(-1) \oplus \mathfrak{k}(0) \oplus \mathfrak{k}(1), \quad \mathfrak{p}_\pm = \mathfrak{p}_\pm(0) \oplus \mathfrak{p}_\pm(\pm 1) \oplus \mathfrak{p}_\pm(\pm 2).$$

Note that the semisimple part of $\mathfrak{k}(0)$ is of type D_5, and that $\mathfrak{p}_\pm(0)$, $\mathfrak{k}(\pm 1) \simeq \mathfrak{p}_\pm(\pm 1)$, and $\mathfrak{p}_\pm(\pm 2)$ give irreducible representations of $\mathfrak{k}(0)$, isomorphic to the natural representation, two half spin representations, and the trivial representation, respectively.

Now we can choose an $X \in \mathcal{O} \cap \mathfrak{p}_+(0)$. Then the centralizer $\mathfrak{k}(X)$ of X in $\mathfrak{k}$, the Lie algebra of $\tilde{K}_{\mathbb{C}}(X)$, is expressed as

$$\mathfrak{k}(X) = \mathfrak{k}(-1) \oplus \mathfrak{k}(X)_{red} \quad \text{with} \quad \mathfrak{k}(X)_{red} := \mathfrak{k}(X) \cap \mathfrak{k}(0) \simeq B_4 \oplus \mathbb{C} \quad \text{(see also [10, §6])}.$$

Using these structural facts, we can deduce that $\mathcal{W}^*_{\varpi_6} \subset V^*_{\varpi_6} \simeq \mathfrak{p}_+ \otimes \mathbb{C}_{-2\zeta}$ is the 9 dimensional $\mathfrak{k}(X)$-submodule of $\mathfrak{p}_+(0)$, which is isomorphic to the natural representation of B_4.

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