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Singular solutions of Nonlinear Fuchsian Equations and Applications to Normal Form Theory (Deformation of differential equations and asymptotic analysis)

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Singular solutions of Nonlinear Fuchsian Equations and Applications to Normal Form Theory

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Motivation and Examples

Vector fields with an isolated singular point

Let us consider the following vector field with an isolated singular point at the origin

\[ \mathcal{X}(x) = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j}, \]

where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) or \( \mathbb{C}^n \), and \( a_j(x) \) is smooth in \( x \). Namely we assume

\[ \mathcal{X}(0) = 0, \]

and \( \mathcal{X} \) does not vanish in some neighborhood of \( x = 0 \) except for the origin.

Linearization and Homology Equation

We want to linearize \( \mathcal{X}(x) \) by a change of variables

\[ x = y + v(y), \quad v = O(|y|^2). \]

We write \( \mathcal{X}(x) \) in the form

\[ \mathcal{X}(x) = x \Lambda \frac{\partial}{\partial x} + R(x) \frac{\partial}{\partial x} \equiv X(x) \frac{\partial}{\partial x}, \]

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\[
\frac{\partial}{\partial x} = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right),
\]
(7)  \[X(x) = x\Lambda + R(x),\]

where

(8)  \[R(x) = (R_1(x), \ldots, R_n(x)), \quad R(x) = O(|x|^2),\]

and \(\Lambda\) is an \(n \times n\) constant matrix.

Noting that

\[
X(x) \frac{\partial}{\partial x} = X(y + v(y)) \frac{\partial y}{\partial y} \frac{\partial}{\partial y}
= X(y + v(y)) \left( \frac{\partial x}{\partial y} \right)^{-1} \frac{\partial}{\partial y},
\]

the linearization condition can be written in the following form

\[X(y + v)(1 + \partial_y v)^{-1} = y\Lambda.\]

Therefore

(9)  \[(y + v)\Lambda + R(y + v) = y\Lambda(1 + \partial_y v) = y\Lambda + y\Lambda\partial_y v.
\]

Hence \(v\) satisfies the so-called homology equation

\[(*) \quad \mathcal{L}v \equiv y\Lambda\partial_y v - v\Lambda = R(y + v(y)), \quad v = (v_1, \ldots, v_n).
\]

Summing up we obtain

The necessary and sufficient condition for that (*) has a solution \(v\) is that \(X\) is linearized by the change of substitution \(x = y + v(y)\).

Expression of a homology equation

We assume that \(\Lambda\) is in a diagonal matrix, namely

(10)  \[\Lambda = \begin{pmatrix}
\lambda_1 & 0 \\
\vdots & \ddots \\
0 & \lambda_n
\end{pmatrix}.
\]

Noting that

\[y\Lambda\partial_y = \sum_{k=1}^{n} \lambda_k y_k \frac{\partial}{\partial y_k}\]
we obtain

\[ \mathcal{L}v = \left( \begin{array}{ccc} \sum \lambda_k y_k \frac{\partial}{\partial y_k} - \lambda_1 & 0 & \cdots \\ 0 & \ddots & \vdots \\ \vdots & \ddots & \sum \lambda_k y_k \frac{\partial}{\partial y_n} - \lambda_n \end{array} \right) \left( \begin{array}{c} v_1 \\ \vdots \\ v_n \end{array} \right). \]

In the following, for the sake of simplicity we always assume that a homology equation has the above expression.

**Non-resonant condition**

The **indicial polynomial** of \( \mathcal{L} \) is given by

\[ \sum_{k=1}^{n} \lambda_k \zeta_k - \lambda_j, \quad (j = 1, \ldots, n). \]

\( \mathcal{L} \) is said to be **non-resonant** if

\[ \sum_{k=1}^{n} \lambda_k \alpha_k - \lambda_j \neq 0 \]

for \( \forall \alpha \in (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n, \ |\alpha| \geq 2, \text{ and } j = 1, \ldots, n. \)

If (13) does not hold we say that \( \mathcal{L} \) is **resonant**. The set of \( y^\alpha \) with \( \alpha \) not satisfying (13) for some \( j \) is called a **resonance**. We have

Under non-resonant condition there exists a formal power series solution.

Indeed, \( \mathcal{L}v = f \) is written in

\[ \mathcal{L}(\sum_\alpha v_\alpha y^\alpha) = \sum_\alpha (\sum_{k=1}^{n} \lambda_k \alpha_k - \Lambda)v_\alpha y^\alpha = \sum_\alpha f_\alpha y^\alpha. \]

Because \( (\sum_{k=1}^{n} \lambda_k \alpha_k - \Lambda) \) is invertible \( \mathcal{L}^{-1} \) exists. Because \( R(x) = O(|x|^2) \) we can determine a formal power series solution by a method of indeterminate coefficients.

**Two theorems for the solvability of a homology equation**
**Poincaré** introduced a famous **Poincaré condition**

\[ \text{Re } \lambda_j > 0, \quad j = 1, \ldots, n \]

and showed the solvability of \((*)\) in a class of analytic functions.

**Solvability of \((*)\) in a real domain**

**Theorem (Sternberg)**  Assume the hyperbolic condition

\[(14) \quad \text{Re } \lambda_k \neq 0, \quad k = 1, \ldots, n.\]

Moreover, suppose the non-resonant condition. Then \((*)\) has a smooth solution.

If resonance occurs we have

**Theorem (Grobman-Hartman)**  Assume the hyperbolicity. Then \((*)\) has a continuous solution.

**Remark** A continuous solution of \((*)\) is defined as a weak solution. The definition of a weak solution is standard. There are extensions of this result to the \(C^k (k \geq 0)\) case by Blitskiy et. al for a certain class of vector fields with resonances.

**Object of Study**

We want to solve \((*)\) in the case of resonances in a class of functions with a "log" type singularity. We also want to solve \((*)\) in a class of functions holomorphic in the domain which is a product of sectors with vertex at the origin.

**Statement of the results**

**Singular solutions**

**Theorem 1.**  Assume the Poincaré condition and

\[ \forall i, j, k, \quad \lambda_i + \lambda_j \neq \lambda_k. \]

Then Eq. \((*)\) has a solution \(v\) of the form

\[ v(y) = \sum_{|\alpha| \geq 2, \alpha \geq \beta} v_{\alpha \beta} y^\alpha (\log y)^\beta, \]
where $(\log y)^{\beta} = \prod_{j=1}^{n}(\log y_{j})^{\beta_{j}}$. $v(y)$ converges in 
$$\{ y \in C^{n}; |y| < \exists \varepsilon, |y_{j} \log y_{j}| < \varepsilon (j = 1, \ldots, n) \}.$$

**Remark.** If there is no resonance the above solution is a classical solution constructed by Poincaré.

If we restrict the solution $v$ to the real domain we obtain a finitely smooth solution of $(\ast)$. Hence a finite smoothness occurs because of the log type singularity caused by the resonance.

**Example** Consider the case $n = 2$. Let $m \geq 2$ be an integer. Let us consider

$$L_{1} = x_{1}\partial_{1} + mx_{2}\partial_{2} - 1, \quad L_{2} = x_{1}\partial_{1} + mx_{2}\partial_{2} - m.$$  

The only resonance is $(\alpha_{1}, \alpha_{2}) = (m, 0)$. The solution $v$ has singularity of $\log x_{1}$ type.

Indeed, the resonance $\alpha = (\alpha_{1}, \alpha_{2}) \in \mathbb{Z}^{2}_{+}$ satisfies $\alpha_{1} + \alpha_{2} \geq 2$ and

$$\alpha_{1} + m\alpha_{2} - 1 = 0, \quad \text{or} \quad \alpha_{1} + m\alpha_{2} = m.$$  

Since $\alpha_{1} + m\alpha_{2} - 1 \neq 0$ by assumption we obtain $\alpha_{1} + m\alpha_{2} = m$ and $\alpha_{1} + \alpha_{2} \geq 2$. It follows that $(\alpha_{1}, \alpha_{2}) = (m, 0)$.

**Sketch of the proof of Theorem 1.** For the sake of simplicity we will prove the above example. We will construct a formal solution of $(\ast)$ in the following form

$$u_{j}(x) = \sum_{\alpha \in \mathbb{Z}_{+}^{2}, |\alpha| \geq 2, k} u_{\alpha,k}^{j} x^{\alpha}(\log x_{1})^{k}, \quad j = 1, 2.$$  

The equation $(\ast)$ can be written in the following form

$$(\ast) \quad L_{j}u_{j} = R_{j}(x_{1} + u_{1}, x_{2} + u_{2}), \quad j = 1, 2.$$  

We set $u_{\alpha,k} = (u_{\alpha,k}^{1}, u_{\alpha,k}^{2})$. We determine $u_{\alpha,k}$ $k = 0, 1, 2, \ldots$ inductively. We determine $u_{\alpha,0}$. By comparing the coefficients we can determine $u_{\alpha,0}$ for $|\alpha| \leq m, \alpha \neq (m, 0)$. On the other hand we note

$$L_{2}(x_{1}^{m}) = 0, \quad L_{2}(x_{1}^{m}\log x_{1}) = x_{1}^{m}.$$
Hence we set $u_{(m,0),0}^{2} = 0$, $u_{(m,0),0} = (u_{(m,0),0}^{1}, 0)$. We note that we can determine $u_{(m,0),0}^{1}$ and $u_{(m,0),1}$ by comparing the coefficients of $x_1^m$ in (*) since $L_1$ has the nonresonance property. It is clear that we can determine $u_{\alpha,0}$ for $|\alpha| > m$ from (*) because there is no resonance for $|\alpha| > m$.

We next determine $u_{\alpha,1}$. We have already determined $u_{(m,0),1} = (0, u_{(m,0),1}^{2})$. By the nonresonance property we can determine $u_{\alpha,1}$ for $|\alpha| > m$. Inductively, $u_{\alpha,2}$ ($|\alpha| = 2m$) can be determined by comparing the coefficients of $x_1^{2m}(\log x_1)^2$. The terms $u_{\alpha,2}$ ($|\alpha| > 2m$) can be determined inductively by the nonresonance property. Inductively, we can determine $u_{\alpha,k}$ ($k = 0, 1, 2, \ldots$). Hence we can determine a formal power series solution.

The convergence can be proved by the method of majorant series. This ends the proof.

Solvability in the sectorial domain

Let $S_0$ be a sector in the complex plane, $S_0 := \{z; |\arg z| < \theta\}$, where $\theta > 0$ is a given small number and the branch of $\arg z$ is taken so that the argument is zero on the real axis. We define a sectorial domain $S$ in $\mathbb{C}^n$ as the product of $n$ copies of $S_0$, $S = S_0 \times \cdots \times S_0$. In the following we consider the solvability of the equation (*) in the sectorial domain $S$.

The typical example of the nonlinear term $R(x)$ is the following:

$$R(x) = A \prod_{j=1}^{n} \frac{x_j^{\alpha_j}}{(x_j - c_j)^{\beta_j}},$$

where $A$, $c_j \in \mathbb{C} \setminus \overline{S}$, $0 < \alpha_j < \beta_j$ ($j = 1, \ldots, n$) are constants. We set $\lambda := (\lambda_1, \ldots, \lambda_n)$. Then we have

**Theorem 2.** Suppose that

$$\lambda_j \in \mathbb{R} \setminus 0 \quad (j = 1, \ldots, n).$$

Let $\Gamma \subset \mathbb{R}^n$ be an open set such that $0 \in \Gamma$ and

$$\Gamma \cap \{\eta; (\lambda, \eta) = \lambda_j\} = \emptyset,$$

for every $j = 1, \ldots, n$, where $(\lambda, \eta) = \sum_{k=1}^{n} \lambda_k \eta_k$. Suppose that, for every $\eta \in \Gamma$,

$$R(x) = O(x^{-\eta}), \ (\text{when } x \to 0 \text{ or } x \to \infty, x \in S).$$
Then there exists $\epsilon > 0$ such that if $\sup_{x \in S} |R(x)| < \epsilon$ the equation (*) has a solution $u$ holomorphic in $S$. Moreover, for every $\eta \in \Gamma$, $u$ behaves like $O(x^{-\eta})$ when $x \to 0$ or $x \to \infty$.

Example. For $R(x)$ in the above example the conditions in the theorem are fulfilled if $\Gamma$ is a sufficiently small neighborhood of the origin and $A$ is sufficiently small.

References