Minimax Theorems of Convexlike Functions

Introduction

In this study we consider fuzzy numbers with bounded supports due to [3] and we treat some type of fuzzy optimization problems, which arise from linear optimization problems and are analyzed under assumptions of the fuzzy goal and fuzzy constraints of decision makers. [5] gives an existence criterion for optimal solutions of the fuzzy optimization problems. In Section 2 the existence of optimal solutions means that there exists at least one solution for systems of inequalities concerning concave functions by applying Ky Fan’s theorem. In Section 3 we show an extension of Ky Fan’s theorem, in which functions are not convex but quasiconvex. In proving the extension we apply fixed point theorems for set-valued mappings. In Section 4 we deal with definitions of convexlike or concavelike functions in the similar way to Chapter 6 in [7] as well as we get minimax theorems under conditions that functions of two variables are lower semi-continuous and quasiconvex in one variable and concavelike in the other.

Existence Criterion

Let us denote by \( \mathbb{R} \) the set of real numbers and \( I = [0,1] \). In [3] the set of fuzzy numbers is characterized by membership functions as follows:

**Definition 1** Let \( \mathcal{F}(\mathbb{R}) \) be the set of fuzzy numbers \( u : \mathbb{R} \rightarrow I \) satisfying the following conditions (i) - (iii) (see [3]):

(i) \( u(\cdot) \) is upper semi-continuous on \( \mathbb{R} \);

(ii) the \( \alpha \)-cut set \( L_\alpha(u) = \{ y \in \mathbb{R} : u(y) \geq \alpha \} \) is bounded for \( \alpha > 0 \) and \( L_0(u) = \bigcup_{0<\alpha \leq 1} L_\alpha(u) \) is bounded;

(iii) \( u(\cdot) \) is fuzzy convex, i.e.,

\[
  u(\lambda y_1 + (1-\lambda)y_2) \geq \min[u(y_1), u(y_2)]
\]

for \( y_i \in \mathbb{R}, i = 1,2 \) and \( \lambda \in \mathbb{R} \) with \( 0 \leq \lambda \leq 1 \);

(iv) there exists one and only one \( m \in \mathbb{R} \) such that \( u(m) = 1 \).

The \( \alpha \)-cut set \( L_\alpha(u) \) is compact in \( \mathbb{R} \) for each \( \alpha \in I \) from the above Conditions (i) and (ii), since (i) means that \( L_\alpha(u) \) is closed for \( \alpha \in I \).
Remark 1 Under the above Conditions (ii) and (iv) the following statements (a)-(d) concerning the the function $u : R \rightarrow I$ are equivalent each other:

(a) $u(\cdot)$ is fuzzy convex;

(b) $L_\alpha(u) \subset L_\beta(u)$ for any $\alpha \in I$;

(c) $u(\cdot)$ is non-decreasing on $(-\infty, m]$ and that $u(\cdot)$ is non-increasing on $[m, \infty)$;

(d) $L_\alpha(u) \subset L_\beta(u)$ for $\alpha > \beta$.

From (a) it is clear that (b) holds. If we suppose that (a) doesn't hold but (b) holds, this leads to a contradiction. It can be shown that (c) leads to (d) and the converse holds. Suppose that for any $m_1 \in R$ with $m_1 > m$ there exist $y_1 < y_2 \leq m_1$ such that $u(y_1) > u(y_2)$ under Condition (ii) and (a). Then it leads to a contradiction. From (c), it follows that (a) holds.

In the following definition we give the quasiconvexity of functions.

Definition 2 Let $C$ be a convex set in a linear space and $f$ a mapping from $C$ to $R$. It is said that $f$ is quasiconvex if $f(\lambda y_1 + (1 - \lambda)y_2) \leq \min[f(y_1), f(y_2)]$ for $y_1 \in C$, $i = 1, 2$ and $0 \leq \lambda \leq 1$. It is said that $f$ is quasiconcave if $f(\lambda y_1 + (1 - \lambda)y_2) \geq \max[f(y_1), f(y_2)]$ for $y_1 \in C$, $i = 1, 2$ and $0 \leq \lambda \leq 1$.

Remark 2 In the same way as in Remark 2.1 it is easily seen that $f : C \rightarrow R$ is quasiconvex if and only if the lower level set $L(f; \gamma) = \{x \in C : f(x) \leq \gamma\}$ is convex for any $\gamma \in R$.

Next we consider the following linear optimization problem (e.g. [4]):

\begin{align*}
\alpha^T x &\leq b_0 & \text{subject to } \alpha^T x \leq b_i, & i = 1, 2, \cdots, m, \ x \geq 0, \quad (2.1)
\end{align*}

where the symbol " $\leq $ " denotes a relaxed or fuzzy version of the ordinary inequality " $\leq $ ".

The first fuzzy inequality (fuzzy goal) means that " the objective function $\alpha^T x$ should be essentially smaller than or equal to an aspiration level $b_0 \in R$ of the decision maker (DM)" and the second (fuzzy constraints of DM) means that " the constraints $\alpha^T x$ should be essentially smaller than or equal to $b_i \in R$, $i = 1, \cdots, m$". Membership functions $u_i \in \mathcal{F}(R)$, $i = 0, 1, \cdots, m$, and it follows that $u_i(y)$ is non-decreasing in $y \in [C_i, b_i]$ and $u_i(y)$ is non-increasing in $y \in [b_i, D_i]$. For $u(x)$ is non-decreasing in $y \in [C_i, b_i]$ and $u_i(y)$ is non-increasing in $y \in [b_i, D_i]$ and $u_i(y) \equiv 0$ elsewhere. Here $C_i \leq b_i \leq D_i$ are constants. Let $u_i$ be concave on the set $[C_i, D_i]$. Put $S_i = \{x \in R^n : C_i \leq \alpha^T x \leq D_i\}$ and $S = \cap_{i=0}^m S_i$.

Then, in order to solve the above problem, we have the following optimization problem:

\begin{align*}
\text{maximize } u(x), \quad (2.3)
\text{where } \quad u(x) = \min_{0 \leq i \leq m} u_i(\alpha^T x). \quad (2.4)
\end{align*}

In [5] we showed the existence criterion for optimal solutions of fuzzy optimization problems as follows:

Theorem 1 Let $u_i(\cdot) \in \mathcal{F}$ for $i = 0, 1, \cdots, m$. The following statements (i) and (ii) hold:

(i) Let $\mu_0 = \max_{x} \min_{i} u_i(\alpha^T x)$. Then we have

\begin{align*}
\mu_0 & = \max \{0 < \alpha \leq 1 : \cap_{i=0}^m L_\alpha(u_i) \neq \emptyset\} \\
& = \sup \{0 < \alpha \leq 1 : \cap_{i=0}^m L_\alpha(u_i) \neq \emptyset\}. 
\end{align*}
(ii) We have at least one optimal solution $z^*$ for $(2.3), (2.4)$, if and only if there exists an $\alpha_0 > 0$ such that

$$\cap_{i=0}^{n}L_{\alpha_0}(u_i) \neq \emptyset.$$ 

The above condition (ii) can be reduced to another type of condition by applying Ky Fan's theorem in [2] as follows:

**Theorem K** Let $C$ be a compact and convex set in a topological linear space. Suppose that functions $f_i : C \rightarrow \mathbb{R}, i = 1, 2, \ldots, n,$ are lower semi-continuous and convex. Let $d \in \mathbb{R}$. Then the following (i) and (ii) are equivalent each other:

(i) There exists an $x_0 \in C$ such that

$$f_i(x_0) \leq d$$

for $i = 1, 2, \ldots, n$;

(ii) for $c = (c_1, \cdots, c_n)$ such that $c_i \geq 0, i = 1, 2, \ldots, n,$ and $\sum_{i=1}^{n} c_i = 1,$ there exists a $y_c \in C$ satisfying

$$\sum_{i=1}^{n} c_i f_i(y_c) \leq d.$$ 

From the above theorem, Problem $(2.3), (2.4)$ has an optimal solution $z^*$ if and only if there exist $0 < \alpha_0 \leq 1$ and $x_0$ such that

$$u_i(a_i^T x_0) \geq \alpha_0$$

for $i = 0, 1, \cdots, m$. Then Problem $(2.3), (2.4)$ has an optimal solution $z^*$, if and only if for some $\alpha_0$ with $0 < \alpha_0 \leq 1$ and $c = (c_0, \cdots, c_m) \in \mathbb{R}^{m+1}$ with $c_i \geq 0, i = 0, 1, \cdots, m,$ there exists a $y_c \in S$ such that

$$\sum_{i=0}^{m} c_i u_i(a_i^T y_c) \geq \alpha_0.$$ 

### 3 Quasiconvex Functions

In this section we suppose the quasiconvexity of membership functions and we show an extension of Ky Fan's theorem by applying the following lemma.

**Lemma 1** Let $C$ be a compact and convex set in a topological linear space $E$. Suppose that a set $A \subset C \times C$ satisfies the following conditions (a) - (c):

(a) The set $\{x \in C : (x, y) \in A\}$ is closed for any $y \in C$;

(b) the set $\{y \in C : (x, y) \not\in A\}$ is convex for any $x \in C$;

(c) for $x \in C$, the point $(x, x) \in A$.

Then there exists some $x_0 \in C$ such that $\{x_0\} \times C \subset A$.

The above Lemma can be proved by applying the following type of fixed points theorem for a class of set-valued mappings (e.g., Theorem 10.3.6 in [1]).

**Theorem 3** Let $E$ be a topological linear space and $C$ a non-empty, compact and convex set in $E$. Let $T$ be a mapping from $C$ to the set of all subsets of $C$. Assume that the image $T(x)$ is non-empty and convex for
each $x \in C$. If for each $y \in C$, the inverse $T^{-1}(y) = \{x \in C : T(x) \ni y\}$ is open, then $T$ has a fixed point in $C$, i.e., there exists an $x_0 \in C$ such that $x_0 \in T(x_0)$.

Proof of Lemma 1

Suppose that for any $x \in C$ there exists a $y \in C$ such that $(x,y) \not\in A$. Denote a set-valued mapping $T$ from $C$ to the set of all subsets of $C$ by $T(x) = \{y \in C : (x,y) \not\in A\}$. The image $T(x) \subset C$ is non-empty and convex from Condition (b) for any $x \in C$. From Condition (a) the set $T^{-1}(y) = \{x \in C : (x,y) \not\in A\}$ is open set in $E$. Then, by applying Theorem 3, $T$ has a fixed point $x_0 \in C$, i.e., $x_0 \in T(x_0)$. It follows that $(x_0,x_0) \not\in A$, which contradicts Condition (c). Thus the conclusion holds.

Q.E.D.

By utilizing the above lemma we think that the following results of an extension of Theorem K can be shown as the below outline of proof.

Extension of Theorem K(ETK)

- Let $f_i : C \rightarrow R$ for $i = 1, \cdots, n$, be lower semi-continuous and quasiconvex, where $C$ is a compact and convex set in a topological linear space $E$ and let $d \in R$. Then the following (i) and (ii) are equivalent each other:
  
  (i) There exists an $x_0 \in C$ such that 
  $$f_i(x_0) \leq d$$
  for $i = 1, 2, \cdots, n$;
  
  (ii) for $c = (c_1, \cdots, c_n)$ such that $c_i \geq 0, i = 1, 2, \cdots, n$, and $\sum_{i=1}^n c_i = 1$, there exists a $y_c \in C$ such that 
  $$\sum_{i=1}^n c_if_i(y_c) \leq d.$$

In the similar way to the discussion of Chapter 6 in [7], we expect that we can prove the above extension.

4 Extensions of Minimax Theorems

[7] gives definitions of convexlike or concavelike functions, which play an important role in proving an extension of minimax theorems under that ETK holds.

Definition 3 Let $C, D$ be two sets and $F$ a mapping from $C \times D$ to $R$. It is said that $F$ is concavelike on $D$ for $x \in C$ if for each $y_1, y_2 \in D$ and $0 < \lambda < 1$, there exists an $y_0 \in D$ such that $F(x, y_0) \geq \lambda F(x, y_1) + (1 - \lambda)F(x, y_2)$. It is said that $F$ is convexlike on $C$ for $y \in D$ if for each $x_1, x_2 \in C$ and $0 < \lambda < 1$, there exists an $x_0 \in C$ such that $F(x_0, y) \leq \lambda F(x_1, y) + (1 - \lambda)F(x_2, y)$.

In what follows we show an extension of minimax theorems concerning concavelike functions.

Extension of Minimax Theorems (ETM)

- Let $C$ be a convex and compact set in a topological linear space and $D$ an arbitrary non-empty set. A function $F : C \times D \rightarrow R$ satisfies the following conditions (i) and (ii).

(i) $F(\cdot,y)$ is lower semi-continuous and quasiconvex on $C$ for $y \in D$.
(ii) $F(x,\cdot)$ is concavelike on $D$ for $x \in C$.

Then it follows that
\[
\sup_{y \in D} \min_{x \in C} F(x,y) = \min_{x \in C} \sup_{y \in D} F(x,y).
\]

Proof. From (i) and the compactness of $C$ there exists $\min_{x \in C} F(x,y)$. Let $c = \sup_{y \in D} \min_{x \in C} F(x,y) < +\infty$. For any $x \in C$, $\{y_1, y_2, \ldots, y_n\} \subset D$ and $\{\lambda_i \geq 0 : \sum_{i=1}^{n} \lambda_i = 1\}$, Condition (ii) means that there exists a $y_0 \in D$ such that $\sum_{i=1}^{n} \lambda_i F(x,y_i) \leq F(x,y_0)$. From (i) there exists an $x_0 \in C$ such that $F(x_0,y_0) = \min_{x \in C} F(x,y_0) \leq c$ and also we have $\sum_{i=1}^{n} \lambda_i F(x,y_i) \leq c$ for any $x \in C$. By Condition (i) and ETK, there exists an $x_1 \in C$ such that $F(x_1,y_i) \leq c$ for any $i$. Then we get $n \cap_{i=1}^{n} \{x \in C : F(x,y_i) \leq c\} \neq \emptyset$. From the compactness of $C$, we have $\cap_{y \in D} \{x \in C : F(x,y) \leq c\} \neq \emptyset$, which means that there exists an $x_2 \in C$ and any $y \in D$ such that $F(x_2,y) \leq c$, or $\min_{x} \sup_{y} F(x,y) \leq \sup_{y} \min_{x} F(x,y)$. Since $F(x,y) \geq \min_{x} F(x,y)$ for $y \in D$, we have $\sup_{y} F(x,y) \geq \sup_{y} \min_{x} F(x,y)$ and also $\min_{x} \sup_{y} F(x,y) \geq \sup_{y} \min_{x} F(x,y)$. Therefore $\sup_{y} \min_{x} F(x,y) = \min_{x} \sup_{y} F(x,y)$. If $\sup_{y \in D} \min_{x \in C} F(x,y) = \infty$, it can be seen that the conclusion holds.

Q.E.D.

The above theorem is an extension of Sion's minimax theorem and Tuy's one. In the following remark an example illustrates EMT.

Remark 3 (a) In [6] Sion assumes that $F$ is upper semi-continuous and quasiconcave on $D$ under the condition that $D$ is compact, in addition to the conditions of EMT. He gets the conclusion that
\[
\min_{x \in C} \max_{y \in D} F(x,y) = \max_{y \in D} \min_{x \in C} F(x,y).
\]
Thus EMT is an extension of Sion's theorem.

(b) Tuy [8] assumes that $C$ and $D$ are convex. He shows that the conclusion
\[
\inf_{x \in C} \sup_{y \in D} F(x,y) = \sup_{x \in C} \inf_{y \in D} F(x,y)
\]
under the condition that $F$ is upper semi-continuous in $y$ in addition to conditions of EMT.

(c) Let $F(x,y) = f(x)g(y)$ for $(x,y) \in [-n,n] \times (-1,1)$, where $n \geq 1$ is integer, $f$ denotes the largest integer which is less than $|x|$. Here
\[
g(y) = y^2 + |y \sin \frac{\pi}{2y}|,
\]
where $y \in (-1,1)$. Then function $F$ satisfies Conditions (i) and (ii) of EMT. Since $\min_{y \in D} F(x,y) = 0$ and $\sup_{y \in D} F(x,y) = f(x)$, It follows that the conclusion of EMT holds.

References


