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Spiral spline interpolation to a planar spiral

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Abstract

We derive regions for T-cubic and arc/T-cubic spirals to two-point Hermite interpolation in terms of unit tangent vectors with help of *Mathematica*. The use of spirals gives the designer an excellent and speedy control over the shape of curve that is produced because there are no internal curvature maxima, curvature minima, inflection points, loops and cusps in a spiral segment.
Key words: Spiral, G^1 Hermite data, T-cubic spline, Arc/T-cubic, Curvature;

1 Introduction and Preliminary results

Smooth curve representation is required for visualization of the scientific data. Smoothness is one of the most important requirements for the visual pleasing display. Fair curves are also important in computer-aided design (CAD) and computer-aided geometric design (CAGD). Cubic splines, although smoother, are not always helpful since they might have *unwanted* inflection points and singularities (see [6], [7]). Spirals are visually pleasing curves of monotone curvature; and they have the advantage of not containing curvature maxima, curvatue minima, inflection points and singularities. Many authors have advocated their use in the design of fair curves (see [5]). These spirals are desirable for applications such as the design of highway or railway routes and trajectories of mobile robots. The benefit of using such curves in the design of surfaces, in particular surfaces of revolution and swept surfaces, is the control of unwanted flat spots and undulations (see [4]). Some advantages of spirals are that they are parametric

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curves, the arc length can be expressed as a polynomial function of the parameter, the curvature can be expressed as a rational function of the parameter and the offset curve is a rational function of the parameter. These last three properties result from the fact that the T-cubic (the Tschirnhausen cubic) has a Pythagorean hodograph and can be expressed as cubic NURBS for compatibility with existing computer aided design software. Meek & Walton has considered two-point Hermite interpolating spirals by joining T-cubic spirals and/or arc/T-cubic spirals (see [1]). The T-cubic spline has a simple representation for treatment of its curvature while the numerator of the derivative of the curvature is quintic and difficult to treat even for the cubic curve.

In Section 2, we obtain (if necessary, with help of *Mathematica*) the equivalent but very simple results (the spiral regions for T-cubic and arc/T-cubic spirals in terms of the unit tangent vectors at the both endpoints) to the ones in [1]. Without loss of generality, assume the two points of the Hermite data are $(0,0)$ and $(-1,0)$, and the tangent vector rotates counterclockwise as one traverses the spiral. Thus, the tangent vector at $(0,0)$ points above the X-axis, $r_0^2(-\cos\theta, \sin\theta)$ and the tangent vector at $(-1,0)$ points below the X-axis, $-r_1^2(\cos\psi, \sin\psi)$ which intersect at (b,c). We assume that $0 < \psi \leq \theta < \pi, \theta+\psi < \pi$ and $r_0, r_1 > 0$. We presented a flow chart of an efficient algorithm to implement spirals for visualization of planar data. Concluding remarks and future research problems are given in Section 3.

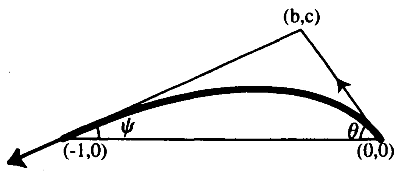

Figure 1: A spiral matching geometric Hermite data in a standard form.

Theorem 1.1
If $4\sin\theta + \sin(2\theta+\psi) \geq$ *(or* $\leq$*)* $7\sin\psi$ *and* $0 < \psi \leq \theta$, *then a unique T-cubic spiral (or arc/T-cubic spiral) with a monotone decreasing curvature exists that matches the Hermite data.*

Proof of above theorem is given in next section and as a consequence of this theorem, we obtain **Corollary** ([1]) If $3b^2 + 4c^2 \geq$ (*or* $\leq$)$12(b^2+c^2)^2, b \geq -1/2$, then a unique T-cubic spiral (or arc/T-cubic spiral) with a monotone decreasing curvature exists that matches the Hermite data.

2 T-cubic and arc/T-cubic Spirals

We consider a T-cubic spline $\boldsymbol{P}(t), 0 \leq t \leq 1$ can be given with piecewise linear functions $u(t), v(t)$ as

$$\boldsymbol{P}(t) = (x(t), y(t)), x'(t) = u(t)^2 - v(t)^2, y'(t) = 2u(t)v(t) \tag{1}$$

where

$$u(t) = (1-t)r_0\sin(w/2) - tr_1\sin(\psi/2)$$

$$v(t) = (1-t)r_0 \cos(w/2) + t r_1 \cos(\psi/2), \quad r_0, r_1 > 0 \tag{2}$$

It is easy to check

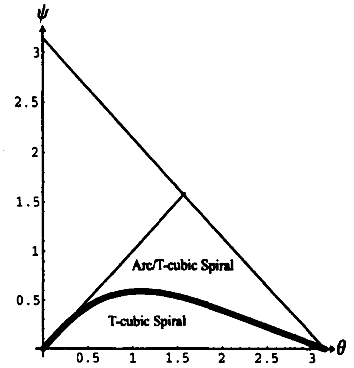

Figure 2: Regions for T-cubic and arc/T-cubic spirals with monotonic decreasing curvature, in terms of unit tangent vectors.

$$\boldsymbol{P}'(0) = r_0^2(-\cos w, \sin w), \quad \boldsymbol{P}'(1) = r_1^2(-\cos\psi, -\sin\psi) \tag{3}$$

Its signed curvature $\kappa(t)$ is given by

$$\kappa(t) = \frac{(\boldsymbol{P}' \times \boldsymbol{P}'')(t)}{\|\boldsymbol{P}'(t)\|^3}, \quad 0 \le t \le 1 \tag{4}$$

where "$\times$" and $\|\bullet\|$ mean the the cross product of the two vectors and the Euclidean norm, respectively, i.e.,

$$\kappa(t) = \frac{2r_0r_1 \sin\{(w+\psi)/2\}}{\{r_0^2(1-t)^2 + r_1^2t^2 + 2r_0r_1t(1-t)\cos\{(w+\psi)/2\}\}^2} \tag{5}$$

from which follows the following spiral condition with the monotone decreasing curvature which attains its maximum at a negative point:

$$\frac{r_1}{r_0}\cos\frac{w+\psi}{2} \ge 1 \tag{6}$$

For T-cubics and arc/T-cubics, we consider conditions: $\boldsymbol{P}(0) = (p, q)$ and $\boldsymbol{P}(1) = (-1, 0)$ as

$$\begin{aligned} 3p &= -3 + r_1^2\cos\psi + r_0r_1\cos\frac{w-\psi}{2} + r_0^2\cos w \\ 3q &= r_1^2\sin\psi - r_0r_1\sin\frac{w-\psi}{2} - r_0^2\sin w \end{aligned} \tag{7}$$

T-cubic spiral: $((p,q) = (0,0), w = \theta; 4\sin\theta + \sin(2\theta+\psi) \ge 7\sin\psi)$:

Solve (7) for r_0, r_1 to obtain

$$r_0 = \frac{2\sqrt{3}\sin\psi}{\sqrt{\cos\theta + \cos\psi - 2\cos(\theta+2\psi) + \sin\{(\theta+\psi)/2\}\sqrt{2 - 2\cos\theta\cos\psi + 14\sin\theta\sin\psi}}}$$

$$\frac{r_1}{r_0} = \frac{2\sin\theta}{\sqrt{\sin^2\{(\theta-\psi)/2\} + 4\sin\theta\sin\psi} - \sin\{(\theta-\psi)/2\}}$$

A combination of (6) and the second equation of (8) for r_1/r_0 gives the spiral conditions for the T-cubic spiral. To prove Corollary of Theorem 1.1 concerning the spiral region for the point (b,c), note

$$(\sin\theta, \cos\theta) = (c/\sqrt{b^2+c^2}, -b/\sqrt{b^2+c^2})$$
$$(\sin\psi, \cos\psi) = (c/\sqrt{(1+b)^2+c^2}, (1+b)/\sqrt{(1+b)^2+c^2})$$

Since

$$4\sin\theta + \sin(2\theta+\psi) - 7\sin\psi = \frac{2c\left(2\sqrt{b^2+c^2}\sqrt{(1+b)^2+c^2} - (b+4b^2+4c^2)\right)}{(b^2+c^2)\sqrt{(1+b)^2+c^2}}$$
$$2\sqrt{b^2+c^2}\sqrt{(1+b)^2+c^2} - (b+4b^2+4c^2)(= N_1 - N_2))$$
$$N_1^2 - N_2^2 = 3b^2 + 4c^2 - 12(b^2+c^2)^2$$

we obtain the T-cubic spiral region for (b,c):

$$3b^2 + 4c^2 \geq 12(b^2+c^2)^2 \tag{9}$$

Arc/T-Cubic Spiral: ($7\sin\psi > 4\sin\theta + \sin(2\theta+\psi)$)
We join the origin to the point (p,q) by a circular arc and the point (p,q) to $(-1,0)$ by a T-cubic spline of the form (1) in a G^3 manner. We assume that the unit tangent vector at the joint (p,q) is $(-\cos w, \sin w)$. Therefore, note that (i) the radius r of the arc is equal to $1/\kappa(0)$, (ii) $\kappa'(0) = 0$ and (iii) its center is $-r(\sin\theta, \cos\theta)$.
From which follows:

$$\begin{aligned} &(i) \quad && r = \frac{1}{2} r_1^2 \cos^2\frac{w+\psi}{2}\cot\frac{w+\psi}{2} \\ &(ii) \quad && r_0 = r_1\cos\frac{w+\psi}{2} \\ &(iii) \quad && (p + r\sin\theta, q + r\cos\theta) = r(\sin w, \cos w) \end{aligned} \tag{10}$$

Letting $W = w+\psi$, (if necessary, with help of *Mathematica*) 10(iii) gives the same two equations ([1]):

$$\begin{aligned} &(i) \quad && f(W)\cos(\psi - W) + g(W)\sin(\psi - W) + h(W)\cos\theta = 0 \\ &(ii) \quad && r_1^2 = \frac{12\sin W}{-f(W)\sin(\psi - W) + g(W)\cos(\psi - W) + h(W)\sin\theta} \end{aligned} \tag{11}$$

where

$$f(W) = 3(1 - 2\cos W - 3\cos^2 W), \quad g(W) = 4(1 + 2\cos W)\sin W$$
$$h(W) = 3(1+\cos W)^2$$

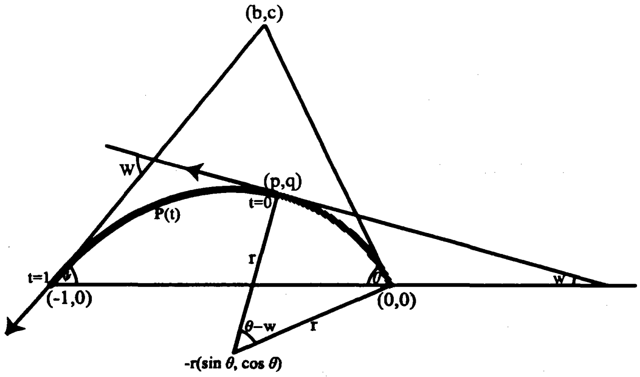

Figure 3: An arc/T-cubic spiral.

Now we consider the existence of the solution W on $(0, \theta+\psi)$ of 11(i) for which the denominator of 11(ii) is positive. Replace $\tan(W/2)$ by z to reduce 11(i) to

$$\frac{4\left\{3(1+z^2)\cos\theta - (3+3z^2+2z^4)\cos\psi + 2z^3(2+z^2)\sin\psi\right\}}{(1+z^2)^3} \left(= \frac{4q(z)}{(1+z^2)^3}\right) \tag{12}$$

We show the existence of the solution of $q(z) = 0$, its uniqueness being proved later. With $z_1 = \tan\frac{\theta+\psi}{2}$,

$$\begin{aligned} q(0) &= 3(\cos\theta - \cos\psi)(<0) \\ q(z_1) &= \frac{z_1(1+z_1^2)^2}{2}\left\{7\sin\psi - 4\sin\theta - \sin(2\theta+\psi)\right\}(>0) \end{aligned} \tag{13}$$

Descartes' rule of signs and intermediate value theorem show the existence of at least one or three positive zeros of $q(z)$ belonging to $(0, z_1)$ for which the denominator d of r_1^2 is positive since

$$d = \frac{4\left\{2z^3(2+z^2)\cos\psi + 3(1+z^2)\sin\theta + (3+3z^2+2z^4)\sin\psi\right\}}{(1+z^2)^3} \tag{14}$$

Next, to show that the number of the positive zero of q is just one, we have only to examine the uniqueness of the positive zero of $q'(z)$.

$$q'(z) = 2z\left\{5z^3\sin\psi - 4z^2\cos\psi + 6z\sin\psi + 3(\cos\theta - \cos\psi)\right\}(= 2zg(z)) \tag{15}$$

First note $g(0) < 0$ and $g(z_1) > 0$ as

$$\begin{aligned} g(0) &= 3(\cos\theta - \cos\psi) < 0 \\ g(z_1) &= \frac{1}{4}z_1(1+z_1^2)\left\{31\sin\psi - 14\sin\theta - 3\sin(2\theta+\psi)\right\} \end{aligned} \tag{16}$$

$$> \frac{1}{2}z_1(1+z_1^2)(5\sin\psi - \sin\theta) > 0 \tag{17}$$

where

$$\begin{aligned} 0 &< 7\sin\psi - 4\sin\theta - \sin(2\theta+\psi) = (6+2\sin^2\theta)\sin\psi - 2(2+\cos\theta\cos\psi)\sin\theta \\ &< 2(4\sin\psi - \sin\theta) \end{aligned}$$

i.e., $4\sin\psi > \sin\theta$

Depending on the shape of the curve $g'(z)$, we consider the following cases:

(i) $g'(z)$ has no real zeros

(ii) $g'(z)$ has two real (positive) zeros α, β $(\alpha < \beta)$ staisfying either

(a) $\alpha > z_1$, (b) $\alpha < z_1 < \beta$, or (c) $0 < \alpha, \beta < z_1$

Since $g'(0) < 0, g'(z_1) > 0$, the unique positive zero of $g'(z)$ can be easily obtained except the case ii(c). For the above ii(c),

$$53\cos 2\psi > 37, \quad \frac{4\cos\psi + \sqrt{-37 + 53\cos(2\psi)}}{15\sin\psi} < \tan\frac{\theta + \psi}{2} \tag{18}$$

A simple calculation gives with $(p, q) = (\tan\frac{\theta}{2}, \tan\frac{\psi}{2})$

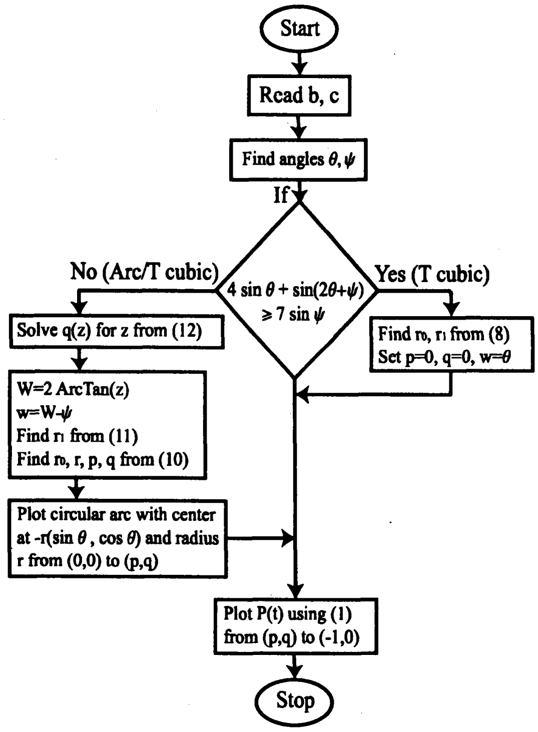

Figure 4: Flow chart to implement T-cubic and Arc/T-cubic spirals.

$$p > \frac{2 - 17q^2 + \sqrt{4 - 98q^2 + 4q^4}}{q(17 - 2q^2 + \sqrt{4 - 98q^2 + 4q^4})} > \frac{2 - 17q^2}{q(17 - 2q^2)}$$

$$q < \frac{53 - 3\sqrt{265}}{2\sqrt{106}} (\approx 0.202199)$$

With help of *Mathematica*,

$$g(\alpha) = 3\cos\theta - \frac{37\cos\psi}{15} + \frac{2\alpha(37 - 53\cos 2\psi)}{45\sin\psi} \tag{19}$$

Note

$$3\cos\theta - \frac{37\cos\psi}{15} = \frac{3(1 - p^2)}{1 + p^2} - \frac{37(1 - q^2)}{15(1 + q^2)}$$

$$< -\frac{8(1-q)(1+q)(41-982q^2+41q^4)}{15(1+q^2)(4+217q^2+4q^4)} < 0 \tag{20}$$

where $41 - 982q^2 + 41q^4 = 0$ has a positive root $\sqrt{41/(491+30\sqrt{266})}(\approx 0.204511)$. Since $g(\alpha) < 0$ and $g(z_1) > 0$, g i.e., q' has just one positive zero, (13) implies the unique positive zero of q on $(0, z_1)$. In this case, we obtain the region for (b, c):

$$3b^2 + 4c^2 \leq 12(b^2+c^2)^2, \quad b > -1/2 \tag{21}$$

where note $\theta > \psi$.

Flow chart for implementation of T-cubic and arc/T-cubic of section 2 is given in figure 4 where as demonstration is referred to [1].

3 Concluding Remarks and Future Direction

Fair curves can be designed interactively using two-point Hermite interpolating spiral. The spiral segments are either spirals taken from the T-cubic curve or spirals created by joining circular arcs to segments of the T-cubic in a G^3 fashion. The main result of this paper is a very simple proof that any geometric Hermite data that can be matched with a (general) spiral can be matched with a unique T-cubic spiral or a unique arc/T-cubic spiral. In other words, the T-cubic spiral or the arc/T-cubic spiral can be used in any situation where a spiral is possible. Due to simple algorithm, these spirals can be easily implemented. Our future work directions are to revise this paper for G^2 case and investigate the existance and uniqueness of Pythagorean hodograph quintic spiral (see [2],[3],[4]) in simple way and develop efficient algorithm for implementation.

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