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ON GALOIS THEORY OF q-DEFORMATIONS OF DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

In this survey paper[1], we consider linear q-difference equations as deformations of linear differential equations, or, what amounts to the same, we consider linear q-difference equations at the limit $q \to 1$. This limit process is sometimes unproperly called a *confluence*.

Typically, a $n \times n$-matrix q-difference equation

$$(*)_q \qquad Y(qx) = A_q(x)Y(x)$$

(where A_q is a given invertible matrix and Y is the unknown matrix function) can be rewritten in the form

$$(*)'_q \qquad d_qY = B_qY, \text{ with } d_qf = \frac{f(qx)-f(x)}{qx-x}, B_q = \frac{A_q - id}{(q-1)x},$$

which "converges" to the differential equation

$$(*)_1 \qquad \frac{d}{dx}Y = B_1Y\,, \quad B_1 = (\lim B_q)$$

when $q \to 1$, when the limit B_1 exists. The standard example is the confluence of hypergeometric q-difference systems to ordinary hypergeometric differential systems [6].

There is a Galois theory for q-difference equations as well as for differential equations. In particular, there is a difference Galois group $G_q \subset GL(n)$ attached to $(*)_q$ and a differential Galois group $G_1 \subset GL(n)$ attached to $(*)_1$. The purpose of this paper is to study *the variation of G_q with q*.

We shall survey two independent approaches of this problem : a purely algebraic approach via non-commutative connections (based on [1]), and an analytic approach based on J. Sauloy's work [7][8][9].

[1]this is the text of my talk at the symposium on "deformations of differential equations and asymtotic analysis" at RIMS Kyoto June 3-7, 2002; I thank Prof. Haraoka for his invitation.

2. ALGEBRAIC APPROACH [1]

2.1. **Difference versus differential modules.** For difference Galois theory as well as for differential Galois theory, there are two different (and non-trivially equivalent) viewpoints : the *Picard-Vessiot viewpoint* and the *tannakian viewpoint*. In either viewpoint, there is an apparent heterogeneity in the definitions of Galois groups in the difference and differential cases respectively, which makes the study of the confluence phenomenon difficult.

Let us recall the tannakian approach in a more general setting.

i) The difference case [11] : one considers a commutative ring R ("of functions") endowed with an automorphism σ (*e.g.* , in the q-difference case, $\sigma = \sigma_q$ is the q-dilatation : $(\sigma_q(f))(x) = f(qx)$). We assume for simplicity that $C := R^\sigma$ is a field (the field of constants).

A more intrinsic way of thinking at a difference equation of type $(*)_q$ consists in introducing a (finite free) R-module M endowed with an invertible σ-linear endomorphism Φ of M (expressed by $A(x)$ in a suitable basis). There are obvious notions of tensor products and duals of such objects. Under suitable conditions, the $\otimes$-category $< (M, \Phi) >$ obtained by performing all standard operations from multilinear algebra on (M, Φ) (and taking suitable subquotients) turns out to be $\otimes$-equivalent to the category $Rep_C G_\sigma$ of finite-dimensional representations of some linear algebraic group G_σ defined over C, the *difference Galois group* of (M, Φ).

ii) The differential case [5],[1, §1.3] : instead of an automorphism σ, one considers a derivation $d : R \to \Omega^1$ into some natural R-module of differentials (in the classical situation like $(*)_1$, $\Omega^1 = Rdx$). We assume for simplicity that $C := \ker d$ is a field (the field of constants).

A more intrinsic way of thinking at a differential equation of type $(*)_1$ consists in introducing a (finite free) R-module M endowed with a *connection* $\nabla : M \to \Omega^1 \otimes_R M$ (in our example, $\nabla(d/dx)$ is expressed by $A(x)$ in a suitable basis). This means a C-linear map satisfying the *Leibniz rule* :

$$\nabla(am) = da \otimes m + a \otimes \nabla(m).$$

There are natural notions of tensor products and duals of connections. For instance, the tensor product is defined by

$$(M_1, \nabla_1) \otimes (M_2, \nabla_2) = (M_1 \otimes M_2, \nabla_1 \otimes id_{M_2} + id_{M_1} \otimes \nabla_2).$$

Under suitable conditions, the $\otimes$-category $< (M, \nabla) >$ obtained by performing all standard operations from multilinear algebra on (M, ∇) turns out to be $\otimes$-equivalent to the category $Rep_C G_d$ of finite-dimensional representations of some linear algebraic group G_d defined over C, the *differential Galois group* of (M, ∇).

2.2. **Unification via the non-commutative world.** [1, §1.4] We propose to *unify the difference and the differential setting using non-commutative connections*. The base ring R will still be commutative, but Ω^1 is now allowed to be a non-commutative R-R-bimodule, *i.e.* the left and right actions by elements of R always commute but do not necessarily coincide. Our modules M are ordinary R-modules, viewed as left R-modules (or commutative R-R-bimodules if one wishes). A *connection* $\nabla : M \to \Omega^1 \otimes_R M$ is as above a C-linear map satisfying the Leibniz rule (on the left) :

$$\nabla(am) = da \otimes m + a \otimes \nabla(m).$$

Let us first explain how non-commutative bimodules naturally occur in our story. The operator d_q introduced above (which $\to \frac{d}{dx}$ when $q \to 1$) is not a derivation, but a σ_q-derivation (σ_q being as above the dilatation by q) : it satisfies the rule

$$\delta(xy) = \delta(x)y + x^\sigma \delta(y).$$

Derivations are "classified" by the module of Kähler differentials, and similarly σ derivations are "classified" by a certain non-commutative R-R-bimodule Ω^1_σ of "twisted Kähler differentials", which satisfies $x\omega = \omega x^\sigma, \forall \omega \in \Omega^1_\sigma$ ([1, §1.4.2.1]).

In this approach, the difference module (M, Φ) is replaced by the non-commutative connection

$$\nabla : M \to \Omega^1_\sigma \otimes_R M, \quad \nabla(m) = dx \otimes \frac{\Phi - id_M}{x^\sigma - x}(m).$$

2.3. **The problem of the tensor product.** The first difficulty which one encounters in this approach is to define the tensor product of two (non-commutative) connections. The problem lies in the fact that in the classical formula

$$(M_1, \nabla_1) \otimes (M_2, \nabla_2) = (M_1 \otimes M_2, \nabla_1 \otimes id_{M_2} + id_{M_1} \otimes \nabla_2),$$

there is an implicit switch $M_1 \otimes_R \Omega^1 \to \Omega^1 \otimes_R M_1$ in the second term of the right hand side. Whereas such a switch goes without saying in the classical (commutative) case, it poses a non-trivial problem[2] in our more general situation due to the non-commutativity of the bimodule Ω^1.

The situation is saved thanks to the following little "miracle" :

Proposition 2.3.1. [1, §2.4.1.1] *There is a canonical R-R-bilinear homomorphism $\phi : M \otimes_R \Omega^1 \to \Omega^1 \otimes_R M$, which expresses a substitute for the Leibniz rule on the right : $\phi(m \otimes da) = \nabla(ma) - \nabla(m)a$.*

[2]related to the fact that there is no braided tensor product on the category of bimodules

In the case of the connection attached to a difference module as above, ϕ is simply given by the formula

$$\phi(m \otimes dx) = dx \otimes \Phi(m).$$

Using ϕ to make sense of the classical formula for the tensor product, one does get a *symmetric monoidal category* [1, §2.4.2.2]. An object (M, ∇) has a dual if and only if M is projective of finite type over R and ϕ is invertible. Under suitable conditions on $R \to \Omega^1$ and (M, ∇), one then gets a tannakian category $< (M, \nabla) >$ and a "differential Galois group".

This theory unifies the difference, differential and mixed cases, and much more. It works independently of any assumption of "integrability", and the Picard-Vessiot viewpoint can be generalized to this degree of generality [1, §3].

2.4. **The theory with parameters; variation of the Galois groups in function of the parameters, and confluence.** One can develop a relative theory, in which one abandons the assumption that $C = \ker d$ is a field. Instead, C is now viewed as a ring of parameters, *cf.* [1, §3.1, §3.2, §3.3]. In our original situation of the confluence of a system of q-difference equations to a system of differential equations, C can be $\mathbf{C}[[q-1]]$, or else, in a more analytic setting, the ring of analytic functions in some neighborhood of 1 in the domain $|q| > 1$.

More precisely, let us assume that the dependence of $A_q(x)$ in q is analytic, while the dependence in x is rational. We fix a point $x = x_0$ and look at solutions around x_0 (this fixes an embedding of every G_q into $GL(n)_C$.

Theorem 2.4.1. [1, §3.3.2.4] *$G_q = G$ is constant except for countably many values $q_1, q_2, \ldots$ of q, and for these values $G_{q_n} \subset G$. In particular, $G_1 \subset G_q$ for "generic" q.*

More generally, when the situation involves several parameters, the result is that the exceptional values of q form a countable set of analytic subvarieties (for which the Galois group is smaller). In fact, there is a stratified situation : these subvarieties admit themselves a countable set of analytic subvarieties for which the Galois group is smaller, and so on[3].

The second assertion in the theorem can be used to compute q-difference Galois groups for generic q, from the knowledge of the differential Galois group for $q \to 1$, provided the latter is big enough. This works notably for hypergeometric q-difference equations, *cf.* [1, §3.3.3.1]. Note that in this case, the differential Galois group is computed using the local monodromy at 1 (which is a pseudo-reflection), a tool which is not available in the q-difference case.

[3]note that this is not exactly a semi-continuity theorem

Remark. The setting of non-commutative connections is also suitable to account for the differential and (contiguity) difference equations satified by hypergeometric series viewed as functions of their arguments and of their parameters simultaneously, and to study the usual phenomena of confluence.

3. ANALYTIC APPROACH [8]

3.1. **Position of the problem.** It is an old and well-known theorem of Schlesinger that the differential Galois group of any *fuchsian* differential system of rank n on a punctured Riemann sphere $\mathbf{CP}^1$ can be computed using monodromy : it is the smallest algebraic subgroup of $GL(n)$ which contains the local monodromy matrices around every puncture (a base point being fixed).

The analytic approach to q-deformations of fuchsian differential systems consists in

i) elucidating the q-analogues of the local monodromies and of Schlesinger's theorem,

ii) studying the limit of the q-analogue of monodromy when $q \longrightarrow 1$.

Point i) had already been tackled by G. Birkhoff long ago (*cf. e.g.* [3])[4], but had been a little forgotten until Sauloy made a fresh start on these questions and also solved point ii).

In order to avoid extreme difficulties with "small divisors", it is essential not to cross the circle $|q| = 1$. Sauloy works exclusively on the domain

$$|q| > 1.$$

3.2. **The local situation at** 0 **or** ∞ **; canonical solutions.** We assume that our q-difference system $(*)_q$ has coefficients in the field $\mathbf{C}(\{x\})$ of germs of meromorphic functions at 0 (or symmetrically, in $\mathbf{C}(\{1/x\})$). The fuchsian condition at 0 means that $A(0)$ is well-defined : on the equivalent system $(*)'_q$, this means that B_q has at worst a simple pole at 0, which reflects the usual notion of fuchsianity at 0 at the limit $q \longrightarrow 1$.

It turns out that, up to "gauge transformations", the local fuchsian differential equations at 0 are equivalent to flat vector bundles on the elliptic curve $E_q = \mathbf{C}^*/q^{\mathbf{Z}}$, *cf.* [2][9].

But for our purpose[5], instead of working up to gauge transformation, it is essential to choose and describe "canonical" solutions of the q-difference system under consideration.

[4] as pointed out in the final chapter of [11], Birkhoff's work on q-difference equations is marred by some mistakes arising from his choice of ramified canonical solutions ; this has now been completely fixed by Sauloy, who uses only unramified solutions instead

[5] for instance, in order to be able to glue the local pictures at 0 and at ∞

Sauloy chooses to work only with functions which are *meromorphic on* $\mathbf{C}^*$. For instance, he uses the theta function

$$\Theta_q(x) = \sum_{m\in\mathbf{Z}} (-1)^m q^{-m(m-1)/2} x^m$$

and the related functions

$$e_{q,c}(x) = \Theta_q(x)/\Theta_q(c^{-1}x), \quad \ell_q(x) = x\Theta_q'(x)/\Theta_q(x).$$

One has the functional equations

$$\sigma_q(e_{q,c}) = ce_{q,c}, \; \sigma_q(\ell_q) = \ell_q + 1,$$

$$\text{or equivalently} \quad d_q(e_{q,c}) = \frac{c-1}{(q-1)x} e_{q,c}, \; d_q(\ell_q) = \frac{1}{(q-1)x},$$

which allow one to consider $e_{q,c}$ and ℓ_q as (unramified !) q-analogues of x^γ and of the logarithm respectively. From these building blocks, one defines the matrix $e_{q,C}$ for any matrix C, using Jordan's multiplicative decomposition of C into its semi-simple and unipotent parts.

Proposition 3.2.1. [7] *Let us assume for simplicity that no two different eigenvalues of* $A(0)$ *have their quotient in* $q^{\mathbf{Z}}$ *(non-resonance at* 0*). Then Sauloy* [7] *shows that there is a canonical solution of* $(*)_q$ *of the form* $X^0 = M^0.e_{q,A(0)}$, *with* $M \in GL(n, \mathcal{M}(\mathbf{C}))$ *(meromorphic entries on* $\mathbf{C}$*),* $M^0(0) = id$.

3.3. **The global (fuchsian) situation.** Let us consider the global fuchsian situation, where the entries of $A_q(x)$ are rational in x, and the fuchsian conditions are fulfilled at 0 and at ∞ (*i.e.* A_q has no pole at 0 and ∞). We also assume the condition of non-resonance at 0 and at ∞ for simplicity.

Let $\{x_{q,i}\}$ denote the set of poles of A_q and zeroes of $\det A_q$. Then M^0 (and symmetrically M^∞) has no pole in $\mathbf{C}^*$ outside the discrete spirals $q^{\mathbf{Z}}.x_i$.

Since all the functions involved here are meromorphic on $\mathbf{C}^*$, there is no monodromy ; *what replaces monodromy are the discrete logarithmic spirals* $q^{\mathbf{Z}}.x_{q,i}$, $q^{\mathbf{Z}}$, and $q^{\mathbf{Z}}.c$ (where c runs among the eigenvalues of $A(0)$ and of $A(\infty)$).

Following Birkhoff's idea, one can compare the canonical solution X^0 at 0 and the canonical solution X^∞ at ∞ : the "ratio"

$$P_q = (X^\infty)^{-1}X^0$$

(*Birkhoff's connection matrix*) is an element of $GL(n, \mathcal{M}(\mathbf{C}^*))$ invariant under the dilatation σ_q, hence an element of $GL(n, \mathcal{M}(E_q))$, where $\mathcal{M}(E_q)$ denotes the field of meromorphic functions on the elliptic curve $E_q = \mathbf{C}^*/q^{\mathbf{Z}}$. It turns out that global fuchsian q-difference system (up to gauge transformations) are classified by the triples $(e_{q,A(0)}, e_{q,A(\infty)}, P_q)$ [7][9].

In the special case where $A(0) = A(\infty) = id$ (so that $e_{q,A(0)} = e_{q,A(\infty)} = id$), the difference Galois group can be computed in terms of P_q, in analogy with Schlesinger's theorem : it is the smallest algebraic subgroup of $GL(n)$ which contains all meaningful evaluations $P_q(a)^{-1}P_q(b)$ [5]. This extends to the general fuchsian case, using two-pointed Galois groupoïds instead of Galois groups [9].

3.4. **Confluence ; connections matrices P_q at the limit.** It is thus tempting to examine what happens to the connection matrices P_q when $q \to 1$ and how monodromy arises from discrete logarithms spirals of poles in the limit. This has been fully elucidated by Sauloy in the striking note [8].

One fixes a value q_0, $|q_0| > 1$, and let q move along a real logarithmic spiral $q = q_0^{\varepsilon}$, $\varepsilon > 0$. One then has the limit formulas

$$\lim_{\varepsilon \to 0^+} e_{q,q^\gamma}(x) = (-x)^\gamma, \quad \lim_{\varepsilon \to 0^+} (q-1)\ell_q(x) = \log(-x).$$

On the other hand, the discrete spirals $q^{\mathbf{Z}}$, $q^{\mathbf{Z}}.x_{q,i}$ become real logarithmic spirals $q_0^{\mathbf{R}}.x_i$ at the limit $\varepsilon \to 0^+$; here, we have set $x_0 = 1$ and $x_1, x_2, \ldots$ denote the singularity of the limit differential system $(*)_1$. These spirals start at 0 and end at ∞. Let us denote by $\mathcal{S}$ the union of them.

We assume that these spirals do not intersect[6], so that, by renumbering the $x_1, x_2, \ldots$, we may assume that $q_0^{\mathbf{R}}.x_i$ and $q_0^{\mathbf{R}}.x_{i+1}$ form the boundary of a domain U_i (a slice of the Riemann sphere), and that $\mathbf{C}^* \setminus \mathcal{S} = \bigcup U_i$.

In order to ensure a clean confluence, we assume, in addition to fuchsianity and non-resonance, that the Jordan structure of $A_q(0)$ and $A_q(\infty)$ moves flatly with q, and that $B_q \to B$ uniformly on every compact of $\mathbf{C}^* \setminus \mathcal{S}$.

Theorem 3.4.1. [8] *Under these assumptions,*

a) the canonical solution X_q^0 (resp. X_q^∞) of $(*)_q$ tends to a canonical solution X^0 (resp. X^∞) of $(*)_1$ uniformly on every compact of $\mathbf{C}^* \setminus \mathcal{S}$,

b) $\lim_{q\to 1} P_q$ *exists on* $\mathbf{C}^* \setminus \mathcal{S}$ *and is a constant matrix $P_{(i)}$ in each slice U_i,*

c) $P_{(i)}^{-1} P_{(i-1)}$ is the local monodromy around x_i (expressed in the basis X^0).

As an illustration, all this is worked out in detail in [8] for q-deformations of ${}_2F_1$.

Remark. Recently, J.P. Ramis, Sauloy and C. Zhang have studied q-analogues of the Stokes matrices in the same spirit, and extended part of the above theory to the non-fuchsian case.

[6]this may be achieved by choosing q_0 sufficiently general

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