<table>
<thead>
<tr>
<th>Title</th>
<th>On the Number of Poles of the First Painleve Transcendents and Higher Order Analogues (Deformation of differential equations and asymptotic analysis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Shimomura, Shun</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2002), 1296: 124-127</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42642">http://hdl.handle.net/2433/42642</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
<tr>
<td>Textversion</td>
<td></td>
</tr>
</tbody>
</table>
On the Number of Poles of the First Painlevé Transcendents and Higher Order Analogues

SHUN SHIMOMURA

Department of Mathematics, Keio University

Let $w(z)$ be an arbitrary solution of the first Painlevé equation

\[(PI) \quad w'' = 6w^2 + z.\]

Then, $w(z)$ is a transcendental meromorphic function, and every pole is double. Denote by $n(r,w)$ the number of poles inside the circle $|z| < r$. In this note, we prove the following:

**Theorem A.** The growth order of $w(z)$ is not less than $5/2$, namely

\[(1) \quad \lim_{r \to \infty} \sup \frac{\log n(r,w)}{\log r} \geq \frac{5}{2}.\]

For another proof of this result, see [2].

It is known that the equations

\[(PI_4) \quad w^{(4)} = 20ww'' + 10(w')^2 - 40w^3 + 16z,\]

\[(PI_6) \quad w^{(6)} = 28ww^{(4)} + 56w'w^{(3)} + 42(w')^2 - 280(w^2w' + w(w')^2 - w^4) + 64z\]

are higher order analogues for (PI). Denote by $w_4(z)$ (resp. $w_6(z)$) an arbitrary meromorphic solution of $(PI_4)$ (resp. $(PI_6)$). It is easy to see that $w_4(z)$ (resp. $w_6(z)$) is transcendental and every pole is double. The following result is proved by the same argument as in the proof of Theorem A.

**Theorem B.** We have

\[(2) \quad \lim_{r \to \infty} \sup \frac{\log n(r,w_4)}{\log r} \geq \frac{7}{3},\]

\[(3) \quad \lim_{r \to \infty} \sup \frac{\log n(r,w_6)}{\log r} \geq \frac{9}{4}.\]

**Remark.** For solutions of $(PI)$, a more precise result is known (see [3], [4]):

\[(4) \quad \frac{r^{5/2}}{\log r} \ll n(r,w) \ll r^{5/2}.\]

(We write $f(r) \ll g(r)$ if $f(r) = O(g(r))$ as $r \to \infty$.)
1. Proof of Theorem A

In what follows, for simplicity, we use the abbreviation \( n(r) := n(r, w) \). To prove (1), we suppose the contrary:

\[
\limsup_{r \to \infty} \frac{\log n(r)}{\log r} < \frac{5}{2},
\]

namely, for some \( \epsilon > 0 \),

\[
n(r) \ll r^{5/2-\epsilon}.
\]

Starting from this supposition, we would like to derive a contradiction. By \( \{a_j\}_{j=1}^{\infty} \) we denote the distinct poles of \( w(z) \) arranged as \( |a_1| \leq \cdots \leq |a_j| \leq \cdots \) (by a Clunie reasoning ([1, §9.2]), \( w(z) \) has infinitely many poles). By virtue of (6), \( w(z) \) is written in the form

\[
w(z) = \Phi(z) + \phi(z),
\]

\[
\Phi(z) = \sum_{a_j} ((z - a_j)^{-2} - a_j^{-2}),
\]

where \( \phi(z) \) is an entire function; in the right-hand side of (8), if \( a_1 = 0 \), the term \( (z - a_1)^{-2} - a_1^{-2} \) should be replaced by \( z^{-2} \). Under supposition (6), we have the following lemmas whose proofs will be given afterward:

**Lemma 1.1.** For arbitrary \( r > 1 \), there exists \( z_0 \) such that

\[
0.7r \leq |z_0| \leq r, \quad \sum_{|a_j| < 2r} |z_0 - a_j|^{-2} \ll r^{1/2-\epsilon/2}.
\]

**Lemma 1.2.** We have, for \( |z| \leq r \),

\[
\sum_{|a_j| \geq 2r} |(z - a_j)^{-2} - a_j^{-2}| \ll r^{1/2-\epsilon}, \quad \sum_{|a_j| \geq 2r} |z - a_j|^{-4} \ll 1,
\]

and

\[
\sum_{|a_j| < 2r} |a_j^{-2}| \ll r^{1/2-\epsilon}.
\]

**Lemma 1.3.** There exists a set \( E^* \subset (0, \infty) \) with finite linear measure such that

\[
\sum_{a_j} |(z - a_j)^{-2} - a_j^{-2}| \ll |z|^9 \quad \text{for } |z| \in (0, \infty) \setminus E^*.
\]

Observing that \( 6w(z) = w''(z)/w(z) - z/w(z) \), we have

\[
m(r, w) \ll m(r, w''/w) + \log r \ll \log r,
\]

where

\[
m(r, w) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |w(re^{i\theta})| d\theta, \quad \log^+ x = \max\{0, \log x\}
\]
(for the notation and basic results in the Nevanlinna theory, see [1]). By Lemma 1.3, for \( r \in (0, \infty) \setminus E^* \),
\[
T(r, \phi) = m(r, \phi) = m(r, w - \Phi) \leq m(r, w) + m(r, \Phi) \ll \log r.
\]
This implies that \( \phi(z) \in \mathbb{C}[z] \). Note that \( |\Phi(z)| \leq \sum_{|a_j|<2r} + \sum_{|a_j|\geq2r} \). By Lemmas 1.1 and 1.2, for every \( r > 1 \), there exists \( z_0, 0.7r \leq |z_0| \leq r \) such that
\[
|\Phi(z_0)| \ll r^{1/2-\epsilon/2}, \quad |\Phi''(z_0)| \ll r^{-\epsilon}.
\]
Combining \( w(z_0) = (w''(z_0) - z_0)^{1/2}/\sqrt{6} \) with these estimates, we have
\[
|\phi(z_0)| \ll |\Phi(z_0)| + (|w''(z_0)| + |z_0|)^{1/2} \ll r^{1/2} + |\phi(z_0)|^{1/2},
\]
which implies that \( \phi(z) \equiv C \in \mathbb{C} \). Hence, from \( z_0 = w''(z_0) - 6w(z_0)^2 \), it follows that
\[
0.7r \leq |z_0| \ll |w''(z_0)| + 6|w(z_0)|^2 \ll r^{1-\epsilon},
\]
which is a contradiction. We have thus proved Theorem A.

2. Proofs of the lemmas

2.1. Proof of Lemma 1.1. Put \( D_r = \{z \mid |z| < r \} \) and \( \Delta_0^\delta = \mathbb{C} \setminus (\bigcup_{j \geq 0} U_j^\delta) \); where \( U_j^\delta = \{z \mid |z - a_j| < \delta|a_j|^{-1/4}\} \) if \( a_j \neq 0 \), and \( U_0^\delta = \{z \mid |z| < \delta\} \) if \( a_0 = 0 \). Since, by (6),
\[
\sum_{0 < |a_j| < r} |a_j|^{-1/2} = \int_0^r \rho^{-1/2}dn(\rho) = \left[ \rho^{-1/2}n(\rho) \right]_0^r + \frac{1}{2} \int_0^r \rho^{-3/2}n(\rho)d\rho \ll r^2,
\]
we can take \( \delta \) so small that \( 3\pi r^2/4 \leq \mu(\Delta_0^\delta \cap D_r) < \pi r^2 \) for every \( r > 1 \), where \( \mu(X) \) denotes the area of a domain \( X \). It is easy to see that
\[
\int\int_{\Delta_0^\delta \cap D_r} \sum_{|a_j|<2r} |z-a_j|^{-2}dxdy \ll n(2r) \log r \leq K_0 r^{5/2-\epsilon/2},
\]
where \( K_0 \) is some positive number. Now consider the set
\[
E_r = \{z \in \Delta_0^\delta \cap D_r \mid \sum_{|a_j|<2r} |z-a_j|^{-2} \leq 4\pi^{-1}K_0 r^{1/2-\epsilon/2}\}.
\]
Suppose that \( \mu(E_r) < \pi r^2/2 \). Then
\[
\int\int_{\Delta_0^\delta \cap D_r \setminus E_r} \sum_{|a_j|<2r} |z-a_j|^{-2}dxdy > 4\pi^{-1}K_0 r^{1/2-\epsilon/2} \left( \frac{3\pi r^2}{4} - \frac{\pi r^2}{2} \right) = K_0 r^{5/2-\epsilon/2},
\]
which contradicts (9). Hence \( \mu(E_r) \geq \pi r^2/2 \). Since \( \mu(\{z \mid |z| < 0.7r\}) = 0.49\pi r^2 \), we have \( \{z \mid 0.7r \leq |z| \leq r\} \cap E_r \neq \emptyset \), which implies the conclusion.
2.2. Proof of Lemma 1.2. For $|a_j| \geq 2r$, and for $z \in D_r$, observe that $|z/a_j| \leq 1/2$. Since

$$|(z - a_j)^{-2} - a_j^{-2}| = 2|z||a_j|^{-3}|1 - (z/a_j)/2||1 - z/a_j|^{-2} \leq 10r|a_j|^{-3},$$

we have, by (6), that

$$\sum_{|a_j| \geq 2r} |(z - a_j)^{-2} - a_j^{-2}| \ll r \sum_{|a_j| \geq 2r} |a_j|^{-3} \ll r \int_{2r}^{\infty} t^{-3} dt
\ll r \int_{2r}^{\infty} t^{-4} n(t) dt \ll r^{1/2-\epsilon},$$

and that

$$\sum_{|a_j| < 2r} |a_j^{-2}| = \int_0^{2r} t^{-2} dt \ll r^{1/2-\epsilon} + \int_0^{2r} t^{-3} n(t) dt \ll r^{1/2-\epsilon}.$$

2.3. Proof of Lemma 1.3. We put

$$E^* = (0, |a_1| + 1) \cup \left( \bigcup_{j=2}^{\infty} (|a_j| - |a_j|^{-3}, |a_j| + |a_j|^{-3}) \right).$$

By (6), the total length of $E^*$ is finite. If $|z| \notin E^*$, then

$$\left( \sum_{0< |a_j| < 2|z|} + \sum_{|a_j| \geq 2|z|} \right) |(z - a_j)^{-2} - a_j^{-2}| \ll (|z|^6 + 1)n(2|z|) + |z|^{1/2} \ll |z|^9.$$

References