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<th>Title</th>
<th>Enbeddings of derived functor modules into degenerate principal series (Representations of noncomutative algebraic systems and harmonic analysis)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2002, 1294: 72-75</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-11</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42581">http://hdl.handle.net/2433/42581</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Enbeddings of derived functor modules into degenerate principal series

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§ 1. Formulation of the problem

Let $G$ be a real linear reductive Lie group and let $G_{\mathbb{C}}$ its complexification. We denote by $g_0$ (resp. $g$) the Lie algebra of $G$ (resp. $G_{\mathbb{C}}$) and denote by $\sigma$ the complex conjugation on $g$ with respect to $g_0$. We fix a maximal compact subgroup $K$ of $G$ and denote by $\theta$ the corresponding Cartan involution. We denote by $\mathfrak{f}$ the complexified Lie algebra of $K$.

We fix a parabolic subgroup $P$ of $G$ with $\theta$-stable Levi part $M$. We denote by $N$ the nilradical of $P$. We denote by $p$, $m$, and $n$ the complexified Lie algebras of $P$, $M$, and $N$, respectively. We denote by $P_\mathbb{C}$, $M_\mathbb{C}$, and $N_\mathbb{C}$ the analytic subgroups in $G_\mathbb{C}$ with respect to $p$, $m$, and $n$, respectively.

For $X \in m$, we define

$$\delta(X) = \frac{1}{2} \text{tr} (\text{ad}_g(X)|_n).$$

Then, $\delta$ is a one-dimesional representation of $m$. We see that $2\delta$ lifts to a holomorphic group homomorphism $\xi_{2\delta} : M_{\mathbb{C}} \to \mathbb{C}^\times$. Defining $\xi_{2\delta}|_{N_{\mathbb{C}}}$ trivial, we may extend $\xi_{2\delta}$ to $P_{\mathbb{C}}$. We put $X = G_{\mathbb{C}}/P_{\mathbb{C}}$. Let $\mathcal{L}$ be the holomorphic line bundle on $X$ corresponding to the canonical divisor. Namely, $\mathcal{L}$ is the $G_{\mathbb{C}}$-homogeneous line bundle on $X$ associated to the character $\xi_{2\delta}$ on $P_{\mathbb{C}}$. We denote the restriction of $\xi_{2\delta}$ to $P$ by the same letter.

For a character $\eta : P \to \mathbb{C}^\times$, we consider the unnormalized parabolic induction $\text{Ind}_P^G(\eta)$. Namely, $\text{Ind}_P^G(\eta)$ is the $K$-finite part of the space of the $C^\infty$-sections of the $G$-homogeneous line bundle on $G/P$ associated to $\eta$. $\text{Ind}_P^G(\eta)$ is a Harish-Chandra $(g, K)$-module.

If $G/P$ is orientable, then the trivial $G$-representation is the unique irreducible quotient of $\text{Ind}_P^G(\xi_{2\delta})$. If $G/P$ is not orientable, there is a character $\omega$ on $P$ such that $\omega$ is trivial on the identical component of $P$ and the trivial $G$-representation is the unique irreducible quotient of $\text{Ind}_P^G(\xi_{2\delta} \otimes \omega)$.

Let $\mathcal{O}$ be an open $G$-orbit on $X$. We put the following assumption:
Assumption 1.1 There is a \( \theta \)-stable parabolic subalgebra \( q \) of \( g \) such that \( q \in \mathcal{O} \).

Under the above assumption, \( q \) has a Levi decomposition \( q = l + u \) such that \( l \) is a \( \theta \) and \( \sigma \)-stable Levi part. In fact \( l \) is unique, since we have \( l = \sigma(q) \cap q \).

For each open \( G \)-orbit \( \mathcal{O} \) on \( X \), we put

\[
\mathcal{A}_{\mathcal{O}} = H^{\dim \mathfrak{u}\otimes}(O, \mathcal{L})_{K\text{-finite}}.
\]

Namely, in the terminology in [Vogan-Zuckerman 1984], we have \( \mathcal{A}_{\mathcal{O}} = \mathcal{A}_q = \mathcal{A}_q(0) \).

We consider the following problem:

**Problem 1.2** Is there an embedding: \( \mathcal{A}_{\mathcal{O}} \hookrightarrow \text{Ind}^{G}_{P}(\xi_{t\delta}) \) or \( \mathcal{A}_{\mathcal{O}} \hookrightarrow \text{Ind}^{G}_{P}(\xi_{t\delta} \otimes \omega) \)?

§ 2. Complex groups

Let \( G \) be a connected real split reductive linear Lie group. Here, we consider Problem 1.2 for the complexification \( G_{\mathbb{C}} \) rather than \( G \) itself. Embedding \( G_{\mathbb{C}} \) into \( G_{\mathbb{C}} \times G_{\mathbb{C}} \) via \( g \mapsto (g, \sigma(g)) \), we may regard \( G_{\mathbb{C}} \times G_{\mathbb{C}} \) as a complexification of \( G_{\mathbb{C}} \). Each parabolic subgroup of \( G_{\mathbb{C}} \) is the complexification of a parabolic subgroup of \( G \). Let \( P \) be a parabolic subgroup of \( G \).

Then, the complexification of \( P_{\mathbb{C}} \) can be identified with \( P_{\mathbb{C}} \times P_{\mathbb{C}} \) via the above embedding \( G_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}} \times G_{\mathbb{C}} \). Hence, the complex generalized flag variety for \( G_{\mathbb{C}} \) is \( X \times X \). We fix a \( \theta \) and \( \sigma \)-stable Cartan subalgebra \( \mathfrak{h} \) of \( g \) such that \( \mathfrak{h} \subseteq p \). We denote by \( w_0 \) (resp. \( w_p \)) the longest element of the Weyl group with respect to \( (g, \mathfrak{h}) \) (resp. \( (m, \mathfrak{h}) \)).

We easily have:

**Proposition 2.1.** \( X \times X \) has a unique \( G_{\mathbb{C}} \)-orbit (say \( \mathcal{O}_{\mathbb{C}} \)). \( \mathcal{O}_{\mathbb{C}} \) satisfies the Assumption 1.1 if and only if \( w_0 w_p = w_p w_0 \).

We consider \( \xi_{t\delta} \) for \( G \). Then the character \( \xi_{t\delta} \otimes \xi_{t\delta} \) on \( P_{\mathbb{C}} \times P_{\mathbb{C}} \) is the \( \xi_{t\delta} \) for \( G_{\mathbb{C}} \).

For characters \( \mu \) and \( \nu \) of \( P_{\mathbb{C}} \), we denote the restriction of \( \mu \otimes \nu \) to \( P_{\mathbb{C}} \) realized as a real form of \( P_{\mathbb{C}} \times P_{\mathbb{C}} \) as above by the same letter.

For the complex case, we have:

**Theorem 2.2.** ([Vogan-Zuckerman 1984])

\[
\mathcal{A}_{\mathcal{O}_{\mathbb{C}}} \cong \text{Ind}^{G_{\mathbb{C}}}_{P_{\mathbb{C}}}(\xi_{t\delta} \otimes 1) \cong \text{Ind}^{G_{\mathbb{C}}}_{P_{\mathbb{C}}}(1 \otimes \xi_{t\delta}).
\]

Therefore, Problem 1.2 reduced to the problem of the existence of intertwining operators.

For \( t \in \mathbb{C} \), we define the following generalized Verma module:

\[
M_p(t\delta) = U(g) \otimes_{U(p)} \xi_{t\delta}.
\]

The following result is well-known.

**Proposition 2.3.** For \( t_1, t_2 \in 2\mathbb{Z} \),

\[
\text{Ind}^{G_{\mathbb{C}}}_{P_{\mathbb{C}}}(\xi_{t_1\delta} \otimes \xi_{t_2\delta}) \cong (M_p(-t_1\delta) \otimes M_p(-t_2\delta))^{K_{\mathbb{C}}\text{-finite}}
\]

So, our Problem 1.2 is seriously related to the existence of homomorphisms between generalized Verma modules. In fact, the following result is known.
Theorem 2.4. ([Matumoto 1993])

Let \( t \) be a non-negative even integer. Then we have

\[
M_p(-(t + 2)\delta) \leftrightarrow M_p(t\delta)
\]

if and only if \( w_0w_p \) is a Duflo involution in the Weyl group for \((\mathfrak{g}, \mathfrak{h})\).

If \( w_0w_p \) is a Duflo involution, using Proposition 2.2 we have:

\[
\begin{align*}
\text{uInd}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(1 \otimes 1) & \rightarrow \text{uInd}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(1 \otimes \xi_{2\delta}) \\
\text{uInd}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}((\xi_{2\delta} \otimes 1) & \rightarrow \text{uInd}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}((\xi_{2\delta} \otimes \xi_{2\delta})).
\end{align*}
\]

In fact, we have:

Theorem 2.5. \( A_0 \leftrightarrow \text{uInd}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}((\xi_{2\delta} \otimes \xi_{2\delta})) \) if and only if \( w_0w_p \) is a Duflo involution in the Weyl group for \((\mathfrak{g}, \mathfrak{h})\).

§ 3. Type A case

As we seen in the case of complex groups, the statement in Problem 1.2 is not correct in general. However, for type A groups, we have affirmative answers.

3.1 \( \text{GL}(n, \mathbb{C}) \)

We retain the notation in §2. We fix a Borel subalgebra \( b \) such that \( \mathfrak{h} \subseteq b \subseteq p \). We denote by \( \Pi \) the basis of the root system with respect to \((\mathfrak{g}, \mathfrak{h})\) corresponding to \( b \). We denote by \( S \) the subset of \( \Pi \) corresponding to \( p \). Assumption 1.1 holds if and only if \( S \) is compatible with the symmetry of the Dynkin diagram. For a Weyl group of the type A, each involution is a Duflo involution. Hence, we have:

Theorem 3.6. Under Assumption 1.1, we have \( A_0 \leftrightarrow \text{uInd}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}((\xi_{2\delta} \otimes \xi_{2\delta})) \).

3.2 \( \text{GL}(n, \mathbb{R}) \)

Speh proved any derived functor module of \( \text{GL}(n, \mathbb{R}) \) is parabolically induced from the external tensor product of some so-called Speh representations and possibly a one-dimensional representation. Using this fact, we can reduce Problem 1.2 to embedding Speh representations into degenerate principal series. More precisely, we consider \( G = \text{GL}(2n, \mathbb{R}) \) and let \( P \) be a maximal parabolic subgroup whose Levi part is isomorphic to \( \text{GL}(n, \mathbb{R}) \times \text{GL}(n, \mathbb{R}) \). Then, \( X = G_{\mathbb{C}}/P_{\mathbb{C}} \) contains a unique open \( G \)-orbit (say \( O \)). In this setting, Assumption 1.1 holds. The fine structure of degenerate principal series for \( P \) has already been studied precisely. ([Sahi 1995], [Zhang 1995], [Howe-Lee 1999], [Barbasch-Sahb-Speh 1988] ) From their results, we have:

\[
A_0 \leftrightarrow \text{uInd}_{P}^{G}((\xi_{2\delta}) \quad \text{if } n \text{ is odd},
A_0 \leftrightarrow \text{uInd}_{P}^{G}((\xi_{2\delta} \otimes \omega) \quad \text{if } n \text{ is even}.
\]

We can deduce an affirmative answer to Problem 1.2 from this.
3.3 $\text{GL}(n, \mathbb{H})$

In this case, we also have an affirmative answer to Problem 1.2. The argument is similar to (and easier than) the case of $\text{GL}(n, \mathbb{R})$.

3.4 $\text{U}(m, n)$

Let $G = \text{U}(m, n)$ and let $P$ be an arbitrary parabolic subgroup of $G$. In this case, Assumption 1.1 automatically holds. We denote by $V$ the set of open $G$-orbits on $X = G_{\mathbb{C}}/P_{\mathbb{C}}$. In fact, we have:

\[
\text{Socle}(\text{Ind}^{G}_{P}(\xi_{2\delta})) = \bigoplus_{\mathcal{O} \in V} \mathcal{A}_{\mathcal{O}}.
\]

References


