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On the Number of Poles of the First Painlevé Transcendents and Higher Order Analogues

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Let \( w(z) \) be an arbitrary solution of the first Painlevé equation

\[
(\text{PI}) \quad w'' = 6w^2 + z.
\]

Then, \( w(z) \) is a transcendental meromorphic function, and every pole is double. Denote by \( n(r, w) \) the number of poles inside the circle \( |z| < r \). In this note, we prove the following:

**Theorem A.** The growth order of \( w(z) \) is not less than 5/2, namely

\[
\lim_{r \to \infty} \sup \frac{\log n(r, w)}{\log r} \geq \frac{5}{2}.
\]

For another proof of this result, see [2].

It is known that the equations

\[
(\text{PI}_4) \quad w^{(4)} = 20ww'' + 10(w')^2 - 40w^3 + 16z,
\]

\[
(\text{PI}_6) \quad w^{(6)} = 28ww^{(4)} + 56w'w^{(3)} + 42(w')^2 - 280(w^2w'' + w(w')^2 - w^4) + 64z
\]

are higher order analogues for (PI). Denote by \( w_4(z) \) (resp. \( w_6(z) \)) an arbitrary meromorphic solution of (PI)_4 (resp. (PI)_6). It is easy to see that \( w_4(z) \) (resp. \( w_6(z) \)) is transcendental and every pole is double. The following result is proved by the same argument as in the proof of Theorem A.

**Theorem B.** We have

\[
\lim_{r \to \infty} \sup \frac{\log n(r, w_4)}{\log r} \geq \frac{7}{3},
\]

\[
\lim_{r \to \infty} \sup \frac{\log n(r, w_6)}{\log r} \geq \frac{9}{4}.
\]

**Remark.** For solutions of (PI), a more precise result is known (see [3], [4]):

\[
\frac{r^{5/2}}{\log r} \ll n(r, w) \ll r^{5/2}.
\]

(We write \( f(r) \ll g(r) \) if \( f(r) = O(g(r)) \) as \( r \to \infty \).

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1. Proof of Theorem A

In what follows, for simplicity, we use the abbreviation $n(r) := n(r, w)$. To prove (1), we suppose the contrary:

\[(5) \quad \limsup_{r \to \infty} \frac{\log n(r)}{\log r} < \frac{5}{2},\]

namely, for some $\varepsilon > 0$,

\[(6) \quad n(r) \ll r^{5/2-\varepsilon}.\]

Starting from this supposition, we would like to derive a contradiction. By $\{a_j\}_{j=1}^{\infty}$ we denote the distinct poles of $w(z)$ arranged as $|a_1| \leq \cdots \leq |a_j| \leq \cdots$ (by a Clunie reasoning ([1, §9.2]), $w(z)$ has infinitely many poles). By virtue of (6), $w(z)$ is written in the form

\[(7) \quad w(z) = \Phi(z) + \phi(z),\]
\[(8) \quad \Phi(z) = \sum_{a_j} ((z - a_j)^{-2} - a_j^{-2}),\]

where $\phi(z)$ is an entire function; in the right-hand side of (8), if $a_1 = 0$, the term $(z - a_1)^{-2} - a_1^{-2}$ should be replaced by $z^{-2}$. Under supposition (6), we have the following lemmas whose proofs will be given afterward:

**Lemma 1.1.** For arbitrary $r > 1$, there exists $z_0$ such that

\[0.7r \leq |z_0| \leq r, \quad \sum_{|a_j| < 2r} |z_0 - a_j|^{-2} \ll r^{1/2-\varepsilon/2}.\]

**Lemma 1.2.** We have, for $|z| \leq r$,

\[\sum_{|a_j| \geq 2r} |(z - a_j)^{-2} - a_j^{-2}| \ll r^{1/2-\varepsilon}, \quad \sum_{|a_j| \geq 2r} |z - a_j|^{-4} \ll 1,\]

and

\[\sum_{|a_j| < 2r} |a_j^{-2}| \ll r^{1/2-\varepsilon}.\]

**Lemma 1.3.** There exists a set $E^* \subset (0, \infty)$ with finite linear measure such that

\[\sum_{a_j} |(z - a_j)^{-2} - a_j^{-2}| \ll |z|^9 \quad \text{for } |z| \in (0, \infty) \setminus E^*.\]

Observing that $6w(z) = w''(z)/w(z) - z/w(z)$, we have

\[m(r, w) \ll m(r, w''/w) + \log r \ll \log r,\]

where

\[m(r, w) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |w(re^{i\theta})| d\theta, \quad \log^+ x = \max\{0, \log x\} \]
(for the notation and basic results in the Nevanlinna theory, see [1]). By Lemma 1.3, for \( r \in (0, \infty) \setminus E^* \),

\[
T(r, \phi) = m(r, \phi) = m(r, w - \Phi) \leq m(r, w) + m(r, \Phi) \ll \log r.
\]

This implies that \( \phi(z) \in \mathbb{C}[z] \). Note that \( |\Phi(z)| \leq \sum |a_j| \leq 2r \) and \( |\sum |a_j| > 2r| \). By Lemmas 1.1 and 1.2, for every \( r > 1 \), there exists \( z_0 \), \( 0.7r \leq |z_0| \leq r \) such that

\[
|\Phi(z_0)| \ll r^{1/2-\epsilon/2}, \quad |\Phi''(z_0)| \ll r^{-\epsilon}.
\]

Combining \( w(z_0) = (w''(z_0) - z_0)^{1/2}/\sqrt{6} \) with these estimates, we have

\[
|\phi(z_0)| \ll |\Phi(z_0)| + (|w''(z_0)| + |z_0|)^{1/2} \ll r^{1/2} + |\phi(z_0)|^{1/2},
\]

which implies that \( \phi(z) \equiv C \in \mathbb{C} \). Hence, from \( z_0 = w''(z_0) - 6w(z_0)^2 \), it follows that

\[
0.7r \leq |z_0| \ll |w''(z_0)| + 6|w(z_0)|^2 \ll r^{1-\epsilon},
\]

which is a contradiction. We have thus proved Theorem A.

2. Proofs of the lemmas

2.1. Proof of Lemma 1.1. Put \( D_r = \{z \mid |z| < r \} \) and \( \Delta^\delta_r = \mathbb{C} \setminus (\bigcup_{j \geq 0} U^\delta_j) \); where \( U^\delta_j = \{z \mid |z - a_j| < \delta |a_j|^{-1/4} \} \) if \( a_j \neq 0 \), and \( U^\delta_0 = \{z \mid |z| < \delta \} \) if \( a_0 = 0 \). Since, by (6),

\[
\sum_{0 < |a_j| < r} |a_j|^{-1/2} = \int_0^r \rho^{-1/2}dn(\rho) = \left[ \rho^{-1/2}n(\rho) \right]_0^r + \frac{1}{2} \int_0^r \rho^{-3/2}n(\rho)d\rho \ll r^2,
\]

we can take \( \delta \) so small that \( 3\pi r^2/4 \leq \mu(\Delta^\delta_r \cap D_r) < \pi r^2 \) for every \( r > 1 \), where \( \mu(X) \) denotes the area of a domain \( X \). It is easy to see that

\[
\iint_{D_r \setminus \bigcup_{j \leq r} U^\delta_j} \frac{dx\,dy}{|z-a_j|^2} \ll \int_0^r \rho^{-1}d\rho d\theta \ll \log r,
\]

if \( |a_j| < 2r \), and if \( r > 1 \); and hence

(9)

\[
\iint_{\Delta^\delta_r \cap D_r \setminus \{a_j| < 2r\}} |z-a_j|^{-2}dxdy \ll n(2r\log r) \leq K_0 r^{5/2-\epsilon/2},
\]

where \( K_0 \) is some positive number. Now consider the set

\[
E_r = \{z \in \Delta^\delta_r \cap D_r \mid \sum_{\{|a_j| < 2r\}} |z-a_j|^{-2} \leq 4\pi^{-1} K_0 r^{1/2-\epsilon/2} \}.
\]

Suppose that \( \mu(E_r) < \pi r^2/2 \). Then

\[
\iint_{\Delta^\delta_r \cap D_r \setminus E_r \setminus \{|a_j| < 2r\}} |z-a_j|^{-2}dxdy > 4\pi^{-1} K_0 r^{1/2-\epsilon/2} \left( \frac{3\pi r^2}{4} - \frac{\pi r^2}{2} \right) = K_0 r^{5/2-\epsilon/2},
\]

which contradicts (9). Hence \( \mu(E_r) \geq \pi r^2/2 \). Since \( \mu(\{z \mid |z| < 0.7r\}) = 0.49\pi r^2 \), we have \( \{z \mid 0.7r \leq |z| \leq r\} \cap E_r \neq \emptyset \), which implies the conclusion.
2.2. Proof of Lemma 1.2. For $|a_j| \geq 2r$, and for $z \in D_r$, observe that $|z/a_j| \leq 1/2$. Since
\[
| (z - a_j)^{-2} - a_j^{-2} | = 2|z||a_j|^{-3}|1 - (z/a_j)/2||1 - z/a_j|^{-2} \leq 10r|a_j|^{-3},
\]
we have, by (6), that
\[
\sum_{|a_j| \geq 2r} | (z - a_j)^{-2} - a_j^{-2} | \ll r \sum_{|a_j| \geq 2r} |a_j|^{-3} \ll r \int_{2r}^{\infty} t^{-3} \, dn(t)
\]
\[
\ll r \int_{2r}^{\infty} t^{-4} n(t) \, dt \ll r^{1/2 - \epsilon},
\]
and that
\[
\sum_{|a_j| < 2r} |a_j^{-2}| = \int_{0}^{2r} t^{-2} \, dn(t) \ll r^{1/2 - \epsilon} + \int_{0}^{2r} t^{-3} n(t) \, dt \ll r^{1/2 - \epsilon}.
\]

2.3. Proof of Lemma 1.3. We put
\[
E^* = (0, |a_1| + 1) \cup \left( \bigcup_{j=2}^{\infty} (|a_j| - |a_j|^{-3}, |a_j| + |a_j|^{-3}) \right).
\]
By (6), the total length of $E^*$ is finite. If $|z| \not\in E^*$, then
\[
\left( \sum_{0 < |a_j| < 2|z|} + \sum_{|a_j| \geq 2|z|} \right) | (z - a_j)^{-2} - a_j^{-2} | \ll (|z|^6 + 1)n(2|z|) + |z|^{1/2} \ll |z|^9.
\]

References