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Γ - PERIODIC WAVELETS AND $L^2(\mathbb{R}^N/\Gamma)$

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ABSTRACT. In this paper, we introduce Γ-Periodic Wavelets and give a decomposition of $L^2(\mathbb{R}^n/\Gamma)$.

1. NOTATIONS AND SOME PRELIMINARIES

Let $A \in \text{GL}(n, \mathbb{R})$ and $A^* = (A^t)^{-1}$.

Define

$$\Gamma = \{ \gamma = Ak; k \in \mathbb{Z}^n \}$$

and

$$\Gamma^* = \{ \gamma^* = A^*k; k \in \mathbb{Z}^n \}$$

We call $\Gamma$ the lattice with basis $A$ and $\Gamma^*$ its dual lattice.

The set $\Omega = \Omega_{\Gamma} = \{ x \in \mathbb{R}^n : x = At, t \in T^n \}$ is called the fundamental domain, where $T = [0, 1]$.

Definition 1. A Multiresolution Analysis with lattice basis (MRALB) of $L^2(\mathbb{R}^n)$ is a family of closed subspaces, $V_j (j \in \mathbb{Z})$ of $L^2(\mathbb{R}^n)$ such that:

1. $V_j (j \in \mathbb{Z})$ is an increasing sequence such that

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \quad \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^n)$$

2. $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$

3. $f(x) \in L^2(\mathbb{R}^n)$ belongs to $V_0$ if and only if $f(x - \gamma) \in V_0$ for all $\gamma \in \Gamma$

4. There exists $g \in V_0$ such that $\{g(x - \gamma); \gamma \in \Gamma\}$ is a Riesz basis of $V_0$. Assume that $\{\varphi(x - \gamma); \gamma \in \Gamma\}$ is an orthonormal basis of $V_0$, then the Fourier transform of the function $\varphi$ satisfies

$$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi), \quad \xi \in \mathbb{R}^n$$

with $2\pi \Gamma^*$ - periodic function $m_0(\xi)$, a filtering function.

For $f, g \in L^2(\mathbb{R}^n)$, define

$$C(f, g)(\xi) = \sum_{\gamma \in \Gamma^*} \hat{f}(\xi + 2\pi \gamma^*)\overline{\hat{g}(\xi + 2\pi \gamma^*)}$$

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we call it the correlation function. For the following Theorem 1 and Theorem 2, see Y. Asoo[1], and [2].

**Theorem 1.** Assume \( \varphi \in L^2(\mathbb{R}^n) \). Then a system
\[
\{2^{\frac{n}{2}} \varphi(2^j x - \gamma) : \gamma \in \Gamma \}
\]
is an orthonormal basis of \( V_j (j \in \mathbb{Z}) \) if and only if
\[
C(f, f)(\xi) = |\text{det}(A)|, \quad \text{a.a. } \xi \in \mathbb{R}^n.
\]

Now, for a Riesz basis \( g \) of \( V_0 \), define the function \( \varphi \) so that
\[
\varphi(\xi) = \sqrt{|\text{det}(A)|} \frac{\hat{g}(\xi)}{\sqrt{C(g, g)(\xi)}},
\]
then \( \{2^{\frac{n}{2}} \varphi(2^j x - \gamma) : \gamma \in \Gamma \} \) is an orthonormal basis of \( V_j \). Let \( E = \{0, 1\}^n \) and \( \psi_\epsilon \in V_1 (\epsilon \in E) \) be such that
\[
(1.1) \quad \hat{\psi}_\epsilon(2 \xi) = m_\epsilon(\xi) \hat{\varphi}(\xi)
\]
where \( \psi_0 = \varphi \) and \( m_\epsilon \) is \( 2\pi \Gamma^* \)-periodic.

**Theorem 2.** The system \( \{\psi_\epsilon(x - \gamma) : \gamma \in \Gamma, \epsilon \in E\} \) is an orthonormal basis of \( V_1 \) if and only if the matrix
\[
U(\xi) = (m_\epsilon(\xi + \pi A^* \eta))_{(\epsilon, \eta) \in E^2}
\]
is unitary for almost all \( \xi \in \mathbb{R}^n \).

For \( j \in \mathbb{Z}, \epsilon \in \bar{E} \equiv E \setminus \{0\}, \) and \( \gamma \in \Gamma \),
\[
(1.2) \quad \psi_{j, \epsilon, \gamma}(x) = 2^{\frac{j}{2}} \psi_\epsilon(2^j x - \gamma)
\]
Define \( W_{(j, \epsilon)} = \overline{\langle \psi_{(j, \epsilon, \gamma)} \rangle} := \{\gamma \in \Gamma\} \) and \( W_j = \bigoplus_{\epsilon \in \bar{E}} W_{(j, \epsilon)} \).

Then
\[
L^2(\mathbb{R}^n) = V_0 \bigoplus_{k=0}^{\infty} W_k = \bigoplus_{k=-\infty}^{\infty} W_k
\]

**Definition 2.** We call the system \( \{\psi_{(j, \epsilon, \gamma)} ; j \in \mathbb{Z}, \epsilon \in \bar{E}, \gamma \in \Gamma\} \)
wavelets basis of \( L^2(\mathbb{R}^n) \), and \( \{\psi_{(0, \epsilon, \gamma)} ; \epsilon \in \bar{E}, \gamma \in \Gamma\} \) mother wavelets.

In the next section, we define \( \Gamma \)-Periodic Wavelets and study an orthogonal decomposition of \( L^2(\mathbb{R}^n/\Gamma) \).

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2. Gamma-Periodic Wavelets and $L^2(\mathbb{R}^n/\Gamma)$

In the following, let $\{V_j; j \in \mathbb{Z}\}$ be a MRALB with $A \in GL^+(n; \mathbb{R})$, \{\varphi(x - \gamma); \gamma \in \Gamma\} be an orthonormal basis of $V_0$, and $\int_{\mathbb{R}^n} \varphi(x) \, dx = \sqrt{\det(A)}$.

A function $f \in L^2(\mathbb{R}^n)$ is called $\Gamma$-periodic if
\[
 f(x) = f(x + \gamma) \quad \text{for;} \quad x \in \mathbb{R}^n, \gamma \in \Gamma.
\]

Put
\[
(1) \quad P_j = P_j(\gamma) = \{f \in V_j; f \text{ is } \Gamma - \text{periodic, } j \in \mathbb{Z}\}
\]

Assume that

(1) $P_j's$ are closed subspaces, and $P_j \subset P_{j+1}$;
(2) $f(x) \in P_j$ if and only if $f(2x) \in P_{j+1}$.

**Proposition 1.** For $j \leq 0$, $\dim(P_j) = 1$, and for $j \geq 1$, $\dim(P_j) = 2^{nj}$.

**Proof.** Note that $\sum_{\gamma \in \Gamma} \varphi(x - \gamma) = \frac{1}{\sqrt{\det(A)}} \in P_0$.

For $j \leq 0$ let $f \in P_j$ be
\[
f(x) = \sum_{\gamma \in \Gamma} c(\gamma) \varphi(x - \gamma)
\]

Then, for any $\gamma \in \Gamma$, $c(\gamma) = c(f) = \text{const.}$ and $f(x) = \frac{c(f)}{\sqrt{\det(A)}}$.

For $j \geq 1$ let $f \in P_j$ then $g(x) = f(2^j x) \in P_0$.
Put $g(x) = \sum_{\gamma \in \Gamma} c(\gamma) \varphi(x - \gamma)$, then for any $\gamma, \gamma_0 \in \Gamma$,
\[
c(\gamma + 2^j \gamma_0) = c(\gamma), \quad \text{thus} \quad \dim(P_j) = 2^{nj}.
\]

For $j \in \mathbb{N}$,
\[
(2) \quad \mathbb{Z}_{2^j}^n \equiv (\mathbb{Z} \mod 2^j)^n
\]
\[
(3) \quad \Gamma^{(j)} \equiv \{\frac{Ak}{2^j}; k \in \mathbb{Z}_{2^j}^n\}
\]

$\Gamma^{(j)}$ is a finite additive group of order $2^{nj}$ and $[\Gamma^{(j+1)} : \Gamma^{(j)}] = 2^n$.

For $f \in P_j$ and $\gamma \in \Gamma^{(j)}$, define
\[
(\gamma f)(x) = f(x - \gamma).
\]

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We see that the space $P_j$ is of dimension $2^{nj}$, $\Gamma^{(j)}$-invariant closed subspace of $L^2(\mathbb{R}^n/\Gamma) = L^2(\Omega_{\Gamma})$.

For $j \in \mathbb{N}$,

$$\varphi_j(x) \equiv \sum_{\gamma \in \Gamma} 2^{nj} \varphi(2^j(x - \gamma))$$

The function $\varphi_j$'s are $\Gamma$-periodic.

**Theorem 1.** The system $\{\gamma \varphi_j; \gamma \in \Gamma^{(j)}\}$, is an orthonormal basis of the space $P_j$.

**Proof.** Let $j = 0$, then $\varphi_0(x) = \frac{1}{\sqrt{\det(A)}}$ and

$$\int_{\Omega} |\varphi_0(x)|^2 \, dx = 1.$$

In general, since $\text{dim}(P_j) = |\Gamma^{(j)}| = 2^{nj}$, it is sufficient to show that, for $\gamma_1, \gamma_2 \in \Gamma^{(j)}$,

$$\langle \gamma_1 \varphi_j, \gamma_2 \varphi_j \rangle \equiv \int_{\Omega} (\gamma_1 \varphi_j)(x)(\overline{\gamma_2 \varphi_j})(x) \, dx = \delta(\gamma_1, \gamma_2).$$

Put $\gamma_l = \frac{A}{2^j} k_l, k_l \in \mathbb{Z}^n, l = 1, 2$, then

$$\langle \gamma_1 \varphi_j, \gamma_2 \varphi_j \rangle = \int_{\Omega} \varphi_j(x - \frac{A}{2^j} k_1) \overline{\varphi_j(x - \frac{A}{2^j} k_2)} \, dx$$

$$= \sum_{l,m \in \mathbb{Z}^n} 2^{nj} \int_{\Omega} \varphi(2^j[x - Al] - Ak_1))\overline{\varphi(2^j[x - Am] - Ak_2)} \, dx.$$

Put $x - Al = 2^{-j} y, y \in \mathbb{R}^n$, then we have

$$\langle \gamma_1 \varphi_j, \gamma_2 \varphi_j \rangle = \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \varphi(y - Ak_1)\overline{\varphi(y - A[k_2 + 2^j m])} \, dy$$

$$= \delta(k_1, k_2) = \delta(\gamma_1, \gamma_2).$$

Let $T(\gamma)f = \gamma f, \gamma \in \Gamma^{(j)}, f \in P_j$, then $(T(\gamma), P_j)$ is a unitary representation of the group $\Gamma^{(j)}$.

Next, we consider the Fourier series expansion of the function $\varphi_j$ and Poisson summation formula. Let

$$\varphi_j(x) = \sum_{l \in \mathbb{Z}^n} c(l) \exp(2\pi i A^* l \cdot x).$$

Then, we get

$$c(l) = \frac{1}{\det(A)} \int_{\Omega} \varphi_j(x) \exp(-2\pi i A^* l \cdot x) \, dx.$$
On the other hand,

\[
\int_{\Omega} \varphi_j(x) \exp(-2\pi i A^* l \cdot x) \, dx
\]

\[= 2 \frac{n_j}{2} \int_{\mathbb{R}^n} \varphi(2^j x) \exp(-2\pi i A^* l \cdot x) \, dx
\]

\[= 2^{-\frac{n_j}{2}} \hat{\varphi}(2\pi \frac{A^*}{2^j} l).
\]

Thus, we get

**Proposition 2.** The Fourier series expansion of the function \( \varphi_j, j \in \mathbb{N} \) is

\[
\varphi_j(x) = \frac{1}{2^{n_2} \det(A)} \sum_{\gamma \in \Gamma} \hat{\varphi}(\frac{2\pi}{2^j} \gamma) \exp(2\pi \gamma \cdot x)
\]

In particular, taking \( x = 0 \),

\[
2^n \sum_{\gamma \in \Gamma} \varphi(2^j \gamma) = \frac{1}{2\pi nj \det(A)} \sum_{\gamma \in \tilde{E}} \hat{\varphi}(\frac{2\pi}{2^j} \gamma)
\]

(Poisson Summation Formula)

Now define \( \psi_{(0,0)}(x) = \frac{1}{\sqrt{\det(A)}}(= \varphi_0(x)) \) and for \( j \in \mathbb{N} \) and \( \epsilon \in \tilde{E} \), define \( \psi_{j,\epsilon} \) as

\[
\psi_{j,\epsilon}(x) = \frac{n_j}{2} \sum_{\gamma \in \Gamma} \psi_{\epsilon}'(2^j (x - \gamma))
\]

Let \( Q_j(\Gamma) \) be the orthogonal complement of \( P_j(\Gamma) \) in \( P_{j+1}(\Gamma) \), \( \dim(Q_j(\Gamma)) = 2^n (2^n - 1) \).

**Theorem 2.** For \( j \in \mathbb{N} \) and \( \epsilon \in \tilde{E} \), let \( Q_{j,\epsilon}(\Gamma) \) be the closure of linear span of \( \{\gamma \psi_{j,\epsilon} ; \gamma \in \Gamma^{(j)}\} \).

Then, \( \{\gamma \psi_{j,\epsilon} ; \gamma \in \Gamma^{(j)}\} \) is an orthonormal basis of \( Q_{j,\epsilon}(\Gamma) \) and

\[
Q_j(\Gamma) = \bigoplus_{\epsilon \in \tilde{E}} Q_{j,\epsilon}(\Gamma)
\]

\[
P_{j+1}(\Gamma) = P_j(\Gamma) \bigoplus_{\epsilon \in \tilde{E}} Q_{j,\epsilon}(\Gamma)
\]

\[
L^2(\Omega_\Gamma) = P_0(\Gamma) \bigoplus_{j \in \mathbb{N}} \bigoplus_{\epsilon \in \tilde{E}} Q_{j,\epsilon}(\Gamma)
\]

**Proof.** For \( \epsilon_1 \neq \epsilon_2 \), \( \{\gamma \psi_{\epsilon_1}(x) ; \gamma \in \Gamma\} \) and \( \{\gamma \psi_{\epsilon_2}(x) ; \gamma \in \Gamma\} \) are orthogonal, so that it is sufficient to prove for \( Q_{j,\epsilon}(\Gamma) \).

The rest of the proof is done in the same way to Theorem 1 of this
Definition 1. We call \( \{ \gamma_j \epsilon \; \gamma \in \Gamma^{(j)}, j \in \mathbb{N}, \epsilon \in \tilde{E} \} \)
\( \Gamma \)-periodic wavelets.

Proposition 3. We have the Fourier series expansion

\[
\psi_{j, \epsilon}(x) = \frac{1}{2^{n_2} \det(A)} \sum_{\gamma^{*} \in \Gamma^{*}} m_{\epsilon}(\frac{\pi \gamma^{*}}{2^j}) \hat{\varphi}(\frac{\pi \gamma^{*}}{2^j}) \exp(2\pi \iota \gamma^{*} \cdot x)
\]

and in particular, taking \( x = 0 \),

\[
2^{\frac{n_2}{2}} \sum_{\gamma \in \text{Gamma}} \psi(2^j \gamma) = \frac{1}{2^{n_2} \det(A)} \sum_{\gamma^{*} \in \Gamma^{*}} m_{\epsilon}(\frac{\pi \gamma^{*}}{2^j}) \hat{\varphi}(\frac{\pi \gamma^{*}}{2^j})
\]

(Poisson Summation Formula)

REFERENCES


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